PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 55, Number 2, March 1976

IDEMPOTENTS OF NORM ONE AND BANACH ALGEBRA REPRESENTATIONS OF COMPACT GROUPS

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ABSTRACT. Let G be a finite group of order n and let A be a (real or complex) Banach algebra. Rudin and Schneider [3] ask whether a mapping $f\colon G \to A$ satisfying $\|f(x)\| = 1$ and $f(x) = (1/n)\Sigma_{y\in G}f(xy^{-1})f(y)$ is necessarily a homomorphism (Question 1, p. 602). They give an affirmative answer if A is either commutative and semisimple or strictly convex.

Here, we prove this result for general Banach algebras, and at the same time prove the natural generalization to compact groups. This allows us to characterize norm one idempotents in generalized group algebras.

Suppose that G is a compact group with identity e and that X is a normed space. A *representation* of G on X is a homomorphism $T\colon G \to B(X)$. T is an *isometric representation* if in addition each T_x is an isometry on X; in this case $T_e = I$ and each T_x is invertible. T is *semi-isometric* if $\|T_x\| \leqslant 1$ for $x \in G$; in this case T_e is a projection of norm one and $T_x = S_xT_e$ where S is an isometric representation of G on $T_e(X)$.

We equip G with its left-invariant Haar measure λ, normalized so that $\lambda(G) = 1$; we shall abbreviate $d\lambda(x)$ to dx. If X is any Banach space (or Banach algebra) then a map $\phi\colon G \to X$ is *Bochner measurable* if it is the almost everywhere limit of a sequence of simple functions, and *Bochner integrable* if in addition $\int\|\phi(x)\|\,dx < \infty$. An operator valued function $T\colon G \mapsto B(X)$ is *strongly measurable* if for each $\xi \in X$, the map $x \to T_x\xi$ is Bochner measurable (see Hille-Phillips [2, pp. 72–74]). If for each $\xi \in X$, $T_x\xi$ is Bochner integrable and $T\xi = \int_G T_x\xi\,dx$, then we shall write $T = \int_G T_x\,dx$.

LEMMA 1. *If $x \to T_x$ is a strongly measurable mapping from G into $B(X)$, then so is the map $y \mapsto T_{xy^{-1}}T_y$.*

We omit the proof of Lemma 1, which follows by approximation by simple functions.

LEMMA 2. *If $\phi\colon G \to X$ is Bochner integrable then*

$$\lim_{x\to e}\int_G \|\phi(ux - \phi(u))\|\,du = 0.$$

PROOF. See [2, Theorem 3.8.3] for the case $G = \mathbf{R}^n$; the same proof applies here. The lemma is proved first for simple functions and follows in general by approximation. Again we omit the details.

Received by the editors August 6, 1975.
AMS (MOS) subject classifications (1970). Primary 43A20; Secondary 43A65.

LEMMA 3. *Suppose $x \to T_x$ is a strongly measurable mapping from G into $B(X)$ such that* $\sup_{x \in G}\|T_x\| = M < \infty$. *Suppose for $x \in G$,*

$$T_x = \int_G T_{xy^{-1}} T_y \, dy.$$

Then $x \mapsto T_x$ is strongly continuous (i.e. for $\xi \in X$, $x \to T_x \xi$ is continuous).

PROOF. For $\xi \in X$,

$$T_x \xi = \int_G T_{xy^{-1}} T_y \xi \, dy,$$

$$T_{xu} \xi = \int_G T_{xuy^{-1}} T_y \xi \, dy = \int_G T_{xy^{-1}} T_{yu} \xi \, dy,$$

by the invariance of Haar measure. Hence

$$\|T_x \xi - T_{xu} \xi\| \leq \int_G \|T_{xy^{-1}}(T_y \xi - T_{yu} \xi)\| \, dy$$

$$\leq M \int_G \|T_y \xi - T_{yu} \xi\| \, dy \to 0 \quad \text{as } u \to e \text{ by Lemma 2.}$$

LEMMA 4. *If $x \mapsto T_x$ is a strongly continuous map from G into $B(X)$ then*
(i) *$(x, y) \to T_x T_y$ is strongly continuous on $G \times G$,*
(ii) *$(x, y) \to T_x^* T_y^*$ is weak*-continuous on $G \times G$.*

PROOF. Since G is compact, the continuous function $x \to \|T_x \xi\|$ is bounded for every $\xi \in X$. Hence the Uniform Boundedness Theorem shows that $\sup_{x \in G}\|T_x\| = M < \infty$. Then

$$\|T_x T_y \xi - T_{x_0} T_{y_0} \xi\| \leq \|T_x T_y \xi - T_x T_{y_0} \xi\| + \|T_x T_{y_0} \xi - T_{x_0} T_{y_0} \xi\|$$

$$\leq M\|T_y \xi - T_{y_0} \xi\| + \|(T_x - T_{x_0})(T_{y_0} \xi)\|$$

$$\to 0 \quad \text{as } x \to x_0 \text{ and } y \to y_0.$$

(ii) follows immediately by duality.

