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ON NONATOMIC BANACH LATTICES AND HARDY SPACES

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ABSTRACT. We are interested in the question when a Banach space X with an unconditional basis is isomorphic (as a Banach space) to an order-continuous nonatomic Banach lattice. We show that this is the case if and only if X is isomorphic as a Banach space with $X(\ell_2)$. This and results of Bourgain are used to show that spaces $H_1(\mathbf{T}^n)$ are not isomorphic to nonatomic Banach lattices. We also show that tent spaces introduced by Coifman, Meyer, and Stein are isomorphic to $\operatorname{Rad} H_1$.

1. INTRODUCTION

There is a natural distinction between sequence spaces and function spaces in functional analysis; as an example, let us point out the subtitles of two volumes of [15] and [16]. In this paper we use the term *sequence space* to indicate a space with the structure of an atomic Banach lattice and the term *function space* to indicate a space with the structure of a nonatomic Banach lattice. Many classical function spaces (e.g., the spaces $L_p[0, 1]$ for $1 < p < \infty$ [22] or [16]) have unconditional bases and hence are isomorphic as Banach spaces to sequence spaces (atomic Banach lattices). On the other hand, $L_1[0, 1]$ has no unconditional basis ([22] or [16]) and in the other direction the sequence spaces ℓ_p for $p \neq 2$ are not isomorphic to any nonatomic Banach lattice [1]. In this note we discuss a general criterion for deciding whether a Banach space with an unconditional basis (i.e., a sequence space) can be isomorphic to a nonatomic Banach lattice (i.e., a function space). Our main result (Theorem 2.4) gives a simple necessary and sufficient condition for an atomic Banach lattice X to be isomorphic to an order-continuous nonatomic Banach lattice; of course, if X contains no copy of c_0, every Banach lattice structure on X is order-continuous.

Our main motivation is to study the Hardy space $H_1(\mathbf{T})$. After the discovery that the space $H_1(\mathbf{T})$ has an unconditional basis [17] it becomes natural to

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investigate if $H_1(\mathbf{T})$ is isomorphic to a nonatomic Banach lattice. Applying Theorem 2.4 to H_1 and using some previous results of Bourgain [2, 3] we show that H_1 is not isomorphic to any nonatomic Banach lattice and furthermore that $H_1(\mathbf{T}^n)$ is not isomorphic to a nonatomic Banach lattice for any natural number n.

We conclude by showing that the space $\operatorname{Rad} H_1$ or $H_1(\ell_2)$ is isomorphic to the tent spaces T^1 introduced by Coifman, Meyer, and Stein [4].

2. Lattices with unconditional bases

Our terminology about Banach lattices will agree with [16]; we also refer the reader to [9, 10] for the isomorphic theory of nonatomic Banach lattices.

A (real) Banach lattice X is called *order-continuous* if every order-bounded increasing sequence of positive elements is norm convergent. Any Banach lattice not containing c_0 is automatically order-continuous.

For any order-continuous Banach lattice X we can define an associated Banach lattice $X(\ell_2)$ (using the Krivine calculus [16, pp. 40–42]) as the space of sequences $(x_n)_{n=1}^{\infty}$ in X such that $(\sum_{k=1}^{n}|x_k|^2)^{1/2}$ is order-bounded (and hence is a convergent sequence) in X. $X(\ell_2)$ becomes an order-continuous Banach lattice when normed by $\|(x_n)\| = \|(\sum_{n=1}^{\infty}|x_n|^2)^{1/2}\|$.

If X has nontrivial cotype then $X(\ell_2)$ is naturally isomorphic to the space $\operatorname{Rad} X$ which is the subspace of $L_2([0,1];X)$ of functions of the form $\sum_{n=1}^{\infty} x_n r_n$ where (r_n) is the sequence of Rademacher functions. The space $\operatorname{Rad} X$ is clearly an isomorphic invariant of X; so if two Banach lattices X and Y with nontrivial cotype are isomorphic, it follows easily that $X(\ell_2)$ and $Y(\ell_2)$ are isomorphic. However, this result holds in general by a result of Krivine [13] or [16, Theorem 1.f.14].

Theorem 2.1. *If X, Y are order-continuous Banach lattices and $T: X \longrightarrow Y$ is a bounded linear operator, then if $(x_n) \in X(\ell_2)$ we have $(Tx_n) \in Y(\ell_2)$ and*

$$\|(T(x_n))\|_{Y(\ell_2)} \le K_G\|T\|\|(x_n)\|_{X(\ell_2)}.$$

Here, as usual, K_G denotes the Grothendieck constant.

