BANACH SPACES EMBEDDING ISOMETRICALLY INTO $L_p$ WHEN $0 < p < 1$

N. J. KALTON AND A. KOLDOBSKY

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ABSTRACT. For $0 < p < 1$ we give examples of Banach spaces isometrically embedding into $L_p$ but not into any $L_r$ with $p < r < 1$.

1. INTRODUCTION

It is a consequence of the Maurey-Nikishin factorization theory that every Banach space that embeds isomorphically into $L_p(0, 1)$ for some $0 < p < 1$ embeds into every $L_p(0, 1)$ for $0 < p < 1$ (see [10], [11] and [15] pp. 257ff.). It is, however, an open problem whether every Banach space that embeds isomorphically into $L_p$ for some $0 < p < 1$ must also embed isomorphically into $L_1$. This problem was formulated by Kwapien [8] in 1969; see [4] where it is shown that $X$ embeds into $L_1$ if and only if $\ell_1(X)$ embeds into $L_p$ for some $p < 1$. The isometric version of the problem asks: if $X$ isometrically embeds into $L_p$ for some $p < 1$ does it follow that $X$ isometrically embeds into $L_1$? This problem was solved negatively by the second author in 1996 [6] who showed that there is a Banach space embedding into $L_{1/2}$ but not into $L_1$. The construction also yielded an example of a Banach space embedding into $L_{1/4}$ but not $L_{1/2}$. Later, J. Borwein and the Center for Computational Mathematics at Simon Fraser University (unpublished) showed by computer methods that this algorithm yields examples of Banach spaces embedding into $L_{a/64}$ but not into $L_{(a+1)/64}$ for $a = 1, 2, \ldots, 63$.

The purpose of this note is to show that for every $0 < p < 1$ we can find a (real) Banach space $X$ embedding isometrically into $L_p$ but not into any $L_r$ for $p < r < 1$. The example constructed in [6] is finite-dimensional and is obtained by a perturbation method. By contrast, our spaces are infinite-dimensional and we use probabilistic ideas to construct them. It is, of course, true that an infinite-dimensional space $X$ embeds isometrically into $L_p$ if and only if every finite-dimensional subspace does, and so our methods also imply the existence of finite-dimensional examples.

We start in Section 2 by discussing the Plotkin-Rudin Equimeasurability and Uniqueness Theorems, which we need for our applications. In Section 3 we construct a very basic example, which we denote by $E_p$. This is the subspace of $L_p(0, 1)$
spanned by a constant function and a sequence of symmetric 1-stable random variables. It turns out that this space is a Banach space that is an absolute direct sum of a one-dimensional space and an isometric copy of $\ell_1$. The spaces $E_p$ provide our first family of examples. We show this by establishing that they have a certain extremal property (see Proposition 3.5).

In Section 4 we provide a second family of examples that are renormings of Hilbert spaces. For each $0 < p < 1$ we construct an example of such a space $X_p$ that embeds isometrically into $L_p$ but not into any $L_r$ for $r > p$. These spaces are absolute direct sums of two infinite-dimensional Hilbert spaces. We observe that these examples have the additional property that no subspace of finite codimension can be embedded into any $L_r$ where $r > p$.

2. REMARKS ON THE PLOTKIN-RUDIN THEOREM

In this section we discuss some essentially known results based on the Plotkin-Rudin theorems on isometric embeddings ([12], [13], [14]). See [7] for a discussion of these results.

We will always work in the setting of a Polish space $\Omega$ equipped with a nonatomic Borel probability measure $\mu$; we then say that $(\Omega, \mu)$ is a standard probability space. All functions are assumed to be Borel; if $f_1, \ldots, f_n$ are real Borel functions, then their joint distribution is the Borel measure on $\mathbb{R}^n$ given by $\mu \circ (f_1, \ldots, f_n)^{-1}$, and this will be denoted by $\rho_{f_1, \ldots, f_n}$.

We say that if $(\Omega_1, \mu_1)$ and $(\Omega_2, \mu_2)$ are two standard probability spaces, then a Borel map $\tau: \Omega_1 \to \Omega_2$ is a measure isomorphism if there is a Borel map $\sigma: \Omega_2 \to \Omega_1$ (an essential inverse) such that

- $\tau \sigma(\omega_1) = \omega_1$, $\mu_1$-a.e.;
- $\sigma \tau(\omega_2) = \omega_2$, $\mu_2$-a.e.;
- $\mu_2 \circ \sigma^{-1} = \mu_1$ and $\mu_1 = \mu_2 \circ \sigma^{-1}$.

