Suppose that $G$ is a compact group with identity $e$ and that $X$ is a normed space. A representation of $G$ on $X$ is a homomorphism $T: G \to B(X)$. $T$ is an isometric representation if in addition each $T_x$ is an isometry on $X$; in this case $T_e = I$ and each $T_x$ is invertible. $T$ is semi-isometric if $\|T_x\| < 1$ for $x \in G$; in this case $T_e$ is a projection of norm one and $T_x = S_x T_e$ where $S$ is an isometric representation of $G$ on $T_e(X)$.

We equip $G$ with its left-invariant Haar measure $\lambda$, normalized so that $\lambda(G) = 1$; we shall abbreviate $d\lambda(x)$ to $dx$. If $X$ is any Banach space (or Banach algebra) then a map $\phi: G \to X$ is Bochner measurable if it is the almost everywhere limit of a sequence of simple functions, and Bochner integrable if in addition $\int \|\phi(x)\| \, dx < \infty$. An operator valued function $T: G \to B(X)$ is strongly measurable if for each $\xi \in X$, the map $x \to T_x \xi$ is Bochner measurable (see Hille-Phillips [2, pp. 72-74]). If for each $\xi \in X$, $T_x \xi$ is Bochner integrable and $T_\xi = \int_G T_x \xi \, dx$, then we shall write $T = \int_G T_x \, dx$.

**Lemma 1.** If $x \to T_x$ is a strongly measurable mapping from $G$ into $B(X)$, then so is the map $y \mapsto T_{xy^{-1}} T_y$.

We omit the proof of Lemma 1, which follows by approximation by simple functions.

**Lemma 2.** If $\phi: G \to X$ is Bochner integrable then

$$\lim_{x \to e} \int_G \|\phi(ux - \phi(u))\| \, du = 0.$$  

**Proof.** See [2, Theorem 3.8.3] for the case $G = \mathbb{R}^n$; the same proof applies here. The lemma is proved first for simple functions and follows in general by approximation. Again we omit the details.
Lemma 3. Suppose $x \mapsto T_x$ is a strongly measurable mapping from $G$ into $B(X)$ such that $\sup_{x \in G} \|T_x\| = M < \infty$. Suppose for $x \in G$,

$$T_x = \int_G T_{xy^{-1}}T_y \, dy.$$ 

Then $x \mapsto T_x$ is strongly continuous (i.e. for $\xi \in X$, $x \mapsto T_x\xi$ is continuous).

Proof. For $\xi \in X$,

$$T_x\xi = \int_G T_{xy^{-1}}T_x\xi \, dy,$$

$$T_{xy}\xi = \int_G T_{xy^{-1}}T_y\xi \, dy = \int_G T_{xy^{-1}}T_{y_0}\xi \, dy,$$

by the invariance of Haar measure. Hence

$$\|T_x\xi - T_{y_0}\xi\| \leq \int_G \|T_{xy^{-1}}(T_x\xi - T_{y_0}\xi)\| \, dy,$$

$$\leq M \int_G \|T_y\xi - T_{y_0}\xi\| \, dy \to 0 \quad \text{as } u \to e \text{ by Lemma 2}.$$ 

Lemma 4. If $x \mapsto T_x$ is a strongly continuous map from $G$ into $B(X)$ then

(i) $(x, y) \mapsto T_xT_y$ is strongly continuous on $G \times G$,

(ii) $(x, y) \mapsto T_xT_y^*$ is weak*-continuous on $G \times G$.

Proof. Since $G$ is compact, the continuous function $x \mapsto \|T_x\xi\|$ is bounded for every $\xi \in X$. Hence the Uniform Boundedness Theorem shows that $\sup_{x \in G} \|T_x\| = M < \infty$. Then

$$\|T_xT_y\xi - T_{x_0}T_{y_0}\xi\| \leq \|T_xT_y\xi - T_xT_{y_0}\xi\| + \|T_xT_{y_0}\xi - T_{x_0}T_{y_0}\xi\|,$$

$$\leq M\|T_y\xi - T_{y_0}\xi\| + \|(T_x - T_{x_0})(T_{y_0}\xi)\|,$$

$$\to 0 \quad \text{as } x \to x_0 \text{ and } y \to y_0.$$

(ii) follows immediately by duality.

Remark. It is not true that $x \mapsto T_x^*$ is strongly continuous from $G$ into $B(X^*)$. For example let $G = \mathbb{N} \{ -1, +1 \}$ and consider the representation on $l_1$ given by $(T_x)_{n} = x, n$ for $\xi = (\xi_n) \in l_1$ and $x = (x_n) \in G$.

