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TAUBERIAN OPERATORS ON BANACH SPACES

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ABSTRACT. A Tauberian operator: $E \to F$ (Banach spaces) is one which satisfies $T''g \in F$, $g \in E''$ imply $g \in E$. The action of such operators and their pre-images on compact sets is studied in order to compare "Tauberian" with "weakly compact", an opposite property. Properties related to range closed are introduced which force operators with Tauberian-like properties to be Tauberian. Classes of spaces appear for which Tauberian is equivalent to semi-Fredholm. One example of this is the historical reason for the definition of these operators.

1. Tauberian operators (§2) appeared in response to a problem in summability (see [5]). Results on range closed Tauberian and co-Tauberian operators are given in [10]; our main results do not assume range closed.

We use standard Banach space notation: $B(E, F)$ is the set of bounded linear maps from E to F, E', $\hat{E}$ are the dual of E and the natural embedding of E in E'', T': $F' \to E'$ is the adjoint of T, and RT, NT are the range and null-space of T. We write $E \subset E''$, identifying E with $\hat{E}$ so that $T''|E = T$.

2. We call $T \in B(E, F)$ *Tauberian* if $T''^{-1}[F] \subset E$, i.e. $g \in E''$, $T''g \in F$ imply $g \in E$. It is immediate that a Tauberian operator has the property

$$(\mathrm{N}): g \in E'', T''g = 0 \text{ imply } g \in E.$$

It is known that (N) implies

$$(\mathrm{R}): NT \text{ is reflexive.}$$

For range closed operators the three conditions are equivalent [5], [10]. Parts of this equivalence hold more generally as follows: (We omit the proofs.) (i) The following three conditions are equivalent. (a) T is Tauberian, (b) T has property (N) and $T[D]$ is closed (D is the unit disc), (c) T has property (N) and the closure of $T[D]$ is included in the range of T. (ii) T has property (N) if and only if it has property (R) and the range of T' has norm closure equal to its w^* closure.

For T one-to-one these results apply to the map given in [3, II, Lemma 1(iii)]. It also follows that any adjoint map with property (N) must be Tauberian. By considering the quotient map (which is automatically Tauberian) we see that if $T \in B(E, F)$ is Tauberian and $S \subset F$ is reflexive, then $T^{-1}[S]$ is reflexive. This generalizes (R).

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3. Tauberian operators are, in a sense, opposite to weakly compact operators since $T \in B(E, F)$ is weakly compact if and only if $RT'' \subset F$. Thus the set of Tauberian operators lies in the complement of a closed subspace of $B(E, F)$, a closed ideal if $F = E$. Other nonweakly compact operators can be obtained by taking adjoints of Tauberian operators. There are examples of pairs of Banach spaces, E, F, neither one reflexive, such that *no operator in $B(E, F)$ is Tauberian*). It suffices to consider a pair such that every operator is weakly compact.

We now characterize Tauberian operators (internally) in terms of their mapping of compact sets. This may be compared with the fact that T is weakly compact if and only if it maps all bounded sets into relatively weakly compact sets. We begin with a criterion for property (N).

3.1. THEOREM. *Let E, F be Banach spaces, $T \in B(E, F)$. Then T has property* (N) *if and only if every bounded sequence $\{x_n\}$ in E with $Tx_n \to 0$ in F has a weakly convergent subsequence.*

Necessity. For any w^* cluster point z of $\{x_n\}$, $T''z = 0$ since $T''x_n = Tx_n \to 0$ in norm. Hence $z \in E$ and it follows that $\{x_n\}$ is relatively weakly compact, hence relatively weakly sequentially compact, by the Eberlein-Smulian theorem.

