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L_0-VALUED VECTOR MEASURES ARE BOUNDED

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ABSTRACT. Every vector measure taking values in $L_0(0,1)$ has bounded range.

The question of whether every vector measure taking values in the space $L_0(0,1)$ is bounded was first raised by Turpin [**17**]. Turpin showed the existence of an unbounded vector measure with range in a certain nonlocally convex F-space. Shortly afterwards, Fischer and Scholer [**3, 4**] and Labuda [**9**] demonstrated that a vector measure taking values in an Orlicz space L_ϕ with ϕ unbounded will be necessarily bounded. The purpose of this note is to show every L_0-valued measure is bounded. This result has applications to stochastic integrals [**1, 13, 14, 18**].

We shall denote by I the unit interval $(0,1)$ and $\mathcal{B}$ is the family of Borel subsets of I. λ will denote Lebesgue measure on $\mathcal{B}$. The space $L_0 = L_0(I;\mathcal{B},\lambda)$ consists of all real Borel functions on I with functions agreeing almost everywhere identified. This space is equipped with convergence in measure, which is F-normed by

$$\|f\| = \int_0^1 \frac{|f(t)|}{1+|f(t)|}\,d\lambda(t).$$

A base of neighborhoods for 0 is given by sets of the form $V(\epsilon, M)$ for $\epsilon > 0$, $M > 0$ where

$$V(\epsilon, M) = \{f \in L_0\colon \lambda(|f| > M) < \epsilon\}.$$

Let (S,Σ) be any measurable space. Then a (*continuous*) *submeasure* $\nu\colon \Sigma \to \mathbf{R}_+$ is a set-function satisfying

$$\nu(A) \le \nu(A\cup B) \le \nu(A) + \nu(B), \qquad A, B \in \Sigma,$$
$$\nu(A_n) \downarrow 0, \quad \text{whenever } A_n \downarrow \emptyset.$$

It is an unsolved problem (Maharam [**10**]) whether every continuous submeasure has an equivalent measure, i.e. a measure giving the same null sets. A continuous submeasure μ induces a pseudo-metric d on Σ given by $d(A,B) = \mu(A\Delta B)$. We say Σ is μ-separable if (Σ, d) is separable; if ν is a measure on a σ-algebra Σ' then a map $h\colon \Sigma \to \Sigma'$ is continuous if it is continuous with respect to the induced psuedo-metrics.

If X is an F-space and $\phi\colon \Sigma \to X$ is a vector measure, then a continuous submeasure μ is said to be a control submeasure for ϕ if it is equivalent to the

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submeasure

$$\|\phi\|(A) = \sup(\|\phi(B)\| : B \in \Sigma,\ B \subset A)$$

for $A \in \Sigma$. Maharam's problem is equivalent to the problem of whether every vector measure with values in an F-space has a control measure (cf. [**2**, p. 14]).

Some further notation will be required. If $A \in \Sigma$ (or $\mathcal{B}$) then 1_A denotes the indicator function of A, i.e.

$$1_A(s) = \begin{cases} 1, & s \in A, \\ 0, & s \notin A. \end{cases}$$

If $\mathcal{G}$ is a partition of a set $A \in \Sigma$ into sets from Σ, then $\Sigma(\mathcal{G})$ denotes the family of all unions of sets from $\mathcal{G}$.

Note. Shortly after the preparation of this paper, the authors learned that the same results have been obtained independently and somewhat earlier by M. Talagrand [**19**]. Talagrand's proof of Theorem 1 is slightly different in character although it has some ideas in common.

THEOREM 1. *Every vector measure taking values in L_0 is bounded.*

PROOF. The proof will be accomplished via several reductions of the problem. We shall start from the assumption that there exists an unbounded vector measure ϕ: $\Sigma \to L_0$ defined on some measurable space (S, Σ), and derive a contradiction. The idea of the argument is to show that we can assume certain properties and these eventually lead to a contradiction.

We denote a control submeasure for ϕ by μ: $\Sigma \to \mathbf{R}_+$. Our first simplifying assumption is

(A1) Σ *is μ-separable and has no μ-atoms.*

Clearly (A1) is justified by the fact that if ϕ is unbounded it is also unbounded on some μ-separable sub-σ-algebra; atoms can be discarded.