Lemma 5. Suppose $T: G \to B(X)$ is strongly continuous and satisfies

(i) $\|T_x\| \leq 1$, \quad $x \in G$, \quad (ii) $T_x = \int_G T_{xy^{-1}}T_y \, dy$, \quad $x \in G$.

Then, for $\xi \in X$, $\|T_x\xi\|$ is independent of $x$ and

$$\|T_x\xi\| = \|T_yT_x\xi\| \quad \text{whenever } x, y, z \in G.$$

Proof.

$$\|T_x\xi\| \leq \int_G \|T_{xy^{-1}}T_y\xi\| \, dy$$

$$< \int_G \|T_x\xi\| \, dy \quad \text{for any } x \in G.$$ 

Hence $\|T_x\xi\| = \int_G \|T_y\xi\| \, dy$ for almost every $x \in G$. Strong continuity of $T_x$ ensures that equality holds everywhere. Referring back to inequality (1) we see that $\|T_x\xi\| = \|T_{xy^{-1}}T_y\xi\|$ for almost every $y \in G$. Again by continuity
equality holds everywhere and the result follows.

We are now able to prove the first version of our main result.

**Theorem 1.** Suppose \( T: G \to B(\mathcal{X}) \) is strongly measurable and satisfies

(i) \( \| T_x \| < 1 \) \( (x \in G) \),

(ii) \( T \) is an isometry, i.e. \( \| T_x \xi \| = \| \xi \| \) for \( \xi \in \mathcal{X} \),

(iii) \( T_x = \int_G T_{xy^{-1}} T_y \, dy \) \( (x \in G) \).

Then \( T \) is a strongly continuous isometric representation of \( G \).

**Proof.** \( T \) is strongly continuous by Lemma 3. Let \( U \) be the closed unit ball of \( \mathcal{X}^* \) and \( \phi \) be any extreme point of \( U \). Since \( T_\phi \) is an isometry, it follows by the Hahn-Banach Theorem that there exists \( \psi \in U \) such that \( T_\phi^* \psi = \psi \).

For \( \xi \in \mathcal{X} \),

\[
\phi(\xi) = \psi(T_\phi \xi) = \int_G \psi(T_x T_{x^{-1}} \xi) \, dx.
\]

For each measurable subset \( A \) of \( G \) with \( \lambda(A) > 0 \) define \( \phi_A \in \mathcal{X}^* \) by

\[
\phi_A(\xi) = \lambda(A)^{-1} \int_A \psi(T_x T_{x^{-1}} \xi) \, dx.
\]

Clearly \( \phi_A \in U \) and \( \phi = \lambda(A) \phi_A + \lambda(G - A) \phi_{G - A} \). As \( \phi \) is an extreme point of \( U \), \( \phi = \phi_A = \phi_{G - A} \). Thus

\[
\int_A (\phi(\xi) - \psi(T_x T_{x^{-1}} \xi)) \, dx = 0
\]

for every measurable \( A \subseteq G \) and \( \xi \in \mathcal{X} \). Hence for \( \xi \in \mathcal{X} \), \( \psi(T_x T_{x^{-1}} \xi) = \phi(\xi) \) almost everywhere, and by the strong continuity of the map \( x \mapsto T_x T_{x^{-1}} \) (see Lemma 4), equality holds everywhere. Hence \( T_\phi^* T_x \psi = \psi \) for \( x \in G \). The choice of \( \psi \) shows that \( T_\phi^* \phi = (T_\phi^*)^2 \psi = \phi \). As \( T_\phi^* \) is weak*-continuous and, by the Krein-Milman theorem, \( U \) is the weak*-closed convex cover of its extreme points we have \( T_\phi^* = I \). Thus \( \phi = \psi \) and we have also proved that \( T_\phi^* T_x \phi = \phi \) for any extreme point \( \phi \), i.e. \( T_x T_{x^{-1}} = I \) by the same argument as above. Hence each \( T_x \) is an isometric isomorphism of \( \mathcal{X} \).

Again if \( \phi \) is any extreme point of \( U \), so is \( T_\phi^* \phi \) and

\[
(T_\phi^* \phi)(\xi) = \int_G \phi(T_{xy^{-1}} T_y \xi) \, dy \quad (\xi \in \mathcal{X}).
\]

Arguing as before we conclude that

\[
T_x^* = T_y^* T_{xy^{-1}}, \quad x, y \in G,
\]

i.e. \( T \) is an isometric representation.