Sufficiency. Suppose $g \in E''$, $T''g = 0$. We may assume $\|g\| = 1$. There exists a net x in D, the unit disc of E, with $x \to g$, w^*. Then $Tx = T''x \to T''g = 0$, w^* and so $Tx \to 0$, weakly. Thus if C is any convex subset of E such that $x \in C$ eventually, it follows that 0 is in the norm closure of TC. Writing $x = \{x^\alpha: \alpha \in A\}$; for each $\alpha \in A$, let C_α be the convex hull of $\{x^\gamma: \gamma \geqslant \alpha\}$. As just mentioned, each C_α contains a sequence $\{c_\alpha^n\}$ with $\|Tc_\alpha^n\| \to 0$. By hypothesis $\{c_\alpha^n\}$ has a subsequence converging weakly to c_α, say. Clearly $Tc_\alpha = 0$. Now $\{c_\alpha: \alpha \in A\}$ is relatively weakly sequentially compact [each sequence $\{v_n\}$ in it has $Tv_n = 0$], hence, by Eberlein-Smulian, relatively weakly compact. Thus c_α has a weak cluster point c. The proof is concluded by showing $g = c$. This is clear since $c_\alpha \to g$, w^*, and c is a w^* cluster point of c_α.

3.2. THEOREM. *Let E, F be Banach spaces, $T \in B(E, F)$. The following are equivalent*:

(i) *T is Tauberian.*

(ii) *For every bounded set $B \subset E$ such that TB is relatively weakly compact, B is relatively weakly compact.*

(iii) *For every bounded set $B \subset E$ such that TB is relatively compact, B is relatively weakly compact.*

(i) implies (ii). Let T be Tauberian, B bounded $\subset E$, TB relatively weakly compact. Let x be a net in B. Then x, being a bounded net in E'' has a w^* convergent subnet which we may assume to be x itself; say $x \to g \in E''$, w^*. Also x has a subnet, which we may again assume to be x itself, such that $Tx \to y \in F$ weakly. Then $T''g = w^* \lim T''x = w^* \lim Tx = y$. Since T is Tauberian, $g \in E$. Then $x \to g$ weakly. Thus B is relatively weakly compact.

(ii) implies (iii). If TB is relatively norm compact its norm closure is a compact, hence weakly compact set which includes TB; hence TB is relatively weakly compact.

(iii) implies (i). Let D be the unit disc of E and $y \in \overline{TD}$. Choose $\{x_n\}$ in D with $Tx_n \to y$. By hypothesis $\{x_n\}$ has a weak cluster point x; clearly $x \in D$ and $y = Tx$, so $y \in RT$. By §2, (i), it remains to prove that T has property (N). This follows from 3.1 and the Eberlein-Smulian theorem.

4. We call $T \in B(E, F)$ a *semi-Fredholm* operator (as in [8]) and write $T \in \Phi_+$ if T is range closed and dim $NT < \infty$. Such operators are Tauberian and in the eponymic case the converse is true, see 4.3 and [5]. Such operators are also discussed in [4], [9]. For the next result, due to Yood, Wolf, Basley, Schubert et al., see [6, 4.11, 4.12].

4.1. THEOREM. *The following are equivalent for* $T \in B(E, F)$:

(i) $T \in \Phi_+$.

(ii) *For every bounded set* $B \subset E$ *such that* TB *is compact,* B *is relatively compact.*

(iii) *Every bounded sequence* $\{x_n\}$ *in* E *with* $Tx_n \to 0$ *has a convergent subsequence.*

4.2. THEOREM. *Let* $T \in B(E, F)$ *be Tauberian. Then* $T \in \Phi_+$ *if and only if* $T|R \in \Phi_+$ *for all reflexive subspaces* R *of* E.

That $T|R \in \Phi_+$ for any closed R is by 4.1. Conversely suppose that $T \notin \Phi_+$. The hypothesis implies that dim $NT < \infty$ so we may restrict T to the complement of NT, i.e. we may assume T is one-to-one. By 4.1 we can find $\{x_n\}$, $\|x_n\| = 1$, $Tx_n \to 0$. By [2, p. 156], $\{x_n\}$ has a basic subsequence, which we may assume to be $\{x_n\}$ itself, with $\|Tx_n\| \leqslant 2^{-n}$. Let X be the linear closure of $\{x_n\}$. Then $T|X$ is compact and Tauberian. Now let D be the unit disc of X. By 3.2, D is relatively weakly compact hence X is reflexive. But $T|X$ is not range closed.