Proof. Essentially this is Krivine's theorem, but we do need to show that if $(x_n) \in X(\ell_2)$ then $(Tx_n) \in Y(\ell_2)$. To see this we show that $(\sum_{k=1}^{n}|Tx_k|^2)^{1/2}$ is norm-Cauchy. In fact, if $m > n$ then

$$\left\| \left(\sum_{k=1}^{m}|Tx_k|^2\right)^{1/2} - \left(\sum_{k=1}^{n}|Tx_k|^2\right)^{1/2} \right\|_Y \le \left\| \left(\sum_{k=n+1}^{m}|Tx_k|^2\right)^{1/2} \right\|_Y$$
$$\le K_G\|T\| \left\| \left(\sum_{k=n+1}^{m}|x_k|^2\right)^{1/2} \right\|_X \le K_G\|T\| \left\| \left(\sum_{k=n+1}^{\infty}|x_k|^2\right)^{1/2} \right\|_X$$

which converges to zero as $n \to \infty$ by the order-continuity of X. □

Corollary 2.2. *If two order-continuous Banach lattices X and Y are isomorphic as Banach spaces, then $X(\ell_2)$ and $Y(\ell_2)$ are isomorphic as Banach spaces.*

If X is a separable order-continuous nonatomic Banach lattice then X can be represented as (i.e., is linearly and order isomorphic with) a Köthe function space on $[0, 1]$ in such a way that $L_\infty[0, 1] \subset X \subset L_1[0, 1]$ and inclusions are continuous. It will then follow that L_∞ is dense in X, and the dual of X can be represented as a space of functions, namely, $X^* = \{f \in L_1 : \int |fg|\, dt < \infty$ for every $g \in X\}$.

Now we are ready to state our main result. Let us observe that for rearrangement invariant function spaces on $[0, 1]$ this result was proved in [9] (cf. also [16, 2.d]) by a quite different technique.

Theorem 2.3. *Let X be an order-continuous, nonatomic Banach lattice with an unconditional basis. Then X is isomorphic as a Banach space to $X(\ell_2)$.*

Proof. We will represent X as a Köthe function space on $[0, 1]$ as described above. Suppose $(\phi_n)_{n=1}^\infty$ is a normalized unconditional basis of X. Then there is an order-continuous atomic Banach lattice Y which we identify as a sequence space and operators $U: X \longrightarrow Y$ and $V: Y \longrightarrow X$ such that $UV = I_Y$, $VU = I_X$, and $U(\phi_n) = e_n$ for $n = 1, 2, \dots$, where e_n denotes the canonical basis vectors in Y. We can regard Y^* as a space of sequences and further suppose that $\|e_n\|_{Y^*} = \|e_n\|_Y = 1$. We will identify $Y(\ell_2)$ as a space of double sequences with canonical unconditional basis $(e_{mn})_{m,n=1}^\infty$; thus for any finitely nonzero sequence we have $\|\sum a_{mn}e_{mn}\|_{Y(\ell_2)} = \|\sum_m(\sum_n |a_{mn}|^2)^{1/2}e_m\|_Y$.

Let r_n denote the Rademacher functions and for each fixed $f \in X$ note that $(r_n f)$ converges weakly to zero, since for $g \in X^*$ we have $\lim_{n\to\infty} \int r_n f g\, dt = 0$. In particular, we have for each $m \in \mathbf{N}$ that $(r_n\phi_m)$ converges weakly to zero. It follows by a standard gliding hump technique that if $\eta = (2\|U\|\|V\|)^{-1}$ then we can find for each $(m, n) \in \mathbf{N}^2$ an integer $k(m, n)$ and disjoint subsets (A_{mn}) of $\mathbf{N}$ so that $\|U(\phi_m r_{k(m,n)})\chi_{A_{mn}} - U(\phi_m r_{k(m,n)})\|_Y \le \eta$.

Identifying Y^* as a sequence space, we let $\psi_m = U^*(e_m)$ and then define $v_{m,n} = \chi_{A_{mn}}U(\phi_m r_{k(m,n)}) \in Y$ and $v^*_{m,n} = \chi_{A_{mn}}V^*(\psi_m r_{k(m,n)}) \in Y^*$. Now suppose (a_{mn}) is a finitely nonzero double sequence. Then

$$\begin{aligned}
\left\|\sum_{m,n} a_{mn}v_{mn}\right\|_Y &\le \left\|\left(\sum_{m,n}|a_{mn}|^2|U(\phi_m r_{k(m,n)})|^2\right)^{1/2}\right\|_Y \\
&\le K_G\|U\|\left\|\left(\sum_{m,n}|a_{mn}|^2|\phi_m r_{k(m,n}|^2\right)^{1/2}\right\|_X \\
&= K_G\|U\|\left\|\left(\sum_m\left(\sum_n|a_{mn}|^2\right)|\phi_m|^2\right)^{1/2}\right\|_X \\
&= K_G\|U\|\left\|\left(\sum_m\left(\sum_n|a_{mn}|^2\right)|Ve_m|^2\right)^{1/2}\right\|_X \\
&\le K_G^2\|U\|\|V\|\left\|\left(\sum_m\left(\sum_n|a_{mn}|^2\right)|e_m|^2\right)^{1/2}\right\|_Y \\
&= K_G^2\|U\|\|V\|\left\|\sum_{m,n}a_{mn}e_{mn}\right\|_{Y(\ell_2)}.
\end{aligned}$$