If $\sigma$ is a measure isomorphism, then it may be modified on a set of $\mu_1$-measure zero to become a Borel isomorphism (i.e., an invertible Borel map). If $(\Omega, \mu)$ is a standard probability space, then there is always a Borel isomorphism $\sigma: \Omega \to [0, 1]$ such that $\lambda = \mu \circ \sigma^{-1}$, where $\lambda$ is Lebesgue measure.

We shall need the following fact.

**Proposition 2.1.** Let $(\Omega, \mu)$ be a standard probability space and suppose $K$ is a Polish space. Suppose $\sigma: \Omega \to K$ is a Borel map and $\nu = \mu \circ \sigma^{-1}$. Suppose there exists a Borel function $f$ on $\Omega$ such that $\rho_f = \mu \circ f^{-1}$ is nonatomic and $f$ is independent of $\sigma$ (i.e., $f$ is independent of the $\sigma$-algebra of sets of the form $\sigma^{-1}B$ for $B$ a Borel subset of $K$). Then there is a Borel map $\tau: \Omega \to [0, 1]$ so that $\sigma \times \tau$ is a measure isomorphism of $\Omega$ onto $(K \times [0, 1], \nu \times \lambda)$.

**Proof.** This is surely well known, but we do not know an explicit reference. It follows, for example, from Proposition 2.2 of [3] once one observes that $\sigma$ is anti-injective (i.e., if $B$ is a Borel set such that $\sigma$ is injective on $B$, then $\mu(B) = 0$)). It suffices by Lusin’s theorem to consider the case when $B$ is compact and $\sigma$ is continuous on $B$; then $\sigma$ is a Borel isomorphism of $B$ onto $\sigma(B)$. To see this, suppose $C_1, \ldots, C_N$ form a partition of $\mathbb{R}$ so that $\rho_f(C_k) = N^{-1}$. Let $B_k = B \cap f^{-1}(C_k)$. Then $\sigma(B_k)$ is Borel and $\mu(f^{-1}(C_k) \cap \sigma^{-1}(\sigma(B_k))) = N^{-1} \nu(\sigma(B_k))$. Hence $\mu(B) \leq N^{-1} \sum_{k=1}^N \nu(\sigma(B_k)) \leq N^{-1}$. $\square$
Let $X$ be a separable normed space, and $T : X \to L_p(\Omega, \mu)$ an isometric embedding. We say that $T$ is in canonical position if it satisfies the following two conditions:

- There exists $x \in X$ so that $Tx$ has full support, i.e., $\mu(Tx \neq 0) = 1$.
- There exists a function $f$ with $\rho f$ nonatomic such that $f$ is independent of the smallest $\sigma$-algebra $\Sigma$ such that each $Tx$ is $\Sigma$-measurable.

It is well known that if $X$ embeds into $L_p$, then there is also an embedding in canonical position.

Let us say that two embeddings $S : X \to L_p(\Omega_1, \mu_1)$ and $T : X \to L_p(\Omega_2, \mu_2)$ are equivalent if

$$\rho_{Sx_1, \ldots, Sx_n} = \rho_{Tx_1, \ldots, Tx_n} \quad x_1, \ldots, x_n \in X.$$ 

**Theorem 2.2** ([12], [13], [14]). (1) Suppose $p$ is not an even integer and $(\Omega, \mu_1)$ and $(\Omega_2, \mu_2)$ are two standard probability spaces. If $S : X \to L_p(\Omega, \mu_1)$ and $T : X \to L_p(\Omega, \mu_2)$ are isometric embeddings such that for some $x_0$ we have $Sx_0 = \chi_{\Omega_1}$ and $Tx_0 = \chi_{\Omega_2}$, then $S$ and $T$ are equivalent.

(2) If, in addition, $S$ and $T$ are in canonical position, then there exists a measure isomorphism $\sigma : \Omega_1 \to \Omega_2$ such that $\mu_2 = \mu_1 \circ \sigma^{-1}$ and $Tx \circ \sigma = Sx$ for $x \in X$.

