New York Journal of Mathematics

New York J. Math. **11** (2005) 563–595.

Rademacher series and decoupling

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Abstract. We study decoupling in quasi-Banach spaces. We show that decoupling is permissible in some quasi-Banach spaces (e.g., L_p and L_p/H_p when $0 < p < 1$) but fails in other spaces such as the Schatten ideal $\mathcal{S}_p$ when $0 < p < 1$. We also relate our ideas to a possible extension of the Grothendieck inequality.

Contents

1. Introduction

This paper was inspired by the work of de la Peña and Montgomery-Smith on decoupling of random variables taking values in Banach space ([7],[8] and [6]; see also [22] and [23]). It is natural to wonder if similar results might hold for random variables taking values in a quasi-Banach space (e.g., in $L_p(0,1)$ when $0 < p < 1$). As we shall see the main results on decoupling extend to some but not all quasi-Banach spaces. Thus decoupling can be defined as a property of quasi-Banach spaces and it is of some interest to decide which spaces enjoy the decoupling property.

Suppose $\xi = (\xi_1, \dots, \xi_n)$ is a sequence of independent real-valued random variables and that $\xi' = (\xi'_1, \dots, \xi'_n)$ is an independent copy of ξ. Suppose $F = (f_{jk})_{j,k=1}^n$ is an array of Borel functions $f_{jk} : \mathbb{R}^2 \to X$ where X is some quasi-Banach space with $f_{jk}(s,t) = f_{kj}(t,s)$ for $j \neq k$ and $f_{jj} \equiv 0$ for $1 \le j \le n$. If X is a Banach space

Received July 18, 2004.

Mathematics Subject Classification. Primary: 46A16, 60B11.

Key words and phrases. decoupling, quasi-Banach space.

The author was supported by NSF grant DMS-0244515.

ISSN 1076-9803/05

then it is shown in [7] that the random variables $F(\xi,\xi)=\sum_{j=1}^n\sum_{k=1}^n f_{jk}(\xi_j,\xi_k)$ and $F(\xi,\xi')=\sum_{j=1}^n\sum_{k=1}^n f_{jk}(\xi_j,\xi'_k)$ satisfy the distributional inequalities:

$$\mathbb{P}(\|F(\xi,\xi)\|>t)\le C\mathbb{P}(\|F(\xi,\xi')\|>t/C)\qquad t>0, \tag{1.1}$$

$$\mathbb{P}(\|F(\xi,\xi')\|>t)\le C\mathbb{P}(\|F(\xi,\xi)\|>t/C)\qquad t>0. \tag{1.2}$$

Here C is an absolute constant. (More general results for higher-order decoupling are given in [8] but we will consider only decoupling of order two.)

For X a quasi-Banach space we may define X to have the decoupling property if both (1.1) and (1.2) hold with C a constant which may depend on X.

By using techniques similar to those of [7] we show in §3 that the decoupling property holds if it holds for the special case when F is of the form

$$F(\epsilon,\epsilon)=\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_k x_{jk}$$

where $(x_{jk})_{j,k=1}^n$ is a symmetric X-valued matrix with zero diagonal and $\epsilon=(\epsilon_j)_{j=1}^n$ is a sequence of independent Rademachers. Indeed it is only necessary to establish inequalities of the form

$$\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_k x_{jk}\Big\|\le C\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon'_k x_{jk}\Big\|, \tag{1.3}$$

$$\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon'_k x_{jk}\Big\|\le C\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_k x_{jk}\Big\|. \tag{1.4}$$

Using this criterion it is quite easy to see that the spaces $L_p(0,1)$ for $0<p<1$ have the decoupling property, while the Schatten ideals $\mathcal{S}_p$ fail to have the decoupling property when $0<p<1$.

In §4 we study more complicated examples of spaces with and without decoupling. We show that Pisier's property (α) [29] implies decoupling and use this to show that L_p/H_p and L_p/R when R is a reflexive subspace have property (α); this uses techniques very similar to earlier work of Pisier [35] and Kisliakov [24]. We also show any minimal extension of ℓ_1 or L_1 has decoupling.

In §5 we point out that the decoupling property is equivalent to two distinct inequalities (1.3) and (1.4) for Rademacher sums. We do not know whether each individual inequality is sufficient to imply decoupling. We note that the Gaussian analogue of (1.4) holds in every quasi-Banach space. We then show that the Schatten ideal $\mathcal{S}_p$ fails (1.4) when $0<p<1$; we do not know if (1.3) or its Gaussian analogue holds in $\mathcal{S}_p$ when $0<p<1$. We also show that a certain minimal extension of $\mathcal{S}_1$ fails (1.3) and its Gaussian analogue.

Finally in §6 we discuss a question which to some extent motivated our interest. We point out that if we have a bounded bilinear form $B: X\times Y\to Z$ where X,Y are Banach spaces with type two and Z has decoupling then B factors through a Banach space. We then consider the question whether the assumptions on X and Y may be replaced by the assumptions that X^* and Y^* have cotype two and X and Y have the bounded approximation property. In particular can one take X and Y to be $C(K)-$spaces. If this were to be true it would imply a strengthening of the Grothendieck inequality. We are able to give some simple weaker results which suggest that it is at least plausible.

In §2 we discuss the necessary background on quasi-Banach spaces for our results.

Let us note at this point that we will frequently find it useful to adopt the convention that C denotes a constant which may vary from line to line and which may depend on the spaces being considered (X, Y, Z, etc.) and their parameters (p, q, r, etc.) but not the elements of the spaces (x, y, z, etc.).

Acknowledgements. We would like to thank Stephen Montgomery-Smith and Staszek Kwapień for some very helpful comments.