**Theorem 2.** Suppose \( T: G \to B(\mathcal{X}) \) is strongly measurable and satisfies

(i) \( \| T_x \| < 1 \) \( (x \in G) \),

(ii) \( T_x = \int_G T_{xy^{-1}} T_y \, dy \) \( (x \in G) \). Then \( T \) is a semi-isometric representation of \( G \).

**Proof.** Again we have \( T \) strongly continuous by Lemma 2. Define a seminorm \( \| \cdot \| \) on \( \mathcal{X} \) by

\[
\| \xi \| = \| T_\phi \xi \|,
\]

and let \( N = T_\phi^{-1}(0) \). By Lemma 5, \( \| T_x \xi \| = \| \xi \| \) for \( x \in G \). Hence there is an induced representation on \( \mathcal{X}/N \) satisfying the hypotheses of Theorem 1. By
Theorem 1,

\[ |T_x \xi - \xi| = 0, \quad \xi \in X, \]

and

\[ |T_x T_y \xi - T_{xy} \xi| = 0, \quad \xi \in X, x, y \in G. \]

Hence \( \|T_x^2 \xi - T_x \xi\| = 0 \), i.e. \( T_x \) is a projection. By Lemma 5, \( \|T_x (T_x \xi - \xi)\| = \|T_x (T_x \xi - T_{xy} \xi)\| = 0 \) for \( x \in G \), i.e. \( T_x T_x = T_x \). Also by Lemma 5, for any \( w \in G \),

\[ \|T_w T_x T_y \xi - T_w T_{xy} \xi\| = \|T_x (T_x T_y \xi - T_{xy} \xi)\| = |T_x T_y \xi - T_{xy} \xi| = 0. \]

Thus \( T_w T_x T_y = T_w T_{xy} \) for \( w, x, y \in G \).

Now suppose we have the equation

\[ (2) \quad T_e T_x = T_x \quad (x \in G). \]

Then we have \( T_x T_y = T_e T_x T_y = T_e T_{xy} = T_{xy} \), and the proof is complete. Therefore it remains only to establish \( (2) \). Here the only difficulty is that \( x \rightarrow T_x^* \) need not be strongly continuous. (For, if it were, we could apply the argument above to \( x \rightarrow T_x^* \).) This is circumvented by the construction that follows. We shall assume here that \( X \) is complete, for convenience.

Fix any \( \xi_0 \in X \) and let \( X_0 \) be the smallest closed subspace of \( X \) such that \( \xi_0 \in X_0 \) and \( T_x(X_0) \subset X_0 \), \( x \in G \). It is enough to consider the induced map \( G \rightarrow B(X_0) \).

Let \( C_0 = \{(\xi_0) \cup (T_x \xi_0: x \in G) \cup (T_x T_y \xi_0: x, y \in G)\} \). Then \( C_0 \) is compact and so is its closed absolutely convex hull \( C \). Let \( Y \) be the linear span of \( C \) equipped with the norm whose unit ball is \( C \). Then \( Y \) is a Banach space, since \( C \) is compact. Furthermore since \( T_w T_x T_y = T_w T_{xy} \) for \( w, x, y \in G \) we have \( T_w(C) \subset C \). Thus \( Y \) is invariant for each \( T_x \) and so \( Y \) is dense in \( X_0 \). Let \( \tilde{T} \) denote the restriction of \( T_x \) to \( Y \); then in the norm of \( Y \), \( \|\tilde{T}_x\| < 1 \). Let \( J: Y \rightarrow X_0 \) be the inclusion map. By construction \( J \) is compact and \( J\tilde{T}_x = T_x J \) \( (x \in G) \). Now suppose \( T_e T_w \neq T_w \). Since \( T_e \) is a projection, \( T_w(X_0) \not\subset T_e(X_0) \). Hence there exists \( \psi \in X_0^* \) such that \( T_w^* \psi \neq 0 \) but \( T_e^* \psi = 0 \). Since \( J \) is compact the map \( x \rightarrow J^* T_x^* \psi \) is continuous. Choose \( u \in G \) such that

\[ c = \|J^* T_u^* \psi\| = \max_{x \in G} \|J^* T_x^* \psi\|. \]

Since \( J \) is compact there exists \( \eta \in C \) such that \( T_u^* \psi(\eta) = c \). Then

\[ |\psi(T_{ux} - T_x J\eta)| = |\psi(T_{ux} - J\tilde{T}_x \eta)| = |J^* T_{ux}^* - \psi(\tilde{T}_x \eta)| < c. \]

However

\[ \int_G \psi(T_{ux} - T_x J\eta) \, dx = \psi(T_u \eta) = c. \]

Hence \( \psi(T_{ux} - T_x J\eta) \equiv c \) by continuity. In particular putting \( x = u \), \( \psi(T_u T_x \eta) = c \), i.e. \( T_e^* \psi(T_u \eta) = c \). However \( T_e^* \psi = 0 \) and hence \( c = 0 \). Thus \( J^* T_u^* \psi = 0 \); but \( Y \) is dense in \( X_0 \) and hence \( T_e^* \psi = 0 \), which is a contradiction to our initial assumption. This completes the proof.