**Proof.** (1) is the usual Plotkin-Rudin equimeasurability theorem [12], [13], [14], [7]. (2) is surely well known and follows directly from Proposition 2.1. Let us indicate one proof. Let $(x_n)$ be any dense sequence in $X$ and define, for $j = 1, 2$, $\tau_j : \Omega_j \to \mathbb{R}^n$ by $\tau_j(\omega_j) = (Sx_n(\omega_j))$ and $\tau_2(\omega_2) = (Tx_n(\omega_2))$. Then by (1) $\mu_1 \circ \tau_1^{-1} = \mu_2 \circ \tau_2^{-1} = \nu$, say. By Proposition 2.1 we can define Borel maps $\kappa_j : \Omega_j \to [0, 1]$ so that $\tau_j \times \kappa_j$ is a measure isomorphism of $(\Omega_j, \mu_j)$ onto $(\mathbb{R}^n \times [0, 1], \nu \times \lambda)$. The map $\sigma$ is then the composition $\alpha(\tau_1 \times \kappa_1)$ where $\alpha$ is the essential inverse of $\tau_2 \times \kappa_2$. 

If $T : X \to L_p(\Omega, \mu)$ is an isometric embedding, then we can always construct a new embedding by a change of density. If $\varphi$ is a nonvanishing Borel function, and $\int |\varphi|^p d\mu = 1$, we define $d\nu = |\varphi|^p d\mu$ and $T'x = \varphi^{-1}Tx$; then $T' : X \to L_p(\Omega, \nu)$ is a new isometric embedding. We then say that $T'$ is obtained from $T$ by a change of density.

**Theorem 2.3.** Suppose $p$ is not an even integer and $S : X \to L_p(\Omega, \mu)$ is an isometric embedding of canonical type. Then, if $T : X \to L_p(\Omega, \mu)$ is any other isometric embedding, there exists a nonvanishing Borel function $\varphi$ so that $T'$ is equivalent to $T$ where $T' : X \to L_p(\Omega, |\varphi|^p d\mu)$ is given by $T'x = \varphi^{-1}Sx$. (Thus $T$ is obtained from $S$ by a change of density.)

**Proof.** We assume $S$ is also of canonical type. Pick any $x_0$ with $\|x_0\| = 1$ so that $Sx_0 = f$ and $Tx_0 = g$ have full support. Consider $V_1x = f^{-1}Sx$ and $V_2x = g^{-1}Tx$. Then $V_1 : X \to L_p(\Omega, |f|^p d\mu)$ and $V_2 : X \to L_p(\Omega, |g|^p d\mu_1)$ are isometric embeddings with $V_1x_0 = \chi_{\Omega_1}$ and $V_2x_0 = \chi_{\Omega_2}$. It follows that there is a measure isomorphism $\sigma : \Omega \to \Omega_1$ so that $|g|^p \mu_1 = |f|^p \mu \circ \sigma^{-1}$ and $V_1x = V_2x \circ \sigma$. Now $Tx \circ \sigma = g \circ \sigma V_2x \circ \sigma = g \circ \sigma f^{-1}Sx$, and if $B$ is a Borel subset of $\mathbb{R}^n$ and $x_1, \ldots, x_n \in X$, then $\mu_1((Tx_1, \ldots, Tx_n) \in B) = \int |g \circ \sigma|^{-p} |f|^p \chi((Tx_1, \ldots, Tx_n) \in B) d\mu$ and the conclusion follows with $\varphi = f(g \circ \sigma)^{-1}$. 

\qed
Corollary 2.4. Let $X$ be a (separable) Banach space that embeds into $L_p$ where $p < 1$. Let $E$ be a subspace of $X$ and suppose $T : E \to L_p(\Omega, \mu)$ is a given isometric embedding. Then there is an isometric embedding $S : X \to L_p(\Omega_1, \mu_1)$ such that the restriction of $S$ to $E$ is equivalent to $T$.