2. Background on quasi-Banach spaces

We recall that a quasi-Banach space X is called r-normable for $0 < r \le 1$ if there is a constant C so that

$$\left\| \sum_{j=1}^n x_j \right\| \le C \left(\sum_{j=1}^n \|x_j\|^r \right)^{\frac{1}{r}} \qquad x_1, \ldots, x_n \in X.$$

If X is r-normable then X may be equivalently renormed so that $C = 1$; in this case we refer to the quasi-norm as an r-norm. By the Aoki–Rolewicz theorem [2, 36] every quasi-Banach space is r-normable for some $0 < r \le 1$. From now on we shall assume that every quasi-norm is an r-norm for some r, and this implies that all quasi-norms are continuous functions (allowing us to integrate when computing expectations).

X is said to have (Rademacher) type p where $0 < p \le 2$ if there is a constant C so that

$$\left(\mathbb{E} \left\| \sum_{j=1}^n \epsilon_j x_j \right\|^p \right)^{\frac{1}{p}} \le C \left(\sum_{j=1}^n \|x_j\|^p \right)^{\frac{1}{p}} \qquad x_1, \ldots, x_n \in X.$$

X has (Rademacher) cotype q where $2 \le q < \infty$ if there is a constant C so that

$$\left(\sum_{j=1}^n \|x_j\|^q \right)^{\frac{1}{q}} \le C \left(\mathbb{E} \left\| \sum_{j=1}^n \epsilon_j x_j \right\|^q \right)^{\frac{1}{q}} \qquad x_1, \ldots, x_n \in X.$$

The notion of Rademacher type for $0 < p < 1$ is however, redundant, in the sense that X has type p if and only if X is p-normable [11]. If $1 < p \le 2$ then a quasi-Banach space of type p is 1-normable, i.e., a Banach space [10]; however when $p = 1$ there are examples of spaces which are type one but are not Banach spaces [11]. It is also important to note that quasi-normed spaces also obey the Kahane–Khintchine inequality:

Proposition 2.1 ([11]). *Let X be a quasi-Banach space and suppose $0 < p < q$. Then there is a constant $C = C(p, q, X)$ such that*

$$\left(\mathbb{E} \left\| \sum_{j=1}^n \epsilon_j x_j \right\|^q \right)^{\frac{1}{q}} \le C \left(\mathbb{E} \left\| \sum_{j=1}^n \epsilon_j x_j \right\|^p \right)^{\frac{1}{p}} \qquad x_1, \ldots, x_n \in X.$$

We will need some factorization results. These results are well-known in the context of Banach spaces, but, (perhaps a little surprisingly) they can be extended almost unchanged to quasi-Banach spaces:

Proposition 2.2. *Suppose X is a quasi-Banach space of cotype q, $2 \le q < \infty$. Then if $q < s < \infty$ or $q = s = 2$ there is a constant $C = C(q, s, X)$ such that if we suppose K is a compact Hausdorff space and $T : C(K) \to X$ is a bounded operator then there is a probability measure μ on K so that*

$$\|Tf\| \le C\|T\| \left(\int |f|^s d\mu \right)^{\frac{1}{s}}. \tag{2.1}$$

Proof. This follows from [19, Theorems 4.1 and 4.3]. □

The case $q = 2$ is a special case of the following more general result from [21] which extends Pisier's abstract Grothendieck theorem [30]. We recall that a (separable) quasi-Banach space X has the bounded approximation property if there is a sequence (T_n) of finite-rank operators on X so that $T_n x \to x$ for all $x \in X$. If X has the bounded approximation property then the dual space X^* separates points.

Proposition 2.3. *Suppose X is a quasi-Banach space with the bounded approximation property such that X^* has cotype 2; let Y be a quasi-Banach space with cotype 2. Then there is a constant C so that if $T : X \to Y$ is a bounded operator then T can be factorized $T = UV$ where $U : X \to \ell_2$ and $V : \ell_2 \to Y$ are bounded operators with $\|U\|\|V\| \le C\|T\|$.*

Let us also recall that an operator $T : X \to Y$ where X is a Banach space and Y is a quasi-Banach space is called *p-absolutely summing* if the there is a constant C so that

$$\left(\sum_{k=1}^n \|Tx_k\|^p \right)^{\frac{1}{p}} \le C \sup_{\|x^*\| \le 1} \left(\sum_{k=1}^n |x^*(x_k)|^p \right)^{\frac{1}{p}} \qquad x_1, \dots, x_n \in X.$$

The least constant C is denoted $\pi_p(T)$. The well-known Pietsch factorization theorem extends again to the situation when Y is quasi-normed:

Proposition 2.4. *Let X be a Banach space and suppose Y is a quasi-Banach space. If $T : X \to Y$ is p-absolutely summing then there is a probability measure μ on B_{X^*} such that*

$$\|Tx\| \le \pi_p(T) \left(\int_{B_{X^*}} |x^*(x)|^p d\mu(x^*) \right)^{\frac{1}{p}} \qquad x \in X.$$

In particular if $p = 2$ then T admits a factorization $T = T_0 j_1 j_2$ where $T_0 : L_2(B_{X^*}, \mu) \to Y$ with $\|T_0\| = \pi_2(T)$ and $j_1 : C(B_{X^*}) \to L_2(\mu)$, $j_2 : X \to C(B_{X^*})$ are the natural injections.

Certain special types of quasi-Banach spaces will also interest us. A quasi-Banach lattice X is *p-convex* where $0 < p < \infty$ if there is a constant C so that

$$\left\| \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \right\| \le C \left(\sum_{j=1}^n \|x_j\|^p \right)^{\frac{1}{p}} \qquad x_1, \dots, x_n \in X$$

and *p-concave* if there is a constant C so that

$$\left(\sum_{j=1}^n \|x_j\|^p \right)^{\frac{1}{p}} \le C \left\| \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \right\| \qquad x_1, \dots, x_n \in X.$$

It is not true that every quasi-Banach lattice satisfies a p-convexity condition for some $p > 0$; however every quasi-Banach lattice with nontrivial cotype satisfies a p-convexity and q-concavity condition for some $0 < p \le q < \infty$. See [12] and [19] for details.