**Corollary.** Suppose \( \mathcal{G} \) is a bounded subsemigroup of \( B(X) \) and \( T: G \rightarrow B(X) \) is a strongly measurable mapping satisfying

(i) \( T(G) \subset \mathcal{G} \),
(ii) \( T_x = \int_G T_{xy^{-1}} T_y \, dy \) (\( x \in G \)).

Then \( T \) is a representation of \( G \).

**Proof.** Renorm \( X \) by \( |\xi| = \sup_{A \in \mathcal{A}(f)} \| A \xi \| \).

We can now characterize idempotents of norm one in the generalized group algebra of a locally compact group \( G \). Let \( A \) be a Banach algebra and let \( L^1(G : A) \) denote the space of Bochner integrable functions \( f : G \to A \). \( L^1(G : A) \) is a Banach algebra under the multiplication

\[
 f * g(x) = \int_G f(xy^{-1}) g(y) \, dy
\]

and norm

\[
 \| f \| = \int_G \| f(x) \| \, dx.
\]

It is well known that if \( A = C \), the norm one idempotents of \( L^1(G : C) = L^1(G) \) are of the form \( \lambda(H)^{-1} \rho(x) \chi_H(x) \) where \( H \) is a compact open subgroup, \( \rho \) is a character on \( H \) and \( \chi_H \) is the characteristic function of \( H \). (See [1, 2.1.4].) Since the elements of \( L^1(G : A) \) are equivalence classes, if \( f \) is an idempotent in \( L^1(G : A) \), then we can assume that the representative satisfies \( f(x) = \int_G f(xy^{-1}) f(y) \, dy \) for all \( x \in G \). We make this assumption in the following theorem.

**Theorem 3.** Let \( f \in L^1(G : A) \) and suppose \( \| f \| = 1 \) and \( f * f = f \). Then \( f \) is continuous and there exists a compact open subgroup \( H \) of \( G \) such that

(i) \( f(x) = 0, x \not\in H \),
(ii) \( f(xy) = \lambda(H) f(x) f(y), x, y \in H \),
(iii) \( \| f(x) \| = (\lambda(H))^{-1}, x \in H \).

**Proof.**

\[
 \| f(x) \| \leq \int_G \| f(xy^{-1}) \| \| f(x) \| \, dy
\]

and

\[
 1 = \int_G \| f(x) \| \, dx = \int_G \int_G \| f(xy^{-1}) \| \| f(y) \| \, dy \, dx
\]

so that

\[
 \| f(x) \| = \int_G \| f(xy^{-1}) \| \| f(y) \| \, dy \text{ almost everywhere.}
\]

Hence if \( \gamma(x) = \int_G \| f(xy^{-1}) \| \| f(y) \| \, dy \) then \( \gamma \) is a norm one idempotent in \( L^1(G) \). Hence there is a compact open subgroup \( H \) such that \( \gamma(x) = \lambda(H)^{-1} \chi_H(x) \).

It follows that \( \| f(x) \| = 0 \) if \( x \not\in H \), and that \( \| f(x) \| < \lambda(H)^{-1} \) for all \( x \in G \).

We may suppose \( A \) has an identity and then identify \( A \) as a subalgebra of \( B(A) \). If we define for \( x \in H \),

\[
 T_x a = \lambda(H) f(x) a,
\]

then \( \| T_x \| < 1 \) and \( \int_H T_{xy^{-1}} T_y \, dy = T_x \). By Theorem 2, \( T_x T_y = T_{xy} \) and the result follows.

If \( G \) is compact we may also consider the algebra \( L^p(G : A) \) (\( 1 < p < \infty \)) with the norm
Using a similar approach to that of [4] we obtain

**Theorem 4.** If \( f \in L^p(G : A) \) satisfies \( \| f \|_p = 1 \) and \( f \ast f = f \) then \( f(xy) = f(x)f(y) \) for all \( x, y \in G \).

**References**