We shall also define a set function θ: $\Sigma \to \mathbf{R}$ by setting $\theta(A)$ to be the supremum of all $\alpha \geq 0$ such that if $M > 0$ there exists $B \in \Sigma$, $B \subset A$ with

$$\lambda\{t\colon |\phi(B;t)| \geq M\} \geq \alpha.$$

(Here $\phi(B;t) = \phi(B)(t)$.) Note that $\theta(S) > 0$.

LEMMA 1. *If $A, B \in \Sigma$ are disjoint then*

$$\theta(A \cup B) \leq \theta(A) + \theta(B).$$

PROOF. If $\alpha < \theta(A \cup B)$ and $M > 0$ there exists $C \in \Sigma$ with $C \subset A \cup B$ and $\lambda\{|\phi(C)| \geq 2M\} \geq \alpha$. Hence

$$\lambda\{|\phi(A \cap C)| \geq M\} + \lambda\{|\phi(B \cap C)| \geq M\} \geq \alpha.$$

By letting $M \to \infty$, we see that $\theta(A) + \theta(B) \geq \alpha$ and the lemma follows.

LEMMA 2. *Let $\mathcal{E} \subset \mathcal{B}$ consist of all Borel sets E such that the set $\{1_E \cdot \phi(A)\colon A \in \Sigma\}$ is bounded in L_0. Then $\mathcal{E}$ is a σ-ideal of $\mathcal{B}$; in particular if $E_n \in \mathcal{E}$ $(n \in \mathbf{N})$ then $\bigcup E_n \in \mathcal{E}$.*

PROOF. If $E_n \in \mathcal{E}$ then there exist $0 < c_n < 2^{-n}$ such that

$$\|c_n \cdot 1_{E_n} \cdot \phi(A)\| \leq 2^{-n}, \qquad A \in \Sigma,\ n \in \mathbf{N}.$$

Thus $\sum_{n=1}^{\infty} c_n \cdot 1_{E_n} \cdot \phi(A)$ converges uniformly to $h \cdot \phi(A)$ where $h = \sum c_n \cdot 1_{E_n}$. It follows easily that $\{h \cdot \phi(A)\colon A \in \Sigma\}$ is also bounded. Finally if $g(t) = h(t)^{-1}$ for $h(t) > 0$ and $g(t) = 0$ otherwise, then $\{gh \cdot \phi(A)\colon A \in \Sigma\}$ is bounded. However $gh = 1_{\cup E_n}$.

In view of Lemma 2 we can find a set $F \in \mathcal{E}$ of maximal measure and if $E \in \Sigma$ then $\lambda(E \backslash F) = 0$. We call F, which is unique up to sets of measure zero, the bounded support of ϕ, and let $I \backslash F$ be the *unbounded support* of ϕ. For each $A \in \Sigma$, let A^* be the unbounded support of the measure $B \to \phi(A \cap B)$. We observe some simple properties of the map $A \to A^*$ ($\Sigma \to \mathcal{B}$).

LEMMA 3. (a) $\lambda(A^*) = 0$ *if and only if* $\{\phi(B)\colon B \subset A\}$ *is bounded.*
(b) $(A \cup B)^* = A^* \cup B^*$ *up to sets of* λ*-measure zero for* $A, B \in \Sigma$.
(c) $\theta(A) \leq \lambda(A^*)$, $A \in \Sigma$.
(d) *If* $\mu(A \Delta B) = 0$ *then* $\lambda(A^* \Delta B^*) = 0$, $A, B \in \Sigma$.

The proofs of these statements are almost immediate.

The next lemma is, however, crucial in the development of the proof of the theorem.

LEMMA 4. *Given* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $\mu(A) < \delta$ *implies* $\lambda(A^*) < \epsilon$. *Hence, if* $A, B \in \Sigma$ *and* $\mu(A \Delta B) < \delta$ *then* $\lambda(A^* \Delta B^*) < \epsilon$.

PROOF. Given $\epsilon > 0$ choose $\delta > 0$ such that $\mu(A) < \delta$ implies $\phi(A) \in V(\epsilon/256, 1)$. Fix any $A \in \Sigma$ with $\mu(A) < \delta$ and let $\mathcal{G} = \{B_1, \ldots, B_n\}$ be any partition of A.