We note that in a quasi-Banach lattice with nontrivial cotype, it is easy to show that there exists a C so that for $x_1, \dots, x_n \in X$ we have

$$C^{-1}\left\|\left(\sum_{j=1}^{n}|x_j|^2\right)^{\frac{1}{2}}\right\| \le \left(\mathbb{E}\left\|\sum_{j=1}^{n}\epsilon_j x_j\right\|^2\right)^{\frac{1}{2}} \le C\left\|\left(\sum_{j=1}^{n}|x_j|^2\right)^{\frac{1}{2}}\right\|. \tag{2.2}$$

Indeed for Banach lattices this is proved in [26] and the same proof goes through almost verbatim.

Let $(\gamma_j)_{j=1}^{\infty}$ denote a sequence of independent normalized Gaussians. The following proposition is well-known for Banach spaces (cf., e.g., [39] or [25] Propositions 9.14–9.15).

Proposition 2.5. *Let X be a quasi-Banach space. Then for $0 < p < \infty$ there is a constant $C = C(p, X)$ so that if $x_1, \dots, x_n \in X$,*

$$\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|^p\right)^{\frac{1}{p}} \le C\left(\mathbb{E}\left\|\sum_{k=1}^{n}\gamma_k x_k\right\|^p\right)^{\frac{1}{p}}.$$

If X has nontrivial cotype then there is a constant $C = C(p, X)$ such that

$$\left(\mathbb{E}\left\|\sum_{k=1}^{n}\gamma_k x_k\right\|^p\right)^{\frac{1}{p}} \le C\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|^p\right)^{\frac{1}{p}}.$$

Proof. An important observation here is that a version of the Kahane contraction principle holds, i.e., there exists $C = C(p, X)$ so that

$$\left(\mathbb{E}\left\|\sum_{j=1}^{n}\alpha_j\epsilon_j x_j\right\|^p\right)^{1/p} \le C \max_{1\le j\le n}|\alpha_j|\left(\mathbb{E}\left\|\sum_{j=1}^{n}\epsilon_j x_j\right\|^p\right)^{1/p}$$
$$\alpha_1, \dots, \alpha_N \in \mathbb{R},\ x_1, \dots, x_n \in X.$$

For Banach spaces $C(p, X) = 1$ if $p \ge 1$ (see [25] Theorem 4.4).

Now let σ be the median value for each $|\gamma_j|$ so that $\mathbb{P}(|\gamma_j| \ge \sigma) = 1/2$. Let $\xi_j = I(|\gamma_j| \ge \sigma)$. Let $(\epsilon_1, \dots, \epsilon_n)$ be Rademachers independent of $(\gamma_1, \dots, \gamma_n)$. Then, using the contraction principle,

$$\begin{aligned}\left(\mathbb{E}\left\|\sum_{j=1}^n\epsilon_jx_j\right\|^p\right)^{1/p}&\le C\left(\left(\mathbb{E}\left\|\sum_{j=1}^n\xi_j\epsilon_jx_j\right\|^p\right)^{1/p}+\left(\mathbb{E}\left\|\sum_{j=1}^n(1-\xi_j)\epsilon_jx_j\right\|^p\right)^{1/p}\right)\\&\le C\left(\mathbb{E}\left\|\sum_{j=1}^n\xi_j\epsilon_jx_j\right\|^p\right)^{1/p}\\&\le C\sigma^{-1}\left(\mathbb{E}\left\|\sum_{j=1}^n\sigma\xi_j\epsilon_jx_j\right\|^p\right)^{1/p}\\&\le C\left(\mathbb{E}\left\|\sum_{j=1}^n\epsilon_j|\gamma_j|x_j\right\|^p\right)^{1/p}\\&= C\left(\mathbb{E}\left\|\sum_{j=1}^n\gamma_jx_j\right\|^p\right)^{1/p},\end{aligned}$$

where the constant $C=C(p,X)$ varies from line to line.

If we assume X has cotype q we proceed as in Proposition 9.14 of [25] and obtain an estimate

$$\left(\mathbb{E}\left\|\sum_{j=1}^n\epsilon_jI(|\gamma_j|>t)x_j\right\|^p\right)^{1/p}\le C\mathbb{P}(|\gamma_j|>t)^{1/r}\left(\mathbb{E}\left\|\sum_{j=1}^n\epsilon_jx_j\right\|^p\right)^{1/p},$$

where $r=\max(p,q)$.

At this point the last step in the proof of [25] must be modified as integration is not permissible. However it is clear that if $\xi_j=1+\sum_{j=0}^\infty 2^{j+1}I(|\gamma_j|\ge 2^j)$ then $\xi_j\ge|\gamma_j|$ and we have an estimate

$$\left(\mathbb{E}\left\|\sum_{j=1}^n\epsilon_j\xi_jx_j\right\|^p\right)^{1/p}\le C\left(\mathbb{E}\left\|\sum_{j=1}^n\epsilon_jx_j\right\|^p\right)^{1/p}.$$

The conclusion follows from the contraction principle. □

A quasi-Banach space X is called *natural* if it is isomorphic to a closed subspace of a p-convex quasi-Banach lattice for some $p>0$. It may be shown that X is natural if and only if it is isomorphic to a subspace of an $\ell_\infty-$product of L_p-spaces for some $p>0$ (Theorem 3.1 of [13]).