Here we have used Krivine's theorem twice. It follows that we can define a linear operator $S : Y(\ell_2) \to Y$ by $Se_{mn} = v_{mn}$ and then $\|S\| \le K_G^2\|U\|\|V\|$.

Similar calculations yield that for any finitely nonzero double sequence (b_{mn}) we have

$$\left\|\sum_{m,n} b_{mn}v_{mn}^*\right\|_{Y^*} \le K_G^2\|U\|\|V\| \left\|\sum_m \left(\sum_n |b_{mn}|^2\right)^{1/2} e_m\right\|_{Y^*}.$$

Suppose then $y \in Y$ and set $a_{mn} = \langle y, v_{mn}^*\rangle$. Let F be a finite subset of $\mathbf{N}^2$. Let $\alpha_m = (\sum_n \chi_F(m,n)|a_{mn}|^2)^{1/2}$, and suppose the finitely nonzero sequence (β_m) is chosen so that $\|\sum \beta_m e_m\|_{Y^*} = 1$ and $\sum \beta_m\alpha_m = \|\sum \alpha_m e_m\|_Y$. Then, with the convention that $0/0 = 0$,

$$\begin{aligned}\left\|\sum_{(m,n)\in F} a_{mn}e_{mn}\right\|_{Y(\ell_2)} &= \sum_m \beta_m\alpha_m \\ &= \sum_{(m,n)\in F} \beta_m\alpha_m^{-1}|a_{mn}|^2 = \left\langle y, \sum_{(m,n)\in F} \beta_m\alpha_m^{-1}a_{mn}v_{mn}^*\right\rangle \\ &\le \|y\|_Y \left\|\sum_{(m,n)\in F} \beta_m\alpha_m^{-1}a_{mn}v_{mn}^*\right\|_{Y^*} \le K_G^2\|U\|\|V\|\|y\|_Y.\end{aligned}$$

Thus for each F the map $T_F: Y \to Y(\ell_2)$ given by

$$T_F y = \sum_{(m,n)\in F} \langle y, v_{mn}^*\rangle e_{mn}$$

has norm at most $K_G^2\|U\|\|V\|$. More generally, we have

$$\|T_F y\| \le K_G^2\|U\|\|V\|\|\chi_{A_F}y\|$$

where $A_F = \bigcup_{(m,n)\in F} A_{mn}$.

It follows that for each $y \in Y$ the series $\sum_{m,n}\langle y, v_{mn}^*\rangle e_{mn}$ converges (unconditionally) in $Y(\ell_2)$. We can thus define an operator $T: Y \to Y(\ell_2)$ by $Ty = \sum_{m,n}\langle y, v_{mn}^*\rangle e_{mn}$ and $\|T\| \le K_G^2\|U\|\|V\|$.

Now notice that $TS(e_{mn}) = c_{mn}e_{mn}$ where $c_{mn} = \langle v_{mn}, v_{mn}^*\rangle$. But

$$\begin{aligned}\langle v_{mn}, v_{mn}^*\rangle &= \langle v_{mn}, V^*\psi_m r_{k(m,n)}\rangle \\ &\ge \langle U(\phi_m r_{k(m,n)}), V^*(\psi_m r_{k(m,n)})\rangle - \eta\|V\|\|\psi_m\|_{X^*} \\ &= \langle \phi_m, \psi_m\rangle - \eta\|V\|\|\psi_m\|_{X^*} \ge 1 - \eta\|V\|\|U\| \ge 1/2.\end{aligned}$$

Thus TS is invertible, so it follows that $Y(\ell_2)$ is isomorphic to a complemented subspace of Y. It then follows from the Pełczyński decomposition technique that $Y \sim Y(\ell_2)$; more precisely, $Y \sim Y(\ell_2) \oplus W$ for some W and so $Y \sim Y(\ell_2)\oplus(Y(\ell_2)\oplus W) \sim Y(\ell_2)\oplus Y \sim Y(\ell_2)$. The conclusion follows from Corollary 2.2. □

Remark. The order continuity of the Banach lattice X is essential. In [14] a nonatomic Banach lattice X (actually an M-space) was constructed which is isomorphic to c_0. In particular, X has an unconditional basis but is not isomorphic to $X(\ell_2)$.