Let $f_i = \phi(B_i)$ $(1 \leq i \leq n)$ and let $\{g_j\colon 1 \leq j \leq 2^n\}$ be some ordering of the functions $\sum_{i=1}^n a_i f_i$ over all choices of signs $a_i = \pm 1$. We consider the map $T\colon l_1 \to L_0$ defined by

$$T(\xi) = \sum_{i=1}^{2^n} \xi_i g_i \quad \text{for } \xi = (\xi_i) \in l_1.$$

The set $K = \{T(\xi)\colon \|\xi\| \leq 1\}$ is exactly the absolutely convex hull of the set $\phi(\Sigma(\mathcal{G}))$.

If $h \in K$ then $h = \sum_{j=1}^n c_j f_j$ where $-1 \leq c_j \leq 1$. Now by a lemma of Musial, Wojczyński and Ryll-Nardzewski [**15**] (essentially the same lemma is originally found in Maurey-Pisier [**12**]), there is a probability measure P on the set $\Omega = \{-1, +1\}^n$ so that for any $x_1, \ldots, x_n \in \mathbf{R}$

$$P\left\{\omega\colon \left|\sum X_i(\omega) x_i\right| \geq \tfrac{1}{8} \left|\sum c_i x_i\right|\right\} \geq \tfrac{1}{8}$$

where $X_i\colon \Omega \to \{-1, +1\}$ is the ith coordinate map.

Let $E = \{t\colon |\sum c_i f_i(t)| \geq 16\}$. Then for $t \in E$

$$P\left\{\omega\colon \sum \left|\sum X_i f_i(t)\right| \geq 2\right\} \geq \tfrac{1}{8}$$

and so $P \otimes \lambda\{(\omega, t)\colon |\sum X_i f_i| \geq 2\} \geq \frac{1}{8}\lambda(E)$.

However for each $\omega \in \Omega$, $\sum X_i f_i \in V(\epsilon/128, 2)$ and hence $\frac{1}{8}\lambda(E) \leq \epsilon/128$ or $\lambda(E) \leq \epsilon/16$. Thus $h \in V(\epsilon/16, 16)$.

We now apply Nikišin's theorem [**16**] to the operator T. By examining the proof given in [**5**] it can be seen that there is a Borel set E with $\lambda(E) \geq 1 - \epsilon$ and

$$\lambda[(|T\xi| > \tau) \cap E] \leq 1024/\epsilon\tau, \qquad 0 < \tau < \infty.$$

(An alternative approach to this step may be obtained from results in a forthcoming paper [**6**].)

Let $d_{\mathcal{G}} = 1_E$. Then for $B \in \Sigma(\mathcal{G})$

$$\int d_{\mathcal{G}} |\phi(B;t)|^{1/2}\, dt = \int_E |\phi(B;t)|^{1/2}\, dt \leq 2048/\epsilon.$$

Consider $d_{\mathcal{G}} \in L_\infty(0,1)$ as a net over all partitions of A ordered by refinement. Then $\{d_{\mathcal{G}}\}$ has a cluster point a, $0 \leq a \leq 1$, a.e. $\int a(t)|\phi(B;t)|^{1/2}\, dt \leq 2048/\epsilon$ for $B \in \Sigma$ with $B \subset A$. Now $\int a(t)\, dt \geq 1 - \epsilon$ and so, if $b(t) = a(t)^{-1}$ for $a(t) > 0$ and $b(t) = 0$ otherwise, $b \cdot a = 1_F$ where $\lambda(F) \geq 1 - \epsilon$. The set $\{1_F \cdot \phi(B)\colon B \in \Sigma,\ B \subset A\}$ is thus bounded in L_0 and so $I \backslash F \supset A^*$, i.e. $\lambda(A^*) \leq \epsilon$.