A complex quasi-Banach space is *A-convex* (see [15]) if it has an equivalent plurisubharmonic quasi-norm, i.e., one that satisfies

$$\|x\|\le\int_0^{2\pi}\|x+e^{i\theta}y\|\frac{d\theta}{2\pi}\qquad x,y\in X.$$

Every complex natural space is A-convex but the Schatten ideal $\mathcal{S}_p$ where $0<p<1$ is A-convex but not natural.

If $(\Omega,\mathbb{P})$ is some probability space we denote by $L_p(\Omega;X)$ the space of Bochner measurable functions (or random variables) $\xi:\Omega\to X$ under the quasi-norm $\|\xi\|_p=(\mathbb{E}\|\xi\|^p)^{\frac1p}$.

If $\xi\in L_p(\Omega;X)$ we let

$$V_p(\xi)=(\mathbb{E}\|\xi-\xi'\|^p)^{\frac1p}$$

where ξ' is an independent copy of ξ.

Lemma 2.6. (i) *There is a constant $C = C(p, X)$ so that if $\xi \in L_p(\Omega; X)$ then*

$$\inf_{x\in X}(\mathbb{E}\|\xi - x\|^p)^{\frac{1}{p}} \le V_p(\xi) \le C \inf_{x\in X}(\mathbb{E}\|\xi - x\|^p)^{\frac{1}{p}}.$$

(ii) *There exists $C = C(p, X)$ so that if ξ is a symmetric random variable in $L_p(\Omega; X)$ then*

$$(\mathbb{E}\|\xi\|^p)^{\frac{1}{p}} \le CV_p(\xi).$$

(iii) *There exists $C = C(p, X)$ so that if ξ and η are independent random variables then*

$$V_p(\xi) \le CV_p(\xi + \eta).$$

Proof. If X is r-normed, let $s = \min(p, r)$ then If $x \in X$ then

$$(\mathbb{E}\|\xi - \xi'\|^p)^{\frac{1}{p}} = (\mathbb{E}\|\xi - x - (\xi' - x)\|^p)^{\frac{1}{p}} \le 2^{\frac{1}{s}}(\mathbb{E}\|\xi - x\|^p)^{\frac{1}{p}}.$$

Conversely if we write

$$\mathbb{E}\|\xi - \xi'\|^p = \int_\Omega\int_\Omega \|\xi(\omega) - \xi(\omega')\|^p d\mathbb{P}(\omega)d\mathbb{P}(\omega')$$

we see that there exists $x \in X$ with $\mathbb{E}\|\xi - x\|^p \le \mathbb{E}\|\xi - \xi'\|^p$.

For (ii) observe that if ξ is symmetric then for any $x \in X$ we have $\xi = \frac{1}{2}(\xi + x + \xi - x)$ and hence we have $\mathbb{E}\|\xi\|^p \le 2^{p(\frac{1}{s}-1)}\mathbb{E}\|\xi - x\|^p$.

(iii) follows from (i). Indeed for any $x \in X$ it is clear that

$$\inf_{x'\in X}(\mathbb{E}\|\xi - x'\|^p)^{\frac{1}{p}} \le (\mathbb{E}\|\xi + \eta - x - x'\|^p)^{\frac{1}{p}}.$$

(Note that (ii) and (iii) are special cases of the contraction principle mentioned in Proposition 2.5 above). □

If f, g are positive random variables defined on some probability space it will be convenient to introduce the notation $f \preccurlyeq_C g$ to mean

$$\mathbb{P}(f \ge t) \le C\mathbb{P}(Cg \ge t)$$

and $f \approx_C g$ to both mean $f \preccurlyeq_C g$ and $g \preccurlyeq_C f$.

We note next some simple properties:

Lemma 2.7. *Let X be an r-normed quasi-Banach space.*

(i) *Suppose $\xi_1, \ldots, \xi_n$ are X-valued random variables and f is a positive random variable such that $\|\xi_j\| \preccurlyeq_{C_j} f$ for $1 \le j \le n$. Then $\|\sum_{j=1}^n \xi_j\| \preccurlyeq_C f$ where $C = n^{\frac{1}{r}}\max_{1\le j\le n} C_j$.*

(ii) *If ξ, ξ' are independent identically distributed X-valued random variables then $\|\xi\| \preccurlyeq_C \|\xi + \xi'\|$ where $C = 3^{\frac{1}{r}}$.*

Proof. Clearly

$$\mathbb{P}\left(\left\|\sum_{j=1}^n \xi_j\right\| \ge t\right) \le \sum_{j=1}^n \mathbb{P}(n^{\frac{1}{r}}\|\xi_j\| \ge t) \le \sum_{j=1}^n C_j\mathbb{P}(C_j n^{\frac{1}{r}} f \ge t)$$

and (i) follows. For (ii) we follow the argument of Montgomery-Smith and de la Pena [7]. Let ξ, ξ', ξ'' be three independent copies of ξ. Then we write

$$\xi = \frac{1}{2}((\xi + \xi') + (\xi + \xi'') - (\xi' + \xi''))$$

and hence by (i)

$$\|\xi\| \preccurlyeq_C \|\xi + \xi'\|$$

where $C = 3^{\frac{1}{r}}$. $\square$

3. Decoupling

Suppose X is quasi-Banach space. Let $F = \{f_{jk} : \ 1 \le j, k \le n\}$ be an $n \times n$ matrix of Borel maps $f_{jk} : \mathbb{R}^2 \to X$ such that $f_{jj} = 0$ and $f_{jk}(s,t) = f_{kj}(t,s)$ if $j \neq k$. Suppose $\xi = (\xi_1, \ldots, \xi_n)$ be a sequence of independent real-valued random variables and let $\xi' = (\xi'_1, \ldots, \xi'_n)$ be an independent copy of $(\xi_1, \ldots, \xi_n)$. We consider the X-valued random variables

$$F(\xi, \xi) = \sum_{j=1}^{n} \sum_{k=1}^{n} f_{jk}(\xi_j, \xi_k) \tag{3.1}$$

and

$$F(\xi, \xi') = \sum_{j=1}^{n} \sum_{k=1}^{n} f_{jk}(\xi_j, \xi'_k). \tag{3.2}$$

Then we refer to $F(\xi, \xi')$ as the *decoupled* version of $F(\xi, \xi)$.