Proof. Let $R : X \to L_p(\Omega, \mu)$ be any isometric embedding of canonical type. We note that $R$ is also of canonical type when restricted to $E$. In fact, it is only necessary to note that for every $x \in X$, $Rx$ has full support in $\Omega$. Indeed, if $Rx_0$ has full support, then

$$\int |Rx + tRx_0|^p d\mu \geq \|x\|^p + \|t\|^p \int_{Rx=0} |Rx_0|^p d\mu,$$

which contradicts the convexity of the norm unless $Rx$ has full support. It follows that we can make a change of density so that the new embedding $S$ restricted to $E$ is equivalent to $T$. \qed

A random variable $f$ is called symmetric $p$-stable $0 < p < 2$ if the Fourier transform of $\rho_f$ is of the form $e^{-c|t|^p}$ for some $c > 0$. We recall that there is an isometric embedding $T$ of $L_r(0, 1)$ into $L_p(0, 1)$ when $0 < p < r < 2$ so that each $Tf$ has a symmetric $r$-stable distribution. (See the remarks on p. 213 of [9].) We will call this the $r$-stable embedding. A particular case is that $L_1$ can be embedded into $L_p$ for $p < 1$ by mapping the basic vectors to a sequence of independent 1-stable random variables.

We will also need the following standard lemmas.

Lemma 2.5. Suppose $X$ is a Banach space and $T : X \to L_p(\Omega, \mu)$ is an isometric embedding where $0 < p < 1$. Then $\{\|Tx\|^p : \|x\| \leq 1\}$ is equi-integrable.

Proof. This follows by contradiction: if $\{\|Tx\|^p : \|x\| \leq 1\}$ is not equi-integrable, then (see [15] p. 137) there exists $\delta > 0$, a disjoint sequence of Borel sets $(A_k)$ and $x_k$ with $\|x_k\| \leq 1$ so that $\int_{A_k} |Tx_k|^p d\mu > \delta^p$. Then by an application of Khintchine's inequality we have for suitable $c > 0$,

$$N^p \geq \text{Ave}_{\epsilon_k = \pm 1} \left\| \sum_{k=1}^N \epsilon_k x_k \right\|^p \geq c^p \int \left( \sum_{k=1}^N |Tx_k|^2 \right)^{\frac{p}{2}} d\mu \geq c^p N \delta^p,$$

and for large enough $N$ this gives a contradiction. \qed

Lemma 2.6. Let $F : \mathbb{R}^{m+1} \to \mathbb{R}$ be a continuous function. Suppose $g_1, \ldots, g_m$ are measurable functions on $(\Omega, \mu)$ and that $(f_n)_{n=1}^{\infty}$ is any sequence of identically distributed independent random variables with common distribution $p = p_{f_n}$. If the functions $F(g_1, \ldots, g_m, f_n)$ are equi-integrable for $n = 1, 2, \ldots$, then $F(g_1, \ldots, g_m, f_0)$ is integrable and

$$\lim_{n \to \infty} \int F(g_1, \ldots, g_m, f_n) d\mu = \int_{\Omega} \int_{\mathbb{R}} F(g_1, \ldots, g_m, t) d\rho(t) d\mu.$$
**Proof.** First, suppose that $F, g_1, \ldots, g_m, f_n$ are all bounded functions. Note that for $a_1, \ldots, a_m, b = 0, 1, 2, \ldots$, we have

$$
\lim_{n \to \infty} \int g_1^{a_1} g_2^{a_2} \cdots g_m^{a_m} f_n^b d\mu = \left( \int g_1^{a_1} \cdots g_m^{a_m} d\mu \right) \left( \int t^b d\rho(t) \right)
$$

since the $f_n^b$ converge weakly in $L_2$ to the constant $\int f_n^b d\mu$. Hence for any polynomial $P$,

$$
\lim_{n \to \infty} \int P(g_1, \ldots, g_m, f_n) d\mu = \int_{\Omega} \int_{\mathbb{R}} P(g_1, \ldots, g_m, t) d\rho(t) d\mu.
$$

If $|f_n|, |g_1|, \ldots, |g_m| \leq M$ and $\varepsilon > 0$, we approximate $F$ on the cube $[-M, M]^{m+1}$ by a polynomial $P$ so that the range of

$$
|P(x_1, \ldots, x_m, y) - F(x_1, \ldots, x_m, y)| \leq \varepsilon
$$

then it follows that we have

$$
\lim_{n \to \infty} \int F(g_1, \ldots, g_m, f_n) d\mu - \int F(g_1, \ldots, g_m, t) d\rho(t) d\mu \leq \varepsilon.
$$

Letting $\varepsilon \to 0$ we obtain (2.1) under the assumption that $f, g_1, \ldots, g_m$ are bounded.