We now come to our second reduction of the problem. We can assume

(A2) *μ is a probability measure on Σ.*

Justification of (A2). For each partition $\mathcal{G}$ of S, $\mathcal{G} = \{B_1, \ldots, B_n\}$ define $\{C_i\colon 1 \leq i \leq n\}$ in $\mathcal{B}$ by $C_i = B_i^* \backslash \bigcup_{j<i} B_j^*$. Define for $A \in \Sigma$

$$\nu_{\mathcal{G}}(A) = \left\{\sum \lambda(C_i)\colon B_i \cap A \neq \emptyset\right\}.$$

Then $\nu\mathcal{G}$ is additive on $\Sigma(\mathcal{G})$, monotone and $\nu_{\mathcal{G}}(S) = \lambda(S^*) > 0$. Denote by ν any pointwise cluster point of the net $\{\nu_{\mathcal{G}}\}$ of set functions on Σ. Then $\nu(S) = \lambda(S^*)$, ν is additive and monotone and $\nu(B) \leq \lambda(B^*)$, $B \in \Sigma$. Hence by Lemma 4, ν is μ-continuous. It follows that ν is countably additive and there is a subset $A \in \Sigma$ so that $\nu(A) > 0$, and if $B \subset A$ with $B \in \Sigma$ then $\nu(B) = 0$ if and only if $\mu(B) = 0$, i.e. ν and μ are equivalent on $\Sigma \cap A$.

We now achieve our reduction by replacing ϕ by its restriction to A and μ by $\nu(A)^{-1}\nu$. The new ϕ is still unbounded since $\lambda(A^*) \geq \nu(A) > 0$, and of course assumption (A1) remains in force.

Our third reduction is that we can assume

(A3) $\lambda(A^* \cap B^*) = 0$ *whenever* $A \cap B = \emptyset$.

The justification of (A3) is partially based on an argument of Kwapien [**8**].

Justification of (A3). Let $\{B_{n,k}\colon 1 \leq k \leq 2^n\}$ be, for each n, a partitioning of S into sets of μ-measure 2^{-n} so that

$$B_{n,k} = B_{n+1,2k-1} \cup B_{n+k,2k}, \qquad 1 \leq k \leq 2^n,\ n \in \mathbf{N},$$

and $\{B_{n,k}\colon 1 \leq k \leq 2^n,\ n \in \mathbf{N}\}$ is μ-dense in Σ.

For given $\epsilon > 0$ there exists δ so that $\mu(A) < \delta$ implies $\lambda(A^*) < \epsilon$. For each n let $m = m(n) = [\delta \cdot 2^n]$.

Let $\psi_n \in L_0$ be defined by

$$\psi_n = \sum_{k=1}^{2^n} \chi_{n,k}, \quad \text{where } \chi_{n,k} = 1_{B^*_{n,k}}.$$

Then $\{\psi_n\}$ is monotone increasing in L_0 and integer-valued.

For any m-subset J of $\{1, 2, \ldots, 2^n\}$,

$$\int_0^1 \max_{i \in J} \chi_{n,i}(t)\, dt \le \epsilon$$

and summing over all such sets,

$$\int_0^1 \sum_J \max_{i \in J} \chi_{n,i}(t)\, dt \le \binom{2^n}{m}\epsilon,$$

or

$$\int_0^1 \binom{2^n}{m} - \binom{2^n - \psi_n(t)}{m} dt \le \binom{2^n}{m}\epsilon.$$

$$\binom{2^n - \psi_n(t)}{m} = \binom{2^n}{m} \cdot \frac{2^n - m}{2^n} \cdots \frac{2^n - m - \psi_n(t) + 1}{2^n - \psi_n(t) + 1}$$

$$\le \binom{2^n}{m}\left(1 - \frac{m}{2^n}\right)^{\psi_n(t)} \le \binom{2^n}{m}\left(1 - \frac{\delta}{2}\right)^{\psi_n(t)}$$

whenever $2^n > \delta^{-1}$. Thus

$$\inf_n \int_0^1 \left(1 - \frac{\delta}{2}\right)^{\psi_n(t)} dt \ge 1 - \epsilon.$$

Applying this to every $\epsilon > 0$ we conclude that $\sup \psi_n = \psi < \infty$ a.e.

Of course, since ϕ is unbounded, we must have $\psi > 0$. Hence there exists $F_0 \in \mathcal{B}$ with $\lambda(F_0) > 0$ and $n \in \mathbf{N}$ so that

$$\psi_n(t) = \psi(t) > 0, \qquad t \in F_0.$$

Now there exists k, $1 \le k \le 2^n$ with $\lambda(B^*_{n,k} \cap F_0) > 0$. Let $F = B^*_{n,k} \cap F_0$.