Theorem 2.4. *Let Y be a Banach space with an unconditional basis. Then Y is isomorphic to an order-continuous nonatomic Banach lattice if and only if $Y \sim Y(\ell_2)$.*

Remark. Here again we regard Y as an order-continuous Banach lattice.

Proof. One direction follows immediately from Theorem 2.3 and Corollary 2.2. For the other direction, it is only necessary to show that if $Y \sim Y(\ell_2)$ then Y is isomorphic to order-continuous nonatomic Banach lattice. To this end we introduce the space $Y(L_2)$; this is the space of sequences of functions (f_n) in $L_2[0, 1]$ such that $\sum \|f_n\|_2 e_n$ converges in Y. We set $\|(f_n)\|_{Y(L_2)} = \|\sum \|f_n\|_2 e_n\|_Y$. It is clear that $Y(L_2)$ is an order-continuous Banach lattice. Now if (g_n) is an orthonormal basis of L_2, we define $W : Y(\ell_2) \to Y(L_2)$ by $W(\sum_{m,n} a_{mn} e_{mn}) = (\sum_n a_{mn} g_n)_{m=1}^{\infty}$, and it is easy to see that W is an isometric isomorphism. □

Proposition 2.5. *If X is a nonatomic order-continuous Banach lattice with unconditional basis, then $X \sim X \oplus X$ and $X \sim X \oplus R$.*

Proof. Both facts follow from Theorem 2.3. □

Note that for spaces with unconditional basis both properties do not hold in general (see [5, 6]).

Proposition 2.6. *Let X be an order continuous nonatomic Banach lattice with an unconditional basis, and let Y be a complemented subspace of X. Assume that Y contains a complemented subspace isomorphic to X. Then $X \sim Y$.*

Proof. The proof is a repetition of the proof of Proposition 2.d.5. of [16]. □

3. Hardy spaces

We recall that $H_1(\mathbf{T}^n)$ is defined to be the space of boundary values of functions f holomorphic in the unit disk $\mathbf{D}$ and such that

$$\sup_{0<r<1} \int_{\mathbf{T}^n} |f(re^{it_1}, re^{it_2}, \dots, re^{it_n})| \, dt_1 dt_2 \cdots dt_n < \infty.$$

The basic theory of such spaces is explained in [18].

Let us consider first the case $n = 1$. Then $\Re H_1$ is defined be the space of real functions $f \in L_1(\mathbf{T})$ such that for some $F \in H_1(\mathbf{T})$ we have $\Re F = f$. $\Re H_1$ is normed by $\|f\|_1 + \min\{\|F\|_{H_1} : \Re F = f\}$. Then H_1 is isomorphic to the complexification of $\Re H_1$ and, further, when considered as a real space is isomorphic to $\Re H_1$. Further it was shown in [17] that $\Re H_1$ has an unconditional basis and is isomorphic to a space of martingales $H_1(\delta)$. To define the space $H_1(\delta)$ let $(h_n)_{n\geq 1}$ be the usual enumeration of the Haar functions on $I = [0, 1]$ normalized so that $\|h_n\|_\infty = 1$. Then suppose $f \in L_1$ is of the form $f = \sum a_n h_n$. We define $\|f\|_{H_1(\delta)} = \int (\sum_n |a_n|^2 h_n^2)^{1/2} dt$ and $H_1(\delta) = \{f \colon \|f\|_{H_1(\delta)} < \infty\}$.

These considerations can be extended to the case $n > 1$. In a similar way, $H_1(\mathbf{T}^n)$ is isomorphic to the complexification of, and is also real-isomorphic to, a martingale space $H_1(\delta^n)$. Here we define for $\alpha \in \mathscr{M} = \mathbf{N}^n$ the function $h_\alpha \in L_1(I^n)$ by $h_\alpha(t_1, \dots, t_n) = \prod h_{\alpha_k}(t_k)$. Then $H_1(\delta^n)$ consists of all $f = \sum_{\alpha \in \mathscr{M}} a_\alpha h_\alpha$ such that $\|f\|_{H_1(\delta^n)} = \int (\sum |a_\alpha|^2 h_\alpha^2)^{1/2} dt < \infty$.

It is clear from the definition that the system $(h_\alpha)_{\alpha\in\mathscr{M}}$ is an unconditional basis of $H_1(\delta^n)$. We can thus define a space $H_1(\delta^n, \ell_2) = H(\delta^n)(\ell_2)$ as in §1; since $H_1(\delta^n)$ has cotype two, this space is isomorphic to $\operatorname{Rad} H_1(\delta^n)$. The following theorem is due to Bourgain [2]:

Theorem 3.1. *$H_1(\delta, \ell_2)$ is not isomorphic to a complemented subspace of $H_1(\delta)$.*

In a subsequent paper [3] Bourgain implicitly extended this result to higher dimensions.