Next assume that $|F|$ is bounded by $M$, but allow $f$ and $g_j$ to be unbounded. For any $m \in \mathbb{N}$, let $f_{k,n} = f_n \mathbf{1}_{|f_n| \leq k}$, and $g_{k,j} = g_j \mathbf{1}_{|g_j| \leq k}$. Then for $n \geq 0$, we have

$$
\left| \int F(g_1, \ldots, g_m, f_n) d\mu - \int F(g_{k,1}, \ldots, g_{k,m}, f_{k,n}) d\mu \right| \leq 2M \left( \mu(|f_0| > k) + \sum_{j=1}^m \mu(|g_j| > k) \right).
$$

Since we have (2.1) for bounded $f_n, g_1, \ldots, g_m$, we obtain the result in general for $F$ bounded.

Now assume that $F(g_1, \ldots, g_m, f_n)$ is equi-integrable and let $F_k = \min(F, k)$ if $F > 0$ and $F_k = \max(F, -k)$ if $F \leq 0$. Then

$$
\lim_{k \to \infty} \int_{\Omega} |F_k(g_1, \ldots, g_m, f_n)| d\mu = \int_{\Omega} \int_{\mathbb{R}} |F_k(g_1, \ldots, g_m, t)| d\rho(t) d\mu,
$$

and it follows that $F(g_1, \ldots, g_m, t)$ is integrable with respect to $\mu \times \rho$. We also have

$$
\lim_{k \to \infty} \int F_k(g_1, \ldots, g_m, f_n) d\mu = \int F(g_1, \ldots, g_m, f_n) d\mu
$$

uniformly in $k$, so that the general result follows by uniform convergence.

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**3. THE SPACES $E_p$ FOR $0 < p < 1$**

**Lemma 3.1.** Suppose $0 < p < 1$. Then for $-\pi/2 < \theta \leq \pi/2$,

$$
(3.1) \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|x \cos \theta + \sin \theta|^p}{1 + x^2} dx = \frac{\cos p\theta}{\cos p\pi/2}.
$$

**Proof.** We consider the case $\theta \neq 0$ of (3.1); the other cases are similar. We define $f(z)$ to be the branch of $(z \cos \theta + \sin \theta)^p$ defined in $C \setminus \{ -\tan \theta - it : t \geq 0 \}$ such that $f(x)$ is real and positive if $x \geq -\tan \theta$. Now by a routine contour integration we have

$$
\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{1 + x^2} dx = e^{ip(\frac{\pi}{2} - \theta)}.
$$
Taking imaginary parts gives
\[
\frac{1}{\pi} \int_{-\tan \theta}^{-\tan \theta} \frac{|x \cos \theta + \sin \theta|^p}{1 + x^2} \, dx = \frac{\sin p(\frac{\pi}{2} - \theta)}{\sin p\pi}.
\]
Taking real parts and substituting in, we have
\[
\frac{1}{\pi} \int_{-\tan \theta}^{\tan \theta} \frac{|x \cos \theta + \sin \theta|^p}{1 + x^2} \, dx = \cos p(\frac{\pi}{2} - \theta) - \cot p\pi \sin p(\frac{\pi}{2} - \theta) = \frac{\sin p(\frac{\pi}{2} + \theta)}{\sin p\pi}.
\]
Combining gives (3.1). \(\square\)

**Lemma 3.2.** Let \( M : \mathbb{C} \to [0, \infty) \) be a continuous nonnegative function. Suppose \( M \) is subharmonic and positively homogeneous (i.e., \( M(az) = aM(z) \) for \( a \geq 0 \)). Then \( M \) is convex.

**Proof.** First, we assume that \( M \) is \( C^2 \) on \( \mathbb{C} \setminus \{0\} \). Then for any \( z = x + iy \neq 0 \) the second derivative of \( M \) is given by a symmetric \( 2 \times 2 \) matrix that has rank at most one. To see this, note that the equation \( M(az) = aM(z) \) implies on differentiation by \( a \), and then by setting \( a = 1 \) that
\[
x \frac{\partial^2 M}{\partial x^2} + y \frac{\partial^2 M}{\partial y^2} = 0,
\]
and hence the second derivative has determinant zero. Thus if \( \nabla^2 M \geq 0 \), the second derivative of \( M \) is nonnegative at \( z \). This shows that \( M \) is convex.