Since for $m > n$, $\sum_{j=1}^{2^n} \chi_{m,j} = \psi_m = \psi_n$ on F, we must have (for fixed m),

$$\sum_{B_{m,j} \subset B_{n,k}} \chi_{m,j}(t) = 1, \qquad t \in F,$$

so that the sets $\{B^*_{m,j} \cap F \colon B_{m,j} \subset B_{n,k}\}$ intersect only in sets of λ-measure zero.

It follows quickly from the μ-λ-continuity of the map $A \mapsto A^*$ that if A_1, $A_2 \in \Sigma$ with $A_1 \cap A_2 = \emptyset$ and A_1, $A_2 \subset B_{n,k}$ then

$$\lambda(F \cap A_1^* \cap A_2^*) = 0.$$

Now we achieve our reduction by replacing ϕ by the measure ϕ', restricted to $B_{n,k} \cap F$, $\phi'(A) = 1_F \cdot \phi(A)$, $A \in \Sigma$, $A \subset B_{n,k} \cap F$. It is again clear that ϕ' is unbounded and we can obtain (A2) by renormalizing μ. It is not difficult to see that our procedure replaces (for $A \subset B_{n,k}$), A^* by $F \cap A^*$ (up to sets of measure zero) and so (A3) now holds.

Under the assumptions (A1)–(A3) we now prove

LEMMA 5. *Given any $\epsilon > 0$, disjoint sets $A_1 \cdots A_n \in \Sigma$ and $M > 0$, there exist $B_i \subset A_i$, $B_i \in \Sigma$ so that for every subset J of $\{1, 2, \ldots, n\}$*

$$\left|\phi\left(\bigcup_{i \in J} B_i \cup \bigcup_{i \notin J} (A_i \backslash B_i)\right)\right| \ge M$$

on a set of measure at least $\sum_{i=1}^n \theta(A_i) - \epsilon$.

PROOF. We may choose a constant K so large that
(i) $1_{I-A_i^*}\phi(C_i) \in V(\epsilon/4n^2, K)$, $C_i \subset A_i$,
(ii) $\phi(A_i) \in V(\epsilon/4n, K)$, $1 \leq i \leq n$.

Choose $B_i \subset A_i$, $B_i \in \Sigma$ so that $\lambda\{|\phi(B_i)| \geq nK + M\} \geq \theta(A_i) - \epsilon/4n$. For $J \subset \{1, 2, \ldots, 2^n\}$, let $C = \bigcup_{i\in J} B_i \cup \bigcup_{i\notin J}(A_i \backslash B_i)$. Then for each i let $E_i = \{t\colon |\phi(B_i; t)| \geq nK + M,\ t \in A_i^*\}$. Then $\lambda(E_i) \geq \theta(A_i) - \epsilon/4n - \epsilon/4n^2 \geq \theta(A_i) - \epsilon/2n$. If $t \in E_i$ and $i \in J$ then

$$|\phi(C; t)| \geq |\phi(B_i; t)| - (n-1)K \geq M$$

except on a set of measure at most $(n-1)\epsilon/4n^2 < \epsilon/4n$. (Here we use the fact that the sets A_i^* are almost disjoint and (i)).

If $t \in E_i$ and $i \notin J$ then

$$|\phi(C; t)| \geq |\phi(B_i; t)| - (n-1)K - |\phi(A_i; t)| \geq M$$

except on a set of measure at most $\epsilon/4n$. Hence $\lambda\{|\phi(C)| \geq M\} \geq \sum_{i=1}^n \theta(A_i) - \epsilon$ as the sets $\{E_i\colon 1 \leq i \leq n\}$ are almost disjoint.

LEMMA 6. *θ is a measure on Σ which is μ-continuous.*

REMARK. Of course (A1)–(A3) are in force here.

PROOF. By Lemma 1, $\theta(A \cup B) \leq \theta(A) + \theta(B)$ and by Lemma 5, $\theta(A \cup B) \geq \theta(A) + \theta(B)$ for disjoint A, B. As $\theta(A) \leq \lambda(A^*)$ and by Lemma 4, $A \mapsto A^*$ is continuous, we must have that θ is μ-continuous and countably additive.