REMARK. It is not true that $x \mapsto T_x^*$ is strongly continuous from G into $B(X^*)$. For example let $G = \prod_{i=1}^{\infty}\{-1, +1\}$ and consider the representation on l_1 given by $(T_x \xi)_n = x_n \xi_n$ for $\xi = (\xi_n) \in l_1$ and $x = (x_n) \in G$.

LEMMA 5. *Suppose T: $G \to B(X)$ is strongly continuous and satisfies*

$$\text{(i) } \|T_x\| \leq 1, \quad x \in G, \qquad \text{(ii) } T_x = \int_G T_{xy^{-1}} T_y \, dy, \quad x \in G.$$

Then, for $\xi \in X$, $\|T_x \xi\|$ is independent of x and

$$\|T_x \xi\| = \|T_y T_z \xi\| \quad \textit{whenever } x, y, z \in G.$$

PROOF.

$$\|T_x \xi\| \leq \int_G \|T_{xy^{-1}} T_y \xi\| \, dy$$

$$\leq \int_G \|T_y \xi\| \, dy \quad \text{for any } x \in G. \tag{1}$$

Hence $\|T_x \xi\| = \int_G \|T_y \xi\| \, dy$ for almost every $x \in G$. Strong continuity of T_x ensures that equality holds everywhere. Referring back to inequality (1) we see that $\|T_x \xi\| = \|T_{xy^{-1}} T_y \xi\|$ for almost every $y \in G$. Again by continuity equality holds everywhere and the result follows.

We are now able to prove the first version of our main result.

THEOREM 1. *Suppose* $T\colon G \to B(X)$ *is strongly measurable and satisfies*
(i) $\|T_x\| \leqslant 1$ $(x \in G)$,
(ii) T_e *is an isometry, i.e.* $\|T_e\xi\| = \|\xi\|$ *for* $\xi \in X$,
(iii) $T_x = \int_G T_{xy^{-1}} T_y \, dy$ $(x \in G)$.
Then T *is a strongly continuous isometric representation of* G.

PROOF. T is strongly continuous by Lemma 3. Let U be the closed unit ball of X^* and ϕ be any extreme point of U. Since T_e is an isometry, it follows by the Hahn-Banach Theorem that there exists $\psi \in U$ such that $T_e^*\psi = \phi$.

For $\xi \in X$,

$$\phi(\xi) = \psi(T_e\xi) = \int_G \psi(T_x T_{x^{-1}}\xi)\, dx.$$

For each measurable subset A of G with $\lambda(A) > 0$ define $\phi_A \in X^*$ by

$$\phi_A(\xi) = \lambda(A)^{-1} \int_A \psi(T_x T_{x^{-1}}\xi)\, dx.$$

Clearly $\phi_A \in U$ and $\phi = \lambda(A)\phi_A + \lambda(G - A)\phi_{G-A}$. As ϕ is an extreme point of U, $\phi = \phi_A = \phi_{G-A}$. Thus

$$\int_A (\phi(\xi) - \psi(T_x T_{x^{-1}}\xi))\, dx = 0$$

for every measurable $A \subset G$ and $\xi \in X$. Hence for $\xi \in X$, $\psi(T_x T_{x^{-1}}\xi) = \phi(\xi)$ almost everywhere, and by the strong continuity of the map $x \mapsto T_x T_{x^{-1}}$ (see Lemma 4), equality holds everywhere. Hence $T^*_{x^{-1}} T^*_x \psi = \phi$ for $x \in G$. The choice of ψ shows that $T_e^*\phi = (T_e^*)^2\psi = \phi$. As T_e^* is weak*-continuous and, by the Krein-Milman theorem, U is the weak*-closed convex cover of its extreme points we have $T_e^* = I$. Thus $\phi = \psi$ and we have also proved that $T^*_{x^{-1}} T^*_x \phi = \phi$ for any extreme point ϕ, i.e. $T_x T_{x^{-1}} = I$ by the same argument as above. Hence each T_x is an isometric isomorphism of X.