4.3. COROLLARY. *Suppose* E *has no reflexive infinite dimensional subspace; then for a map* $T \in B(E, F)$, T *is Tauberian if and only if* $T \in \Phi_+$.

5. The result obtained by taking $E = c$ in 4.3 is an extension of the Berg-Crawford-Whitley theorem. (See [5].) We may also take $E = l$ (space of absolutely convergent series) in 4.3 but Theorem 5.1 is better. See also 5.2.

5.1. THEOREM. *For any Banach space* F *and* $T \in B(l, F)$ *the following are equivalent.*

(i) T *is Tauberian.*

(ii) T *has property* (N).

(iii) $T \in \Phi_+$.

By 4.3, (i) and (iii) are equivalent. If (ii) holds, T has property (R) and so its null-space is finite dimensional. Since the restriction of T to a closed subspace satisfies (N), we may assume that T is one-to-one; our assumption is then that T'' is also one-to-one and so T': $F' \to l^\infty$ has dense range. By a remark of Beurling (see [1, Theorem 3]), T' is onto; hence T is an isomorphism. (The equivalence of (i), (ii) also follows from 3.1, 3.2.)

The next result (which is false for $E = c$) generalizes the equivalence of (i) and (ii) in 5.1. The referee has pointed out that this proof may be simplified by citing [7].

5.2. THEOREM. *Let E be a weakly sequentially complete Banach space, F any Banach space, and $T \in B(E, F)$. Then T is Tauberian if and only if it has property* (N).

Applying Eberlein-Smulian to 3.2(iii), it is sufficient to show that if $\{x_n\}$ in E is bounded and $\{Tx_n\}$ is convergent, then $\{x_n\}$ has a weakly convergent subsequence. As in 5.1, we may restrict ourselves to the linear closure of $\{x_n\}$, thus we may assume that E is separable. Let $R = NT$; it is reflexive since T has property (N). Let q: $E \to E/R$ be the quotient map. We first show that $\{q(x_n)\}$ is weakly Cauchy. [If this is false, there exist increasing sequences $\{m(k), n(k)\}$ and $f \in (E/R)'$ with $|f[q(x_{m(k)}) - q(x_{n(k)})]| \geqslant 1$. Let w be a weak cluster point of $\{x_{m(k)} - x_{n(k)}\}$ by 3.1. Then $Tw = 0$, so $w \in R$. Thus $q(w) = 0$ which contradicts $|f[q(w)]| \geqslant 1$.] This means that $\{f(x_n)\}$ is convergent for all $f \in R^{\perp} \subset E'$. Now $R' = E'/R^{\perp}$ is separable since $R'' = R$ is, so there exists a sequence $\{f_n\} \subset E'$ such that the linear closure of $R^{\perp} \cup \{f_1, f_2, \ldots\}$ is E'. Select a subsequence $\{u_n\}$ of $\{x_n\}$ such that $\lim_n f_i(u_n)$ exists for each i. [This is possible since $\{x_n\}$ is bounded.] Since also $\{f(u_n)\}$ is convergent for all $f \in R^{\perp}$ and $\{u_n\}$ is bounded, it follows that $\{u_n\}$ is weakly Cauchy, hence weakly convergent.

6. We ask the following questions.

6.1. When is it true that T is Tauberian if and only if T'' is? (For example this is true if T is range closed.)

6.2. Which Banach spaces E, other than those mentioned in 4.3 have the property that a Tauberian map must be range closed? (Clearly every Tauberian map on E has finite dimensional null-space if and only if E has no infinite dimensional reflexive subspace.)

6.3. When is the induced map T: $E''/E \to F''/F$ an isomorphism? (It is one-to-one if and only if T is Tauberian. For range closed maps see [10].)

6.4. We feel that a Tauberian map T: $C(X) \to C(Y)$ must be close to an isomorphism in some sense.

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