We now make a further reduction; we may assume

(A4) *There is a constant p, $0 < p < 1$, so that $\theta(A) = p\mu(A)$, $A \in \Sigma$.*

Justification of (A4). Since θ is μ-continuous and nonzero (ϕ is unbounded), there is a subset $B \in \Sigma$ so that $\theta(B) > 0$ and θ and μ are equivalent on $\Sigma \cap B$. Restrict ϕ to B and redefine $\mu(A)$ as $\theta(B)^{-1}\theta(A)$ for $A \in \Sigma \cap B$. Let $p = \theta(B)$ and (A4) will hold. Of course since $\theta(B) > 0$, ϕ is still unbounded.

Under assumptions (A1)–(A4) we now prove

LEMMA 7. *Let Σ_0 be a finite subalgebra of Σ and suppose ϵ, $M > 0$. Then there is a set $C \in \Sigma$ independent of Σ_0 with $\mu(C) = \frac{1}{2}$ so that*

$$\lambda\{|\phi(C)| \geq M\} \geq p - \epsilon.$$

PROOF. Let $A_1, \ldots, A_n$ be the atoms of Σ_0. Choose N sufficiently large so that $\mu(B) \leq n/N$ implies $\phi(B) \in V(\epsilon/2, 1)$. Subdivide each A_i into N disjoint sets $(A_{ij}\colon 1 \leq j \leq N)$ of μ-measure $\mu(A_i)/N$. Now use Lemma 5 to produce $B_{ij} \subset A_{ij}$ so that for any subset J of $L = \{(i,j)\colon 1 \leq i \leq n, 1 \leq j \leq N\}$,

$$\lambda\left\{\left|\phi\left(\bigcup_J B_{ij} \cup \bigcup_{L\backslash J}(A_{ij}\backslash B_{ij})\right)\right| \geq M + 1\right\} \geq p - \frac{\epsilon}{2}.$$

By appropriate choice of J we may suppose that if $D = \bigcup_J B_{ij} \cup \bigcup_{L\backslash J}(A_{ij}\backslash B_{ij})$, then

$$\tfrac{1}{2}\mu(A_i) \leq \mu(D \cap A_i) \leq \tfrac{1}{2}\mu(A_i) + N^{-1}$$

for each fixed i. Choose $D_i \in \Sigma$, $D_i \subset D \cap A_i$ so that $\mu(D_i) = \frac{1}{2}\mu(A_i)$. Let $C = \bigcup D_i$. Then $\mu(D\backslash C) \leq n/N$, and $\lambda\{|\phi(C)| \geq M\} \geq p - \epsilon$ as required. Clearly $C \cap A_i = D_i$.

We now are in position for the final step in the theorem. Assumptions (A1)–(A4) remain in force. First we determine $\delta > 0$ so that $\mu(A) < \delta$ implies that $\phi(A) \in V(p/50, 1)$. Next select an integer r so that $(1 - \delta/2)^r \leq 9/25$. Select a further integer N so that $2^N > \delta^{-1}$ and $N > 2^{r+2}/p$ and a constant K, $K > 2^{N+2}$.

We select, by induction, a sequence $\{C_n: 1 \leq n \leq N\}$ of sets in Σ and an increasing sequence of constants $\{M_n: 1 \leq n \leq N\}$ so that

(i) $\mu(C_n) = \frac{1}{2}$, $1 \leq n \leq N$,

(ii) C_n is independent of the algebra generated by $\{C_1, \ldots, C_{n-1}\}$ for $n \geq 2$,

(iii) $\lambda\{|\phi(C_n)| \geq M_n\} \leq p/16N$,

(iv) $\lambda\{|\phi(C_{n+1})| \geq M_n + K\} \geq \frac{1}{2}p$, $n \geq 1$,

(v) $\lambda\{|\phi(C_1)| \geq K\} \geq \frac{1}{2}p$.