Again if ϕ is any extreme point of U, so is $T_x^*\phi$ and

$$(T_x^*\phi)(\xi) = \int_G \phi(T_{xy^{-1}} T_y \xi)\, dy \quad (\xi \in X).$$

Arguing as before we conclude that

$$T_x^* = T_y^* T^*_{xy^{-1}}, \qquad x, y \in G,$$

i.e. T is an isometric representation.

THEOREM 2. *Suppose* $T\colon G \to B(X)$ *is strongly measurable and satisfies*
(i) $\|T_x\| \leqslant 1$ $(x \in G)$,
(ii) $T_x = \int T_{xy^{-1}} T_y \, dy$ $(x \in G)$. *Then* T *is a semi-isometric representation of* G.

PROOF. Again we have T strongly continuous by Lemma 2. Define a seminorm $|\cdot|$ on X by

$$|\xi| = \|T_e\xi\|,$$

and let $N = T_e^{-1}(0)$. By Lemma 5, $|T_x\xi| = |\xi|$ for $x \in G$. Hence there is an induced representation on X/N satisfying the hypotheses of Theorem 1. By

Theorem 1,

$$|T_e\xi - \xi| = 0, \quad \xi \in X,$$

and

$$|T_xT_y\xi - T_{xy}\xi| = 0, \quad \xi \in X, x, y \in G.$$

Hence $\|T_e^2\xi - T_e\xi\| = 0$, i.e. T_e is a projection. By Lemma 5, $\|T_x(T_e\xi - \xi)\| = \|T_e(T_e\xi - \xi)\| = \|T_e^2\xi - T_e\xi\| = 0$ for $x \in G$, i.e. $T_xT_e = T_x$. Also by Lemma 5, for any $w \in G$,

$$\|T_wT_xT_y\xi - T_wT_{xy}\xi\| = \|T_e(T_xT_y\xi - T_{xy}\xi)\| = |T_xT_y\xi - T_{xy}\xi| = 0.$$

Thus $T_wT_xT_y = T_wT_{xy}$ for $w, x, y \in G$.

Now suppose we have the equation

$$T_eT_x = T_x \quad (x \in G). \tag{2}$$

Then we have $T_xT_y = T_eT_xT_y = T_eT_{xy} = T_{xy}$, and the proof is complete. Therefore it remains only to establish (2). Here the only difficulty is that $x \to T_x^*$ need not be strongly continuous. (For, if it were, we could apply the argument above to $x \to T_x^*$.) This is circumvented by the construction that follows. We shall assume here that X is complete, for convenience.

Fix any $\xi_0 \in X$ and let X_0 be the smallest closed subspace of X such that $\xi_0 \in X_0$ and $T_x(X_0) \subset X_0$, $x \in G$. It is enough to consider the induced map $G \to B(X_0)$.

Let $C_0 = \{(\xi_0) \cup (T_x\xi_0\colon x \in G) \cup (T_xT_y\xi_0\colon x, y \in G)\}$. Then C_0 is compact and so is its closed absolutely convex hull C. Let Y be the linear span of C equipped with the norm whose unit ball is C. Then Y is a Banach space, since C is compact. Furthermore since $T_wT_xT_y = T_wT_{xy}$ for $w, x, y \in G$ we have $T_w(C) \subset C$. Thus Y is invariant for each T_x and so Y is dense in X_0. Let $\tilde{T}$ denote the restriction of T_x to Y; then in the norm of Y, $\|\tilde{T}_x\| \leqslant 1$. Let J: $Y \to X_0$ be the inclusion map. By construction J is compact and $J\tilde{T}_x = T_xJ$ $(x \in G)$. Now suppose $T_eT_w \neq T_w$. Since T_e is a projection, $T_w(X_0) \not\subset T_e(X_0)$. Hence there exists $\psi \in X_0^*$ such that $T_w^*\psi \neq 0$ but $T_e^*\psi = 0$. Since J is compact the map $x \to J^*T_x^*\psi$ is continuous. Choose $u \in G$ such that

$$c = \|J^*T_u^*\psi\| = \max_{x \in G} \|J^*T_x^*\psi\|.$$

Since $J(C)$ is compact there exists $\eta \in C$ such that $T_u^*\psi(J\eta) = c$. Then

$$|\psi(T_{ux^{-1}}T_xJ\eta)| = |\psi(T_{ux^{-1}}J\tilde{T}_x\eta)| = |J^*T_{ux^{-1}}^*\psi(\tilde{T}_x\eta)| \leqslant c.$$