If \( M \) is not \( C^2 \), then we may approximate it by functions of the form
\[
\tilde{M}(z) = \int_0^{2\pi} \varphi(\theta)M(ze^{i\theta}) \, d\theta
\]
where \( \varphi \) is smooth and nonnegative. Each such function \( \tilde{M} \) is convex and so \( M \) is convex. \(\square\)

Now, for \( 0 < p < 1 \), let us define a function \( N_p(x, y) \) on \( \mathbb{R}^2 \) by setting
\[
N_p(x, y) = r^p \left( \frac{\cos \theta}{\cos \frac{\pi}{2}} \right)\frac{1}{\frac{\pi}{2}},
\]
whenever \( x \geq 0 \) and \( x = r \cos \theta \), \( y = r \sin \theta \) with \( r \geq 0 \), \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \). Then extend \( N_p \) to be an even function, i.e., so that \( N_p(x, y) = N_p(-x, -y) \) whenever \( x \leq 0 \).

Note also that \( N_p(0, 1) = 1 \) but \( N_p(1, 0) = (\sec \frac{\pi}{2})^{\frac{1}{p}} \).

**Lemma 3.3.** If \( 0 < p < 1 \), \( N_p \) is an absolute norm on \( \mathbb{R}^2 \); i.e., \( N_p \) is a norm so that \( N_p(x, y) = N_p(|x|, |y|) \).

**Proof.** Let \( u(z) = r^p \cos \theta \) when \( z = re^{i\theta} \) with \( -\pi < \theta \leq \pi \). Then \( u \) is subharmonic and \( N_p(x, y) = (\sec \frac{\pi}{2})^{\frac{1}{p}} \left( \max(u(z), u(-z)) \right)^{\frac{1}{p}} \) where \( z = x + iy \). Hence \( N_p \) is a norm by Lemma 3.2. The fact that \( N_p \) is absolute is trivial. \(\square\)
We now define a Banach space $E_p$ for $0 < p < 1$. We define this to be the space $\ell_1 \oplus \mathbb{R}$ with the norm $\| (x, y) \|_{E_p} = N_p(\|x\|, \|y\|)$.

Let $(f_n)$ be a sequence of independent 1-stable random variables on some probability space $(\Omega, \mu)$ so that $\int e^{itf_n} d\mu = e^{-|t|}$. Then for any finitely nonzero sequence $(\xi_n)_{n=1}^{\infty}$ and any $\eta$ we have

$$\| \sum_{n=1}^{\infty} \xi_n f_n + \eta \|_p = N_p(\sum_{n=1}^{\infty} |\xi_n|, |\eta|).$$

It follows that:

**Proposition 3.4.** $E_p$ is isometric to a closed subspace of $L_p$ for $0 < p < 1$.

Next, we show that $E_p$ cannot be embedded into $L_r$ for any $p < r < 1$. To do this we introduce the quantity

$$a_p = \lim_{t \to 0} \frac{N((\cos \frac{p\pi}{2})\frac{1}{t} t, 1) - 1}{\sqrt{\sin \frac{p\pi}{2}}}. $$

**Proposition 3.5.** Suppose $0 < p < 1$ and that $(g_n)$ is a sequence in $L_p(\Omega, \mu)$ that is 1-equivalent to the standard unit vector basis of $\ell_1$. Suppose $h \in L_p$ and $\|h\|_p = 1$. Then

$$\lim_{n \to \infty} \|h + t g_n\|_p \geq N_p((\cos \frac{p\pi}{2})\frac{1}{t} t, 1) \geq 1 + a_p |t|.$$

**Proof.** It follows from Theorem 2.3 and Corollary 2.4 that it suffices to consider the case when $g_n = (\cos \frac{p\pi}{2})\frac{1}{t} f_n$ where $(f_n)$ is a sequence of independent 1-stable random variables with $\int e^{itf_n} d\mu = e^{-|t|}$. We now apply Lemma 2.6:

$$\lim_{n \to \infty} \int |h + \tau f_n|^p d\mu = \frac{1}{\pi} \int_{-\infty}^{\infty} \int |h(\omega) + \tau x|^p \frac{1}{1 + x^2} dx d\mu(\omega)$$

$$= \int N_p(\tau, h(\omega))^p d\mu(\omega).$$

Now since $N_p$ is an absolute norm,

$$\int N_p(\tau, 1)^{1-p} N_p(\tau, h(\omega))^p d\mu \geq \int N_p(\tau, |h(\omega)|^p) d\mu$$

and hence

$$\int N_p(\tau, h(\omega))^p d\mu(\omega) \geq N_p(\tau, 1)^p.$$

This gives us the first inequality.