Theorem 3.2. *For every $n = 1, 2, \dots$ the space $H_1(\delta^n, \ell_2)$ is not isomorphic to any complemented subspace of $H_1(\delta^n)$.*

Sketch of proof. For $n = 1$ this theorem is proved in detail in [2]. The subsequent paper [3] states only the weaker fact that $H_1(\delta^n)$ is not isomorphic to $H_1(\delta^{n+1})$. His proof, however, gives Theorem 3.2 as well. All that is needed is to change in §3 of [3] condition (m+1) and Lemma 4. Before we formulate the appropriate condition we need some further notation. By $BMO(\delta^n)$ we will denote the dual of $H_1(\delta^n)$ and by $BMO(\delta^n, \ell_2)$ we will denote the dual of $H_1(\delta^n, \ell_2)$. The space $H_1(\delta^n, \ell_2)$ has an unconditional basis given by $(h_\alpha \otimes e_k)_{\alpha\in\mathscr{M},\, k\in\mathbf{N}}$. In our notation from §2 $h_\alpha \otimes e_k$ is a sequence of $H_1(\delta^n)$-functions which consists of zero functions except at the kth place where there is h_α. The same element can be treated as an element of the dual space. Note that the natural duality gives

$$\langle h_\alpha \otimes e_k, h_{\alpha'} \otimes e_{k'}\rangle = \begin{cases} \int_{I^n} |h_\alpha| & \text{when } \alpha = \alpha' \text{ and } k = k', \\ 0 & \text{otherwise.} \end{cases}$$

Now we are ready to state the new condition $(m+1)$:

Let $\Phi\colon H_1(\delta^n, \ell_2) \longrightarrow H_1(\delta^n)$ and $\Phi^\times\colon BMO(\delta^n, \ell_2) \longrightarrow BMO(\delta^n)$ be bounded linear operators (note that $\Phi^\times$ is *not* the adjoint of Φ). Then for every $\varepsilon > 0$ there exists a set $A \subset \mathscr{M}$ such that $\sum_{\alpha\in A} |h_\alpha| = 1$ and integers k_α for $\alpha \in A$ such that

$$\sum_{\alpha\in A} \int_{I^n} |\Phi(h_\alpha \otimes e_{k_\alpha})| \cdot |\Phi^\times(h_\alpha \otimes e_{k_\alpha})| < \varepsilon.$$

With this condition one can repeat the proof from [3] and obtain the theorem. □

Corollary 3.3. *We have*

$$\ell_2 \overset{c}{\subset} H_1(\delta) \overset{c}{\subset} H_1(\delta, \ell_2) \overset{c}{\subset} H_1(\delta^2) \overset{c}{\subset} H_1(\delta^2, \ell_2) \overset{c}{\subset} \cdots$$

where $X \overset{c}{\subset} Y$ means that X is isomorphic to a complemented subspace of Y but Y is **not** *isomorphic to a complemented subspace of X.*

Proof. It is well known and easy to check that the map $h_\alpha \otimes e_k \mapsto h_\alpha(t_1, \dots, t_n)\cdot r_k(t_{n+1})$ where r_k is the kth Rademacher function gives the desired complemented embedding. That no smaller space is isomorphic to a complemented subspace of a bigger one is the above theorem of Bourgain. □

Corollary 3.4. *The spaces $H_1(\delta^n)$ is not isomorphic to a nonatomic Banach lattice for $n = 1, 2, \dots$. The spaces $H_1(\delta^n, \ell_2)$ are each isomorphic to a nonatomic Banach lattice.*

Proof. The first claim follows directly from Theorems 3.1, 3.2, and 2.3. We only have to observe that (since $H_1(\delta^n)$ does not contain any subspace isomorphic to c_0 and indeed has cotype two) any Banach lattice isomorphic as a Banach space to $H_1(\delta^n)$ is order continuous (see [16, Theorem 1.c.4]). The second claim follows from Corollary 2.4. □