We now say that X has the *decoupling property* if there is a constant C depending only on X such that for every such F and ξ we have

$$\|F(\xi, \xi)\| \approx_C \|F(\xi, \xi')\|.$$

The fundamental theorem of de la Peña and Montgomery-Smith [7] states that every Banach space has the decoupling property.

If $0 < p < \infty$ let us say that X has the L_p-*decoupling property* if for some C we have

$$C^{-1}(\mathbb{E}\|F(\xi, \xi')\|^p)^{\frac{1}{p}} \le (\mathbb{E}\|F(\xi, \xi)\|^p)^{\frac{1}{p}} \le C(\mathbb{E}\|F(\xi, \xi')\|^p)^{\frac{1}{p}}.$$

Let $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ and $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_n)$ denote two independent sequences of Rademachers. We shall consider a weaker notion of decoupling for functions F of the form

$$F(\epsilon, \epsilon') = \sum_{j=1}^{n} \sum_{k=1}^{n} \epsilon_j \epsilon'_k x_{jk}$$

where $(x_{jk})_{j,k=1}^n$ is a symmetric X-valued matrix with $x_{jj} = 0$ for all j. For decoupling of such functions (Rademacher chaos of dimension two) in Banach spaces, see [22] and [4].

At this point we recall our convention that C denotes a constant depending only the space X and p, but may vary from line to line.

Theorem 3.1. *Let X be a quasi-Banach space. The following conditions on X are equivalent:*

(i) *X has the decoupling property.*
(ii) *For some (respectively, every) $0 < p < \infty$, X has the L_p-decoupling property.*

(iii) *For some (respectively, every) $0 < p < \infty$, there exists C so that if $(x_{jk})_{j,k=1}^n$ is a symmetric X-valued matrix with $x_{jj} = 0$ for $1 \le j \le n$ then*

$$C^{-1}\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' x_{jk}\Big\|^p\Big)^{\frac{1}{p}} \le \Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k x_{jk}\Big\|^p\Big)^{\frac{1}{p}} \le C\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' x_{jk}\Big\|^p\Big)^{\frac{1}{p}}. \tag{3.3}$$

(iv) *There exists C so that if $(x_{jk})_{j,k=1}^n$ is a symmetric X-valued matrix with $x_{jj} = 0$ for $1 \le j \le n$ then for every $x \in X$ we have*

$$\mathbb{P}\Big(\Big\|x + \sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k x_{jk}\Big\| \ge C^{-1}\|x\|\Big) \ge C^{-1}. \tag{3.4}$$

(v) *For some (respectively every) $0 < p < \infty$, there exists C so that if $(x_{jk})_{j,k=1}^n$ is a symmetric X-valued matrix with $x_{jj} = 0$ for $1 \le j \le n$ then for every $x \in X$ we have*

$$\|x\| \le C\Big(\mathbb{E}\Big\|x + \sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k x_{jk}\Big\|^p\Big)^{\frac{1}{p}}. \tag{3.5}$$

Proof. We first note that (i) $\Rightarrow$ (ii) (for every p) and (ii) $\Rightarrow$ (iii) (for any p). We next prove that (i) and (iv) are equivalent and that (ii) and (v) are equivalent (for any p). The directions (i) $\Rightarrow$ (iv) and (ii) $\Rightarrow$ (v) are trivial. We turn to the reverse implications. The proofs of these statements are very similar. Suppose $F = (f_{jk})_{1\le j,k\le n}$ is given as described at the beginning of this section. If $\xi = (\xi_1, \dots, \xi_n)$ and $\xi' = (\xi_1', \dots, \xi_n')$ are two identically distributed and mutually independent sequences of independent random variables, we introduce a sequence $\epsilon_1, \dots, \epsilon_n$ of Rademachers independent of both sequences. We then (as in the argument of de la Peña and Montgomery-Smith [7] define $(\theta_1, \dots, \theta_n, \theta_1', \dots, \theta_n')$ by $(\theta_j, \theta_j') = (\xi_j, \xi_j')$ when $\epsilon_j = 1$ and $(\theta_j, \theta_j') = (\xi_j', \xi_j)$ when $\epsilon_j = -1$.

Then $F(\theta, \theta') \approx F(\xi, \xi')$ and $F(\theta, \theta) \approx F(\xi, \xi)$. However if we let $\xi^{+1} = \xi$ and $\xi^{-1} = \xi'$

$$F(\theta, \theta') = \frac{1}{4}\sum_{j=1}^n\sum_{k=1}^n\sum_{\delta_j,\delta_k=\pm1}(1+\delta_j\epsilon_j)(1-\delta_k\epsilon_k)f_{jk}(\xi^{\delta_j}, \xi^{\delta_k})$$
$$F(\theta, \theta) = \frac{1}{4}\sum_{j=1}^n\sum_{k=1}^n\sum_{\delta_j,\delta_k=\pm1}(1+\delta_j\epsilon_j)(1+\delta_k\epsilon_k)f_{jk}(\xi^{\delta_j}, \xi^{\delta_k}).$$

If we expand these out we obtain formulas

$$F(\theta,\theta')=\frac14(F(\xi,\xi)+2F(\xi,\xi')+F(\xi',\xi'))+\sum_{j=1}^n\epsilon_jG_j(\xi,\xi') \\ +\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_kH_{jk}(\xi,\xi')$$

$$F(\theta,\theta)=\frac14(F(\xi,\xi)+2F(\xi,\xi')+F(\xi',\xi'))+\sum_{j=1}^n\epsilon_jG'_j(\xi,\xi') \\ +\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_kH'_{jk}(\xi,\xi')$$

where $(G_j)_{j=1}^n,(G'_j)_{j=1}^n,(H_{jk})_{j,k=1}^n,(H'_{jk})_{j,k=1}^n$ are X-valued Borel functions on $\mathbb{R}^{2n}$ with H_{jj},H'_{jj} vanishing identically for $1\le j\le n$. Here we have exploited the symmetry assumptions (so that $F(\xi,\xi')=F(\xi',\xi)$).

