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# ON THE DIEUDONNÉ PROPERTY FOR $C(\Omega, E)$

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ABSTRACT. In a recent paper, F. Bombal and P. Cembranos showed that if $E$ is a Banach space such that $E^*$ is separable, then $C(\Omega, E)$, the Banach space of continuous functions from a compact Hausdorff space $\Omega$ to $E$, has the Dieudonné property. They asked whether or not the result is still true if one only assumes that $E$ does not contain a copy of $l_1$. In this paper we give a positive answer to their question. As a corollary we show that if $E$ is a subspace of an order continuous Banach lattice, then $E$ has the Dieudonné property if and only if $C(\Omega, E)$ has the same property.

If $E$ is a Banach space and $\Omega$ is a compact Hausdorff space, then $C(\Omega, E)$ will stand for the Banach space of the $E$-valued continuous functions on $\Omega$ under the supremum norm.

A Banach space $E$ is said to have the *Dieudonné property* if for every Banach space $F$, any bounded linear operator $T: E \to F$ that transforms weakly Cauchy sequences into weakly convergent sequences is weakly compact. In [**3**] F. Bombal and P. Cembranos showed that if $E$ is a Banach space such that $E^*$ is separable, then $C(\Omega, E)$ has the Dieudonné property and they asked whether the same result is true when replacing the assumption that $E^*$ is separable by supposing only that $l_1$ does not embed in $E$. In this paper we give a positive answer to their question.

Recall that a topological space $(X, \gamma)$ is said to be Polish if it is homeomorphic to a separable complete metric space and it is said to be analytic if it is the continuous image of a Polish space. A subset $A$ of a topological space $(X, \gamma)$ is said to be coanalytic if its complement $(X \backslash A, \gamma)$ is analytic. Finally $A$ is said to be PCA if it is the continuous image of a coanalytic space.

The notations and terminology used and not defined can be found in [**5**, **8**, or **10**]. In the proof of Lemma 3 we need the following two results.

THEOREM 1 (M. SREBRNY [**9**]). *Let $X$ and $Y$ be two analytic spaces and let $F$ be a multivalued function from $X$ to the subsets of $Y$, such that its graph is PCA and for which one can prove that for every $x \in X$, $F(x) \neq \emptyset$ using only the axioms of ZFC. Then there exists a universally measurable map $f: X \to Y$ such that $f(x) \in F(x)$ for every $x \in X$.*

THEOREM 2 (I. ASSANI [**1**, **2**]). *Let $E$ be a separable Banach space. The set of weakly Cauchy sequences is a coanalytic subset of $E^{\mathbf{N}}$.*

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LEMMA 3. *Let $E$ be a separable Banach space containing no isomorphic copy of $l_1$. Then there is a sequence of maps $\theta_n : E^* \to E$ so that each $\theta_n$ is universally measurable (for the weak*-topology of $E^*$) and, for each $e^* \in E^*$, we have*
(i) $(\theta_n(e^*))$ *is a weak Cauchy sequence,*
(ii) $\|\theta_n(e^*)\| \leq 1$ *for* $n \geq 1$,
(iii) $\lim_{n\to\infty}\langle\theta_n(e^*)\rangle = \|e^*\|$.

PROOF. Let $B$ be the closed unit ball of $E$ and let $H$ be the subset of $B^{\mathbf{N}}$ consisting of those sequences in $E$ that are weakly Cauchy. By Theorem 2, $H$ is a coanalytic subset of $B^{\mathbf{N}}$. Let $F$ be the multivalued function from $E^*$ to the subsets of $B^{\mathbf{N}}$ defined by

$$F(e^*) = \left\{(e_n) \in B^{\mathbf{N}}; (e_n) \text{ is weak Cauchy and } \lim_{n\to\infty}\langle e_n, e^*\rangle = \|e^*\|\right\}.$$

Rosenthal's Theorem [**7**] guarantees that $F(e^*) \neq 0$ for every $e^* \in E^*$. We endow $E^*$ with its weak*-topology and the graph $G(F)$ of $F$ is the set $\{(e^*, (e_n)); (e_n) \in F(e^*)\}$ which is a subset of $E^* \times H$.

Consider the map $\psi: E^* \times H \to \mathbf{R}$ defined by

$$\psi(e^*, (e_n)) = \lim_{n\to\infty}\langle e_n, e^*\rangle - \|e^*\|.$$

It is easy to see that $\psi$ is a Borel map and $G(F) = \psi^{-1}(0)$. Hence $G(F)$ is a Borel subset of $E^* \times H$ and hence it is coanalytic in $E^* \times B^{\mathbf{N}}$.