Remark. For $H_p(\mathbf{T}^n)$ with $0 < p < \infty$ we have the following situation. When $1 < p < \infty$ the orthogonal projection from $L_p(\mathbf{T}^n)$ onto $H_p(\mathbf{T}^n)$ is bounded so then $H_p(\mathbf{T}^n)$ is isomorphic to $L_p(\mathbf{T}^n)$. This implies in particular that these spaces are isomorphic to nonatomic lattices. When $0 < p < 1$ then $H_p(\mathbf{T}^n)$ admit only purely atomic orders as a p-Banach lattices. To see this observe that if X is not a purely atomic p-Banach lattice then its Banach envelope (for definition and properties see [11]) is a Banach lattice which is not purely atomic. On the other hand it is known that the Banach envelope of $H_p(\mathbf{T}^n)$ is isomorphic to ℓ_1. For $n = 1$ this can be found in [11, Theorem 3.9], for $n > 1$ the proof uses [19, Theorem 2′] but otherwise is the same; alternatively see [11, Theorem 3.5] for a proof using bases. When we compare it with the observation from [1] mentioned in the Introduction, that ℓ_1 is not isomorphic to any nonatomic Banach lattice, we conclude that the spaces $H_p(\mathbf{T}^n)$ cannot be isomorphic to any nonatomic p-Banach lattice.

Remark. For the dual spaces $H_1(\mathbf{T}^n)^* = BMO(\mathbf{T}^n)$ the situation is rather different. We first observe the following proposition:

Proposition 3.5. *For any Banach space X the spaces $\ell_1(X)^*(= \ell_\infty(X^*))$ and $L_1([0, 1], X)^*$ are isomorphic.*

Proof. Clearly $\ell_1(X)^*$ is isomorphic to a one-complemented subspace of $L_1(X)^*$. Now let $\chi_{n,k} = \chi_{((k-1)2^{-n}, k2^{-n})}$ for $1 \leq k \leq 2^n$ and $n = 0, 1, \ldots$. Let $T: \ell_1(X) \to L_1(X)$ be defined by $T((x_n)) = \sum x_n \chi_{m,k}$ where $n = 2^m + k - 1$. Let $L_1(\mathscr{D}_N, X)$ be the subspace of all functions measurable with respect to the finite algebra generated by the sets $((k-1)2^{-N}, k2^{-N})$ for $1 \leq k \leq 2^N$, and define $S_N: L_1(\mathscr{D}_N; X) \to \ell_1(X)$ by setting $S(x \otimes \chi_{N,k})$ to be the element with x in position $2^N + k - 1$ and zero elsewhere. Then applying [22, II.E, Exercise 7] (cf. [8, Proposition 1]), we obtain that $L_1(X)^*$ is isomorphic to a complemented subspace of $\ell_1(X)^*$. Then by the Pełczyński decomposition technique we obtain the proposition. □

Now from Proposition 3.5, observe that, since $H_1(\mathbf{T}^n) \sim \ell_1(H_1(\mathbf{T}^n))$, we have $L_1(H_1(\mathbf{T}^n))^* \sim BMO(\mathbf{T}^n)$, and clearly this isomorphism induces a nonatomic (but not order-continuous) lattice structure on $BMO(\mathbf{T}^n)$. (It is easy to see that a space which contains a copy of ℓ_∞ cannot have an order-continuous lattice structure, because it fails the separable complementation property.)

4. Rad H_1 AND TENT SPACES

The space $H_1(\delta, \ell_2)$ is, as observed in §2, isomorphic to Rad H_1 and has a structure as a nonatomic Banach lattice. The complex space Rad H_1 is easily seen to be isomorphic to the vector-valued space $H_1(\mathbf{T}, \ell_2)$ consisting of the boundary values of the space of all functions F analytic in the unit disk $\mathbf{D}$

with values in a Hilbert space ℓ_2 and such that

$$\sup_{0<r<1} \int_0^{2\pi} \|F(re^{i\theta})\| \frac{d\theta}{2\pi} = \|F\| < \infty.$$

To see this isomorphism just note that $H_1(\mathbf{T}, \ell_2)$ can be identified with the space of sequences (f_n) in H_1 such that

$$\|(f_n)\| = \int_0^{2\pi} \left(\sum_{n=1}^{\infty} |f_n(e^{i\theta})|^2 \right)^{1/2} \frac{d\theta}{2\pi} < \infty.$$

This is in turn easily seen to be equivalent to the norm of $\sum r_n f_n$ in $L_2([0, 1]; H_1)$ (see [16, Theorem 1.d.6]).

We now show that a nonatomic Banach lattice isomorphic to $\operatorname{Rad} H_1$ arises naturally in in harmonic analysis. More precisely we will show that tent space T^1, which was introduced and studied by Coifman, Meyer, and Stein in [4], is isomorphic to $\operatorname{Rad} H_1$. Tent spaces are useful in some questions of harmonic analysis (cf. [7] or [21]). They can be defined over $\mathbf{R}^n$, but for the sake of simplicity we will consider them only over $\mathbf{R}$.