We now prove that (iv) $\Rightarrow$ (i). We adopt again the convention that C is a constant depending only on p and X but which may vary from line to line. We start with observation that, since the transformation $\epsilon_j\to-\epsilon_j$ for all j leaves the distribution of the right-hand unchanged, we have an estimate by averaging two terms,

$$\left\|\frac14(F(\xi,\xi)+2F(\xi,\xi')+F(\xi',\xi'))+\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_kH_{jk}(\xi,\xi')\right\|\preccurlyeq_C\|F(\theta,\theta')\|.$$

Now condition (iv) easily implies that

$$\|F(\xi,\xi)+2F(\xi,\xi')+F(\xi',\xi')\|\preccurlyeq_C\|F(\theta,\theta')\|$$

and from this we obtain

$$\|F(\xi,\xi)+F(\xi',\xi')\|\preccurlyeq_C\|F(\xi,\xi')\|.$$

Finally we appeal to Lemma 2.6 (iii) to obtain $\|F(\xi,\xi)\|\preccurlyeq_C\|F(\xi,\xi')\|$. The other estimate $\|F(\xi,\xi')\|\preccurlyeq_C\|F(\xi,\xi)\|$ is quite similar.

For (v) $\Rightarrow$ (ii) the calculations are quite analogous and we will omit them.

To complete the proof we will show that (iii) (for some p) $\Rightarrow$ (iv). Let us note first that (iii) $\Rightarrow$ (v). Indeed suppose (x_{jk}) is any symmetric X-valued matrix with $x_{jj}=0$. By considering two independent copies, it is clear that (3.3) implies

$$C^{-1}V_p\left(\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon'_kx_{jk}\right)\le V_p\left(\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_kx_{jk}\right)\le CV_p\left(\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon'_kx_{jk}\right).$$

It follows that

$$\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_kx_{jk}\right\|^p\right)^{\frac1p}\le CV_p\left(\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_kx_{jk}\right).$$

Now we also have

$$V_p\left(x+\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_kx_{jk}\right)=V_p\left(\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_kx_{jk}\right)$$

and as

$$\|x\| \le C\Big(\mathbb{E}\Big\|x + \sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k x_{jk}\Big\|^p\Big)^{\frac{1}{p}} + \Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k x_{jk}\Big\|^p\Big)^{\frac{1}{p}}$$

we deduce that (v) holds. Now it trivially follows that (v) holds also in the ℓ_∞-product $Y = \ell_\infty(X)$. We deduce that (ii) holds in Y and hence that (iii) holds with X replaced by Y. Now by the Kahane–Khintchine inequality (applied twice), if $q > p$, there exists $C = C(p, q, X)$ so that

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' y_{jk}\Big\|^q\Big)^{\frac{1}{q}} \le C\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' y_{jk}\Big\|^p\Big)^{\frac{1}{p}} \tag{3.6}$$

for all Y-valued matrices (y_{jk}).

Now suppose $\zeta = \sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k y_{jk}$ and $\zeta' = \sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' y_{jk}$ where (y_{jk}) is symmetric X-valued matrix with $y_{jj} = 0$. We remark that (3.6) implies that we have hypercontractivity for the decoupled random variable ζ', but that we do not at this stage have an equivalent statement for the undecoupled random variable ζ.

Now the set $\{\|\zeta'\|^p : \ (\mathbb{E}\|\zeta'\|^p)^{\frac{1}{p}} \le 1\}$ is equi-integrable and hence, using (iii), for a suitable constant C,

$$\mathbb{P}(\|\zeta\| \ge C^{-1}(\mathbb{E}\|\zeta'\|^p)^{\frac{1}{p}}) \ge C^{-1}$$

(or, equivalently $\|\zeta'\| \preccurlyeq_C (\mathbb{E}\|\zeta\|^p)^{\frac{1}{p}}$ where the right-hand side is a constant).

Again, since (iii) holds in Y, if $\zeta_1, \dots, \zeta_n$ are independent copies of ζ and $\zeta_1', \dots, \zeta_n'$ are independent copies of ζ',

$$\Big(\mathbb{E}\max_{1\le j\le n}\|\zeta_j\|^p\Big)^{\frac{1}{p}} \le C\Big(\mathbb{E}\max_{1\le j\le n}\|\zeta_j'\|^p\Big)^{\frac{1}{p}}.$$

It follows that if $t > 0$

$$1 - (1 - \mathbb{P}(\|\zeta\| > t))^n \le Ct^{-1}\Big(\mathbb{E}\max_{1\le j\le n}\|\zeta_j'\|^p\Big)^{\frac{1}{p}}.$$