By Theorem 1, there exists a universally measurable map $f: E^* \to E^{\mathbf{N}}$ such that $f(e^*) \in F(e^*)$ for every $e^* \in E^*$. For each $n \geq 1$, let $\pi_n : B^{\mathbf{N}} \to E$ be the $n$th projection $\pi_n((e_k)) = e_n$. To finish the proof, it is enough to define $\theta_n$ by $\theta_n(e^*) = \pi_n(f(e^*))$.

The proof of the following theorem follows the steps of Theorem 3 in [**4**]. The following sketch is given for the reader's convenience.

THEOREM 4. *Let $E$ be a Banach space containing no isomorphic copy of $l_1$, and let $\Omega$ be a compact Hausdorff space. Then $C(\Omega, E)$ has the Dieudonné property.*

PROOF. It is enough to prove the theorem for $E$ separable and $\Omega$ metrizable. Let $T: C(\Omega, E) \to X$ be a bounded linear operator that takes weakly Cauchy sequences into weakly convergent sequences.

Let $V$ be the unit ball of $C(\Omega, E)$ and let $W = \overline{T(V)}$ in $X$. We will show that $W$ is weakly compact by proving that every $x^* \in X^*$ attains its maximum on $W$ (James [**6**]).

Fix $x^* \in X^*$, with $\|x^*\| \leq 1$. Then $\sup_{w\in W} x^*(w) = \|T^*x^*\|$.

As in [**4**], let $G$ be the representing measure of $T$, let $\lambda$ be its control measure and let $h: \Omega \to E^*$ be a weak*-Borel $\lambda$-measurable map from $\Omega \to E^*$ such that for each $f \in C(\Omega, E)$, $\langle f, T^*x^*\rangle = \int_\Omega \langle f(\omega), h(\omega)\rangle\, d\lambda(\omega)$ and $\|T^*x^*\| = \int_\Omega \|h(\omega)\|\, d\lambda$.

Now for each $\omega \in \Omega$, let $\psi_n(\omega) = \theta_n(h(\omega))$, where $\theta_n$ are the maps obtained in Lemma 3. Each $\psi_n : \Omega \to E$ is universally measurable and
(i) $\psi_n(\omega)$ is weak-Cauchy in $E$ for every $\omega \in \Omega$,
(ii) $\|\psi_n(\omega)\| \leq 1$ for $n \geq 1$ and $\omega \in \Omega$,
(iii) $\lim_{n\to\infty}\langle\psi_n(\omega), h(\omega)\rangle = \|h(\omega)\|$.

Let $\varepsilon_n > 0$ be any decreasing sequence so that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Choose $\delta_n > 0$ a decreasing sequence so that if $\lambda(B) < \delta_n$, then $\|G\|(B) < \varepsilon_n/2$. For each $n \geq 1$ let

$\Omega_n$ be a compact subset of $\Omega$ so that $\lambda(\Omega_n) > 1 - \delta_n$ and each $\psi_k$ is continuous on $\Omega_n$. Of course we can assume that $\Omega_n$ is an increasing sequence. By the Borsuk-Dugundji theorem [**10**] there is an extension operator $S_n\colon C(\Omega_n, E) \to C(\Omega, E)$ so that $\|S_n\| = 1$ and $S_n(f(\omega)) = f(\omega)$ for $f \in C(\Omega_n, E)$ and $\omega \in \Omega_n$. Let $g_{n,k} = S_n(\psi_k|\Omega_n)$.

By construction of the $\psi_k$, the sequence $(\psi_k(\omega))$ is a weakly Cauchy sequence in $E$ for each $\omega \in \Omega$. This shows that $(\psi_k|\Omega_n)$ is a weak Cauchy sequence in $C(\Omega_n, E)$, hence $(g_{n,k})_{k\geq 1}$ is a weakly Cauchy sequence in $C(\Omega, E)$. Therefore $x_{n,k} = T(g_{n,k})$ is weakly convergent in $X$ and $x_{n,k} \in W$. Let $w_n = \text{weak-}\lim_k x_{n,k}$ for each $n \geq 1$. Hence $w_n \in W$ for each $n \geq 1$.

It follows as in [**4**] that $w = \lim_n w_n$ exists, belongs to $W$ and

$$x^*(w) = \int_\Omega \|h(\omega)\|\, d\lambda = \sup_{u\in W} x^*(u).$$

Hence $x^*$ attains its maximum on $W$ at $w$.

COROLLARY 5. *Let $E$ be a subspace of an order continuous Banach lattice. Then $E$ has the Dieudonné property if and only if $C(\Omega, E)$ has the Dieudonné property.*

PROOF. If $E$ contains a copy of $l_1$, then $E$ would contain a complemented copy of $l_1$ [**4**, **11**], but this is impossible if $E$ has the Dieudonné property. Apply Theorem 4 to finish the proof.

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