Let us fix $\alpha > 0$. For $x \in \mathbf{R}$ we define

$$\Gamma_\alpha(x) = \{(y, t) \in \mathbf{R} \times \mathbf{R}^+ : |x - y| < \alpha t\}.$$

Given a function $f(y, t)$ defined on $\mathbf{R} \times \mathbf{R}^+$ we put

$$\|f\|_\alpha = \int_{\mathbf{R}} \left(\int_{\Gamma_\alpha(x)} |f(y, t)|^2 t^{-2} \, dy \, dt \right)^{1/2} dx.$$

It was shown in [4, Proposition 4] that for different α's the norms $\|\cdot\|_\alpha$ are equivalent; i.e., for $0 < \alpha < \beta < \infty$ there is a $C = C(\alpha, \beta)$ such that for every f we have

$$\|f\|_\alpha \le \|f\|_\beta \le C\|f\|_\alpha. \tag{4.1}$$

This implies that the space $T^1 = \{f(y, t) : \|f\|_\alpha < \infty\}$ does not depend on α. Observe that T^1 is clearly a nonatomic Banach lattice.

The main result of this section is

Theorem 4.1. *The space T^1 is lattice-isomorphic to $H_1(\delta, L_2)$ and, hence, isomorphic to* $\operatorname{Rad} H_1$.

Actually for the proof of this theorem it is natural to work with the dyadic H_1 space on $\mathbf{R}$. This space, which we denote $H_1(\delta_\infty)$, can be defined as follows:

Let $I_{nk} = [k \cdot 2^n, (k+1) \cdot 2^n]$ for $n, k = 0, \pm 1, \pm 2 \dots$, and let h_{nk} be the function which is equal to 1 on the left-hand half of I_{nk}, -1 on the right-hand half of I_{nk}, and zero outside I_{nk}. In other words, h_{nk} is the Haar system on $\mathbf{R}$. The system $\{h_{nk}\}_{n,k=0,\pm 1,\pm 2,\dots}$ is a complete orthogonal system. For a function $f = \sum_{n,k} a_{nk} h_{nk}$ we define its $H_1(\delta_\infty)$-norm by

$$\|f\| = \int_{\mathbf{R}} \left(\sum_{n,k} |a_{nk}|^2 |h_{nk}|^2 \right)^{1/2} dt. \tag{4.2}$$

That this space is isomorphic to the space $H_1(\delta)$ follows from the work of Sjölin and Stromberg [20]. However, slightly more is true:

Lemma 4.2. *The atomic Banach lattices $H_1(\delta)$ and $H_1(\delta_\infty)$ are lattice-isomorphic (or, equivalently the natural normalized unconditional bases of these spaces are permutatively equivalent).*

Proof. For any subset $\mathscr{A}$ of $\mathbf{Z}^2$ write $H_{\mathscr{A}}$ for the closed linear span of $\{h_{nk}: (n, k) \in \mathscr{A}\}$ in $H_1(\delta_\infty)$. For $m \in \mathbf{Z}$ let $\mathscr{A}_m = \{(n, k): I_{nk} \subset [2^{-m-1}, 2^{-m}]\}$ and $\mathscr{B}_m = \{(n, k): I_{nk} \subset [-2^{-m}, -2^{-m-1}]$. Let $\mathscr{D} = \bigcup_{m\in\mathbf{Z}}(\mathscr{A}_m \cup \mathscr{B}_m)$ and $\mathscr{D}_+ = \bigcup_{m\geq 0}\mathscr{A}_m$. Then it is clear that $H_{\mathscr{D}}$ and $H_{\mathscr{D}_+}$ are each lattice isomorphic to $\ell_1(H_1(\delta))$. Now $H_1(\delta_\infty)$ is lattice isomorphic to $H_{\mathscr{D}} \oplus H_{\mathscr{E}}$ where $\mathscr{E} = \{(m, 0), (m, -1) : m \in \mathbf{Z}\}$. It is easy to show that $H_{\mathscr{E}}$ is lattice isomorphic to ℓ_1. Similarly $H_1(\delta)$ is lattice-isomorphic to $H_1(\mathscr{D}_+)\oplus\ell_1$, and this completes the proof of the lemma. □

Remark. Note also that $H_1(\delta)$ is lattice-isomorphic to $\ell_1(H_1(\delta))$.