Pick n to be the smallest integer so that the left-hand side is at least $1/2$. Then

$$\Big(\mathbb{E}\max_{1\le j\le n}\|\zeta_j'\|^p\Big)^{\frac{1}{p}} \ge (2C)^{-1}t$$

and hence

$$\mathbb{P}\Big(\max_{1\le j\le n}\|\zeta_j'\| > t/C\Big) \ge C^{-1}$$

(where C is a different constant depending only on p, X). By choice of n this implies that

$$\mathbb{P}(\|\zeta'\| > t/C) \ge 1 - (1 - C^{-1})^{\frac{1}{n}}$$

but

$$\mathbb{P}(\|\zeta\| > t) \le 1 - 2^{-\frac{1}{n}}$$

and this implies an estimate (for a different choice of C)

$$\mathbb{P}(\|\zeta\| > t) \le C\mathbb{P}(\|\zeta'\| > t/C)$$

or equivalently $\|\zeta\| \preccurlyeq_C \|\zeta'\|$. Once we have this estimate it follows from (3.3) that the set $\{\|\zeta\|^p : \ (\mathbb{E}\|\zeta\|^p)^{\frac{1}{p}} \le 1\}$ is also equi-integrable, and we can reverse the above reasoning to also deduce that $\|\zeta'\| \preccurlyeq_C \|\zeta\|$ and hence $\|\zeta'\| \approx_C \|\zeta'\|$.

Now suppose $y \in Y$ and consider $y + \zeta$. We observe that $\|y\| \preccurlyeq_C \|y+\zeta\| + \|\zeta\|$ for some C and so to show (iv) (for the larger space Y) we need only show that $\|\zeta\| \preccurlyeq_C \|y+\zeta\|$. Now applying Lemma 2.7 repeatedly $\|\zeta\| \preccurlyeq_C \|\zeta'\|$ and $\|\zeta'\| \preccurlyeq_C \|\zeta' - \zeta_1'\|$ where ζ_1, ζ_1' are independent copies of ζ, ζ'. Thus

$$\|\zeta\| \preccurlyeq_C \|\zeta - \zeta_1\| = \|(y+\zeta) - (y+\zeta_1)\| \preccurlyeq_C \|y+\zeta\|. \qquad \square$$

Corollary 3.2. *Every natural quasi-Banach space has the decoupling property. In particular any quasi-Banach lattice with nontrivial cotype has the decoupling property.*

Proof. We note from (iii) that it is trivial that if X has the decoupling property that $L_p(X)$ (for $0 < p < \infty$) has the decoupling property; the fact that $\ell_\infty(X)$ has the decoupling property follows from (v) (and was used in the proof). Thus from the fact that $\mathbb{R}$ (or $\mathbb{C}$) has the decoupling property we obtain that an ℓ_∞-product of L_p-spaces has decoupling and hence every natural space has decoupling. $\square$

Corollary 3.3. *If (Ω, μ) is any probability space then X has the decoupling property if and only if $L_p(X) = L_p(\Omega, \mu; X)$ has the decoupling property.*

Proof. This follows easily from the equivalence of (i) and (ii) in Theorem 3.1. $\square$

Let us now give an example of a space failing decoupling. For $0 < p < 1$, let $\mathcal{S}_p$ be the Schatten ideal of all compact operators x on a separable Hilbert space such that $\|x\|_p = \operatorname{tr}\,(x^*x)^{\frac{p}{2}} < \infty$. Let $e_{jk} = e_j \otimes e_k$ where e_j is an orthonormal basis. Then

$$\mathbb{E}\left\|\sum_{j=1}^n \sum_{k=1}^n \epsilon_j \epsilon_k e_{jk}\right\| = n$$

However

$$\left\|\sum_{j=1}^n e_{jj}\right\| = n^{\frac{1}{p}}$$

and this contradicts (v) of Theorem 3.1. We note that this space has a plurisubharmonic quasi-norm and so is A-convex (see [15]).

4. Property (α) and decoupling

Let us recall the definition of Pisier's property (α) [29]. A quasi-Banach space X has property (α) if there is a constant C so that if $(x_{jk})_{j,k=1}^n$ is an X-valued matrix and $(a_{jk})_{j,k=1}^n$ is a scalar matrix then

$$\left(\mathbb{E}\left\|\sum_{j=1}^n \sum_{k=1}^n a_{jk} x_{jk} \epsilon_j \epsilon_k'\right\|^2\right)^{\frac{1}{2}} \le C \max_{j,k} |a_{jk}| \left(\mathbb{E}\left\|\sum_{j=1}^n \sum_{k=1}^n x_{jk} \epsilon_j \epsilon_k'\right\|^2\right)^{\frac{1}{2}}.$$

It easily follows from this definition that X has (α) if and only if for some constant C we have

$$\begin{aligned} C^{-1}\left(\mathbb{E}\left\|\sum_{j=1}^n \sum_{k=1}^n x_{jk} \epsilon_{jk}\right\|^2\right)^{\frac{1}{2}} &\le \left(\mathbb{E}\left\|\sum_{j=1}^n \sum_{k=1}^n x_{jk} \epsilon_j \epsilon_k'\right\|^2\right)^{\frac{1}{2}} \\ &\le C\left(E\left\|\sum_{j=1}^n \sum_{k=1}^n x_{jk} \epsilon_{jk}\right\|^2\right)^{\frac{1}{2}} \end{aligned}$$

for every X-valued matrix $(x_{jk})_{j,k=1}^n$ where $(\epsilon_{jk})_{j,k=1}^n$ denotes an array of independent Rademachers.