Clearly Lemma 7 implies we can make such a construction. Set $M_0 = 0$ for convenience and

$$E_n = \{t: |\phi(C_n; t)| \geq M_{n-1} + K\}, \qquad n = 1, 2, \ldots, N.$$

Then $\sum_{n=1}^{N} \lambda(E_n) \geq \frac{1}{2}Np$. Hence the set of t which belongs to at least $\frac{1}{4}Np$ of the sets E_n has measure at least $\frac{1}{4}p$. Now use (iii) as well to produce a set $F \subset I$ with $\lambda(F) \geq 3p/16$ such that if $t \in F$, then $t \in E_n$ for at least $\frac{1}{4}Np$ sets E_n and $|\phi(C_n; t)| \leq M_n$ for all n, $1 \leq n \leq N$.

Let $A_1, \ldots, A_{2^N}$ be the atoms of the finite algebra generated by $\{C_1, \ldots, C_N\}$ so that $\mu(A_i) = 2^{-N}$. Let $f_i = \phi(A_i)$. Let $u_i(t)$ $(t \in I)$ be the decreasing rearrangement of the finite sequence $\{|f_1(t)|, |f_2(t)|, \ldots, |f_{2^N}(t)|\}$.

For fixed $t \in F$, let $i_1, \ldots, i_r$ be chosen to be distinct and so that $|f_{i_k}(t)| = u_k(t)$, $1 \leq k \leq r$. Since $\frac{1}{4}Np > 2^r$ there are two distinct indices m and n such that $A_{i_k} \subset C_m$ if and only if $A_{i_k} \subset C_n$ (for $1 \leq k \leq r$), and $t \in E_m \cap E_n$. Hence

$$|\phi(C_n; t) - \phi(C_m; t)| \leq \sum_{i=r+1}^{2^N} u_k(t) \leq 2^N u_r(t).$$

However, if $n > m$, $|\phi(C_n; t)| \geq M_m + K$ and $|\phi(C_m; t)| \leq M_m$ so that we conclude

$$u_r(t) \geq K/2^N \geq 4, \qquad t \in F.$$

Now choose $q \in \mathbf{N}$ so that $\frac{1}{2}\delta \leq q \cdot 2^{-N} \leq \delta$; this is possible since $2^N > \delta^{-1}$. We introduce two sets of random variables $\{X_1, \ldots, X_{2^N}\}$, $\{Y_1, \ldots, Y_{2^N}\}$ defined on some (finite) probability space Ω. The joint distribution of $\{X_i: i \leq 2^N\}$ is such that a q-subset of $\{1, 2, \ldots, 2^N\}$ is chosen at random and $X_i = 1$ or 0 according as i belongs to this subset or i fails to belong to the subset. $\{Y_1, \ldots, Y_{2^N}\}$ are mutually independent and independent of $\{X_1, \ldots, X_{2^N}\}$ with $P(Y_i = 1) = P(Y_i = -1) = \frac{1}{2}$.

For any $\omega \in \Omega$, $\sum_{i=1}^{2^N} X_i(\omega) Y_i(\omega) \phi(A_i) \in V(p/25, 2)$. For fixed $t \in (0, 1)$, suppose as above $i_1, \ldots, i_r$ are distinct indices so that $u_k(t) = |f_{i_k}(t)|$, $1 \leq k \leq r$. Let Ω_k $(1 \leq k \leq r)$ be the event that $X_{i_1} = \cdots = X_{i_{k-1}} = 0$ but $X_{i_k} = 1$. Then by symmetry $P\{\omega \in \Omega_k: |\sum X_i Y_i f_i(t)| \geq u_k(t)\} \geq \frac{1}{2}P(\Omega_k)$. Hence

$$P\left\{\left|\sum X_i Y_i f_i(t)\right| \geq u_r(t)\right\} \geq \frac{1}{2}P\left(\bigcup_{k=1}^{r} \Omega_k\right) \geq \frac{1}{2}\left(1 - \left(1 - \frac{q}{2^N}\right)^r\right)$$
$$\geq \frac{1}{2}\left(1 - \left(1 - \frac{\delta}{2}\right)^r\right) > \frac{8}{25}.$$

Now $P \otimes \lambda\{(\omega, t): |\sum X_i Y_i f_i| \geq 2\} \leq p/25$ and hence $\lambda\{t: u_r(t) \geq 2\} \leq p/8$. Thus $\lambda(F) \leq p/8$. However we originally showed $\lambda(F) \geq 3p/16$ so that we have arrived at the desired contradiction and the proof of the theorem is complete.

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