However

$$\int_G \psi(T_{ux^{-1}}T_xJ\eta)\,dx = \psi(T_uJ\eta) = c.$$

Hence $\psi(T_{ux^{-1}}T_xJ\eta) \equiv c$ by continuity. In particular putting $x = u$, $\psi(T_eT_uJ\eta) = c$, i.e. $T_e^*\psi(T_uJ\eta) = c$. However $T_e^*\psi = 0$ and hence $c = 0$. Thus $J^*T_w^*\psi = 0$; but Y is dense in X_0 and hence $T_w^*\psi = 0$, which is a contradiction to our initial assumption. This completes the proof.

COROLLARY. *Suppose $\mathcal{Q}$ is a bounded subsemigroup of $B(X)$ and T: $G \to B(X)$ is a strongly measurable mapping satisfying*

(i) $T(G) \subset \mathcal{Q}$,

(ii) $T_x = \int_G T_{xy^{-1}} T_y \, dy$ $(x \in G)$.
Then T is a representation of G.

PROOF. Renorm X by $|\xi| = \sup_{A \in \mathcal{Q} \cup \{I\}} \|A\xi\|$.

We can now characterize idempotents of norm one in the generalized group algebra of a locally compact group G. Let A be a Banach algebra and let $L^1(G : A)$ denote the space of Bochner integrable functions f: $G \to A$. $L^1(G : A)$ is a Banach algebra under the multiplication

$$f * g(x) = \int_G f(xy^{-1}) g(y) \, dy$$

and norm

$$\|f\| = \int_G \|f(x)\| \, dx.$$

It is well known that if $A = C$, the norm one idempotents of $L^1(G: C) = L^1(G)$ are of the form $\lambda(H)^{-1}\rho(x)\chi_H(x)$ where H is a compact open subgroup, ρ is a character on H and χ_H is the characteristic function of H. (See [1, 2.1.4].) Since the elements of $L^1(G: A)$ are equivalence classes, if f is an idempotent in $L^1(G: A)$, then we can assume that the representative satisfies $f(x) = \int_G f(xy^{-1})f(y) \, dy$ for all $x \in G$. We make this assumption in the following theorem.

THEOREM 3. *Let $f \in L^1(G : A)$ and suppose $\|f\| = 1$ and $f * f = f$. Then f is continuous and there exists a compact open subgroup H of G such that*

(i) $f(x) = 0, x \notin H$,

(ii) $f(xy) = \lambda(H) f(x) f(y), x, y \in H$,

(iii) $\|f(x)\| = (\lambda(H))^{-1}, x \in H$.

PROOF.

$$\|f(x)\| \leqslant \int_G \|f(xy^{-1})\| \|f(x)\| \, dy$$

and

$$1 = \int_G \|f(x)\| \, dx = \int_G \int_G \|f(xy^{-1})\| \|f(y)\| \, dy \, dx$$

so that

$$\|f(x)\| = \int_G \|f(xy^{-1})\| \|f(y)\| \, dy \quad \text{almost everywhere.}$$

Hence if $\gamma(x) = \int_G \|f(xy^{-1})\| \|f(y)\| \, dy$ then γ is a norm one idempotent in $L^1(G)$. Hence there is a compact open subgroup H such that $\gamma(x) = \lambda(H)^{-1}\chi_H(x)$.

It follows that $\|f(x)\| = 0$ if $x \notin H$, and that $\|f(x)\| \leqslant \lambda(H)^{-1}$ for all $x \in G$.

We may suppose A has an identity and then identify A as a subalgebra of $B(A)$. If we define for $x \in H$,

$$T_x a = \lambda(H) f(x) a,$$

then $\|T_x\| \leqslant 1$ and $\int_H T_{xy^{-1}} T_y dy = T_x$. By Theorem 2, $T_x T_y = T_{xy}$ and the result follows.

If G is compact we may also consider the algebra $L^p(G : A)$ $(1 \leqslant p < \infty)$ with the norm

$$\|f\|_p = \left\{ \int \|f(x)\|^p \, dx \right\}^{1/p}.$$

Using a similar approach to that of [4] we obtain

THEOREM 4. *If $f \in L^p(G : A)$ satisfies $\|f\|_p = 1$ and $f * f = f$ then $f(xy) = f(x)f(y)$ for all $x, y \in G$.*

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