For the second part observe that

$$\lim_{t \to 0^+} \frac{N_p((\cos \frac{p\pi}{2})\frac{1}{t} t, 1) - 1}{t} = a_p$$

and use the fact that $N_p$ is a norm. \qed

**Theorem 3.6.** For $0 < p < 1$ the space $E_p$ is a Banach space isometric to a subspace of $L_p$, which is not isometric to a subspace of any $L_r$ for $r > p$. 
\textbf{Proof.} This is immediate from Proposition 3.5 once we show that the function \( p \to a_p \) is strictly increasing on \((0,1)\). Since \( L_r \) embeds into \( L_p \) when \( p < r \) and \( E_r \) embeds into \( L_r \), it is clear from Proposition 3.5 that \( p \to a_p \) is increasing. This function is non-constant since \( \lim_{p \to 1} a_p = 1 \) and \( a_{1/2} = 1/2 \). Since it is a real-analytic function, it must therefore be strictly increasing. \( \Box \)

\textbf{Remark.} It would be interesting to estimate the smallest integer \( n = n(r,p) \) so that the \( n \)-dimensional subspace of \( E_p \) spanned by the constant function and \( f_1, \ldots, f_{n-1} \) fails to embed into \( L_r \). We also mention that the span of the constant function and the sequence \( \{f_n\} \) is isomorphic to the Ribe space \([2]\); for similar examples involving \( p \)-stable random variables see \([1]\).

4. Perturbed Hilbert Spaces

In this section we give an alternative construction of examples that are isomorphic but not isometric to Hilbert spaces.

\textbf{Lemma 4.1.} Suppose \( 0 < p < 1 \). Then there exists \( \epsilon(p) > 0 \) so that if \( 0 < a < \epsilon(p) \), the following equation defines an absolute norm on \( \mathbb{R}^2 \):

\[ N(x,y)^p = \frac{1}{2}(x^2 + (1 + a)^{\frac{2}{p}}y^2)^{\frac{p}{2}} + (x^2 + (1 - a)^{\frac{2}{p}}y^2)^{\frac{p}{2}}. \]

\textbf{Proof.} This follows easily from Lemma 3.2 since, if \( a \) is small enough, \( (x^2 + (1 + a)^{\frac{2}{p}}y^2)^{\frac{p}{2}} \) and \( (x^2 + (1 - a)^{\frac{2}{p}}y^2)^{\frac{p}{2}} \) are both subharmonic. \( \Box \)

\textbf{Theorem 4.2.} Suppose \( 0 < p < 1 \) and \( N \) is given by (4.1). Then the space \( X = \ell_2 \oplus_N \ell_2 \) embeds into \( L_p \) but does not embed into any space \( L_r \) where \( r > p \).

\textbf{Proof.} We first establish an embedding of \( X \) into \( L_p(\Omega, \mu) \). Let \( (e_n) \) and \( (e'_n) \) be the canonical orthonormal bases of the two factors of \( X \). Let \( (f_n), (g_n) \) be two mutually independent sequences of independent normalized Gaussians; we denote by \( \gamma \) their common distribution so that \( d\gamma(t) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{t^2}{2})dt \). Let \( E \) be a Borel set independent of \( (f_n, g_n) \) with \( \mu E = \frac{1}{2} \). Let \( h = (1 + a)^{\frac{1}{p}} \chi_E + (1 - a)^{\frac{1}{p}} \chi_{\bar{E}} \).