Proof of Theorem 4.1. We will prove that T^1 is lattice-isomorphic to $H_1(\delta_\infty, L_2)$. Let us introduce squares $A_{nk} \subset \mathbf{R} \times \mathbf{R}^+$ defined as $A_{nk} = I_{nk}\times[2^n, 2^{n+1}]$ for $n, k = 0, \pm 1, \pm 2, \dots$. It is geometrically clear that squares $\{A_{nk}\}_{n,k=0,\pm1,\pm2,\dots}$ are essentially disjoint and that they cover $\mathbf{R} \times \mathbf{R}^+$. For $j = 0, 1, 2$ we define

$$A^j_{nk} = [(k + j/3)2^n, (k + (j + 1)/3)2^n] \times [2^n, 2^{n+1}].$$

Note that in this way we divide each A_{nk} into three essentially disjoint rectangles. Let $D^j = \bigcup_{n,k} A^j_{nk}$. Let T^1_j be the subspace of T^1 consisting of all functions whose support is contained in D^j. Clearly $T^1 = T^1_0 \oplus T^1_1 \oplus T^1_2$, so it is enough to show that T^1_j is lattice-isomorphic to $H_1(\delta_\infty, L_2)$.

We write $f^j \in T^1_j$ as $f^j = \sum_{n,k} f^j_{nk}$ where $f^j_{nk} = f^j \cdot \chi_{A^j_{nk}}$. We start with $j = 1$. For any $\alpha > 0$ we have

$$\begin{aligned} \|f^1\|_\alpha &= \int_{\mathbf{R}} \left(\int_{\Gamma_\alpha(x)} |f^1(y, t)|^2 t^{-2} dy\, dt\right)^{1/2} dx \\ &= \int_{\mathbf{R}} \left(\int_{\Gamma_\alpha(x)} \sum_{n,k} |f^1_{nk}(y, t)|^2 t^{-2} dy\, dt\right)^{1/2} dx \qquad (4.3) \\ &= \int_{\mathbf{R}} \left(\sum_{nk} \int_{\Gamma_\alpha(x)} |f^1_{nk}(y, t)|^2 t^{-2} dy\, dt\right)^{1/2} dx. \end{aligned}$$

If we now take $\alpha = \frac{2}{3}$ we have $\Gamma_\alpha(x) \supset A^1_{nk}$ for all $x \in I_{nk}$, so from (4.3) we get

$$\|f^1\|_\alpha \geq \int_{\mathbf{R}} \left(\sum_{nk} \chi_{I_{nk}}(x) \int_{A^1_{nk}} |f^1_{nk}(y, t)|^2 t^{-2} dy\, dt\right)^{1/2} dx. \qquad (4.4)$$

On the other hand, when we take $\alpha = \frac{1}{6}$ we have $\Gamma_\alpha(x) \cap A^1_{nk} = \varnothing$ for all $x \notin I_{nk}$, so from (4.3) we get

$$\|f^1\|_\alpha \leq \int_{\mathbf{R}} \left(\sum_{n,k} \chi_{I_{nk}}(x) \int_{A^1_{nk}} |f^1_{nk}(y, t)|^2 t^{-2} dy\, dt\right)^{1/2} dx. \qquad (4.5)$$

For each (n, k) the subspace of T^1 consisting of functions supported on A^1_{nk} is easily seen to be isometric to the Hilbert space. If we fix an isometry between this space and ℓ_2, we obtain from (4.2)–(4.4) that T^1_1 is lattice-isomorphic to $H_1(\delta_\infty, L_2)$. In order to complete the proof of Theorem 4.1 it is enough to show that T^1_0 and T^1_2 are lattice-isomorphic to T^1_1. This isomorphism can be given by $\sum_{nk} f^j_{nk} \mapsto \sum_{nk} f^1_{nk}$. The fact that this map is really an isomorphism follows from

Lemma 4.3. *Let $\phi(t)$ be a uniformly bounded measurable function on $\mathbf{R}^+$. For a function f defined on $\mathbf{R}\times\mathbf{R}^+$ we define*

$$A_\phi(f)(y, t) = f(y + t\phi(t), t).$$

Then $A_\phi : T^1 \longrightarrow T^1$ is a continuous linear operator.

Proof of Lemma 4.3. Since

$$\begin{aligned}\int_{\Gamma_\alpha(x)} |A_\phi(f)(y, t)|^2 t^{-2} dy\, dt &= \int_{\mathbf{R}^+} \left(t^{-2}\int_{x-\alpha t}^{x+\alpha t} |A_\phi(f)(y, t)|^2 dy\right) dt\\ &= \int_{\mathbf{R}^+} \left(t^{-2}\int_{x-\alpha t - t\phi(t)}^{x+\alpha t - t\phi(t)} |f(y, t)|^2 dy\right) dt\\ &\le \int_{\mathbf{R}^+} \left(t^{-2}\int_{x-(\|\phi\|_\infty+\alpha)t}^{x+(\|\phi\|_\infty+\alpha)t} |f(y, t)|^2 dy\right) dt\\ &= \int_{\Gamma_{\alpha+\|\phi\|_\infty}(x)} |f(y, t)|^2 t^{-2} dy\, dt,\end{aligned}$$

the lemma follows. □

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