We define our embedding by

\[ T e_n = b_1 f_n, \]
\[ T e'_n = b_1 h g_n \]

where \( b_1^{-p} = \|f_n\|_p^p = \int |t|^p d\gamma(t) \). We can and do assume that \( T \) is of canonical type. Suppose \( (\xi_n), (\eta_n) \) are two finitely nonzero sequences of reals. Then

\[ \int_\Omega |\sum_{n=1}^\infty \xi_n T e_n + \sum_{n=1}^\infty \eta_n T e'_n|^p d\mu = b_1^p \int_\Omega |\sum_{n=1}^\infty \xi_n f_n + \sum_{n=1}^\infty \eta_n g_n|^p d\mu \]

\[ = \int_\Omega (\sum_{n=1}^\infty |\xi_n|^2 + h^2 \sum_{n=1}^\infty |\eta_n|^2)^{\frac{p}{2}} d\mu \]

\[ = N((\sum_{n=1}^\infty |\xi_n|^2)^{\frac{p}{2}}, (\sum_{n=1}^\infty |\eta_n|^2)^{\frac{p}{2}})^p. \]

Now assume \( X \) also embeds isometrically into \( L_r \) for some \( p < r < 2 \). Then \( X \) can also be embedded into \( L_p \) by an \( r \)-stable embedding \( S \). In view of Theorem 2.3, it may be assumed that \( S \) is obtained from \( T \) by a change of density, i.e., there exists
a nonvanishing Borel function $\varphi$ with $\|\varphi\|_p = 1$ such that $S : X \to L_p(\Omega, |\varphi|^p d\mu)$ is given by $Sx = \varphi^{-1}Tx$. Fix any $0 < q < p$. It follows for an appropriate choice of $b_2$ that the map $S'x = b_2Sx$ embeds $X$ into $L_q(\Omega, |\varphi|^q d\mu)$. Now we make a further change of density. Let $b_2^q = \int_\Omega |\varphi|^{p-q} d\mu$ and define $\psi = b_3^{-1}\varphi^{-1}$. Let $R : X \to L_q(\Omega, |\psi|^q |\varphi|^p d\mu)$ by $Rx = \psi^{-1}S'x$. Then $Rx = b_3b_2Tx$. Let $b_0 = b_3b_2b_1$.

We now use Lemma 2.5 and Lemma 2.6. Suppose $x, y \in \mathbb{R}$.

$$N(x, y)^q = b_0^q \lim_{m \to \infty} \lim_{n \to \infty} \int_\Omega |xf_m + yh_g|^q |\varphi|^p |\psi|^q d\mu$$

$$= b_0^q \lim_{m \to \infty} \int_\Omega \int_\mathbb{R} |xf_s + yth|^q d\gamma(t) |\varphi|^p |\psi|^q d\mu$$

$$= b_0^q \int_\Omega \int_\mathbb{R} |xs + yth|^q d\gamma(s) d\gamma(t) |\varphi|^p |\psi|^q d\mu$$

$$= b_0^q \int_\mathbb{R} t^q d\gamma(t) \int_\Omega (x^2 + y^2h^2)^{\frac{q}{2}} |\varphi|^p |\psi|^q d\mu.$$  

Since $h$ takes only the values $(1 \pm a)^{\frac{1}{2}}$, this implies that we can find positive constants $c_1, c_2$ so that for all $x, y$,

$$N(x, y)^q = c_1(x^2 + (1 - a)^2y^2)^{\frac{q}{2}} + c_2(x^2 + (1 + a)^2y^2)^{\frac{q}{2}}.$$  

Since $N(1, 0) = N(0, 1) = 1$, this requires

$$c_1 + c_2 = 1,$$

$$c_1(1 - a)^{\frac{q}{2}} + c_2(1 + a)^{\frac{q}{2}} = 1.$$  

Note also that

$$\lim_{t \to 0} \frac{N(1, t)^2 - 1}{t^2} = \frac{1}{2}((1 + a)^{\frac{q}{2}} + (1 - a)^{\frac{q}{2}})$$

$$= c_1(1 - a)^{\frac{q}{2}} + c_2(1 + a)^{\frac{q}{2}}.$$  

It is clearly impossible to satisfy these three conditions. This contradiction shows that we cannot embed $X$ into $L_r$ for any $r > p$. \qed

**Remark.** It is worth remarking in this context that it is unknown if there is an infinite-dimensional space $X$ that embeds isometrically into $L_p$ and $L_r$ where $p < 2 < r$ and is isomorphic but not isometric to a Hilbert space (see [5]).

**References**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI-COLUMBIA, COLUMBIA, MISSOURI 65211
E-mail address: nigel@math.missouri.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI-COLUMBIA, COLUMBIA, MISSOURI 65211
E-mail address: koldobsk@math.missouri.edu