Every quasi-Banach lattice X with nontrivial cotype has property (α). This follows from the facts that X is p-convex and q-concave for some $0 < p \le q < \infty$ and so using (2.2) on X and $X(\ell_2)$ one quickly shows that for some C

$$C^{-1}\Big\|\Big(\sum_{j=1}^n\sum_{k=1}^n |x_{jk}|^2\Big)^{\frac12}\Big\| \le \Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon'_k x_{jk}\Big\|^2\Big)^{\frac12} \le C\Big\|\Big(\sum_{j=1}^n\sum_{k=1}^n |x_{jk}|^2\Big)^{\frac12}\Big\|,$$

and

$$C^{-1}\Big\|\Big(\sum_{j=1}^n\sum_{k=1}^n |x_{jk}|^2\Big)^{\frac12}\Big\| \le \Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_{jk} x_{jk}\Big\|^2\Big)^{\frac12} \le C\Big\|\Big(\sum_{j=1}^n\sum_{k=1}^n |x_{jk}|^2\Big)^{\frac12}\Big\|.$$

Conversely property (α) implies nontrivial cotype. To see this, first observe that there are Banach spaces failing (α). For example, it is easy to see the algebra $\mathcal{K}(H)$ of compact operators on a Hilbert space fails to have (α) (and indeed the same is true for every Schatten ideal $\mathcal{S}_p$ when $1 \le p < \infty$ and $p \ne 2$). Now suppose a quasi-Banach space X has property (α); then so does any quasi-Banach (crudely) finitely representable in X and so ℓ_∞ cannot be finitely representable in X; thus X has nontrivial cotype. (The fact that ℓ_∞ is finitely representable in X if and only X fails to have some cotype is well-known for Banach spaces; for quasi-Banach spaces, it is apparently only known that ℓ_∞ is crudely finitely representable in X if X fails to have some cotype [3].)

Theorem 4.1. *A quasi-Banach space with property* (α) *has the decoupling property.*

Proof. Let us suppose X is r-normed where $r \le 1$. Suppose X has property (α) and is therefore of some nontrivial cotype $q \ge 2$. Now consider any finite-dimensional subspace V of $L_2(\Omega, \mathbb{P}; X)$ spanned by vectors of the form $\epsilon_j\epsilon'_k x_{jk}$ for $1 \le j, k \le n$. By property (α) this space is C-isomorphic to a quasi-Banach lattice where C is independent of V. Since it has nontrivial cotype with constants also independent of V, it follows from Corollary 3.2 that V has the decoupling property with a constant independent of V. Now assume (x_{jk}) is a symmetric X-valued $n \times n$-matrix with $x_{jj} = 0$. Then if η_j is a further sequence of Rademachers on some other probability space $(\Omega', \mathbb{P}')$ we have, by the above remarks,

$$\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon'_k x_{jk}\Big\|_{L_2(\mathbb{P};X)} \le C\Big(\mathbb{E}_\eta\Big\|\sum_{j=1}^n\sum_{k=1}^n (1+\eta_j)(1-\eta_k)\epsilon_j\epsilon'_k x_{jk}\Big\|^2_{L_2(\mathbb{P};X)}\Big)^{\frac12}.$$

Thus

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon'_k x_{jk}\Big\|^2\Big)^{\frac12} \le C\Big(\operatorname*{Ave}_{A\subset[n]}\mathbb{E}\Big\|\sum_{j\in A}\sum_{k\notin A}\epsilon_j\epsilon'_k x_{jk}\Big\|^2\Big)^{\frac12}. \tag{4.1}$$

Now fix $A \subset [n]$ and let $\delta_j = 2\chi_A(j) - 1$. Then

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_{j\in A}\sum_{k\notin A}\epsilon_j\epsilon'_k x_{jk}\Big\|^2\Big)^{\frac12} &= \Big(\mathbb{E}\Big\|\sum_{j\in A}\sum_{k\notin A}\epsilon_j\epsilon_k x_{jk}\Big\|^2\Big)^{\frac12}\\ &= \frac12\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n(1-\delta_j\delta_k)\epsilon_j\epsilon_k x_{jk}\Big\|^2\Big)^{\frac12}\\ &\le 2^{\frac1r-1}\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_k x_{jk}\Big\|^2\Big)^{\frac12}.\end{aligned}$$

Combining with (4.1) we have an estimate

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon'_k x_{jk}\Big\|^2\Big)^{\frac12} \le C\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_k x_{jk}\Big\|^2\Big)^{\frac12}. \tag{4.2}$$

Let us now consider the reverse inequality. We will suppose n is a power of 2, say $n = 2^m$; this can be done without loss of generality by adding rows and columns of zeros. Consider the operator $T : \ell_\infty([n]\times[n]) \to L_2(\mathbb{P}; X)$ defined by

$$T(a_{jk})_{j,k=1}^n = \sum_{j=1}^n\sum_{k=1}^n a_{jk}\epsilon_j\epsilon'_k x_{jk}.$$

Then for some constant C depending only on X we have

$$\|T\| \le C\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon'_k x_{jk}\Big\|^2\Big)^{\frac12}.$$

Hence since $L_2(X)$ has nontrivial cotype q say, we can use Proposition 2.2 to deduce that if $q < s < \infty$ there is a constant $C = C(q,s,X)$ so that we can find $(\lambda_{jk})_{j,k=1}^n$ with $\lambda_{jk} \ge 0,\ \lambda_{jj} = 0$ and $\sum_{j=1}^n\sum_{k=1}^n \lambda_{jk} = 1$ such that if $A \subset [n]\times[n]$ we have

$$\Big(\mathbb{E}\Big\|\sum_{(j,k)\in A}\epsilon_j\epsilon'_k x_{jk}\Big\|^2\Big)^{\frac12} \le C\Big(\sum_{(j,k)\in A}\lambda_{jk}\Big)^{\frac1s}\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon'_k x_{jk}\Big\|^2\Big)^{\frac12}. \tag{4.3}$$

Now define $A_{jk} = \{i : (k-1)2^j + 1 \le i \le k2^j\}$ for $j = 0, 1, \dots, m$ and $1 \le k \le 2^{m-j}$. Let

$$\xi_j = \sum_{k=1}^{2^{m-j}}\sum_{h,i\in A_{jk}}\epsilon_h\epsilon_i x_{hi}.$$

Thus $\xi_0 = 0$ and

$$\xi_m = \sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_k x_{jk}.$$

Now for $1 \le j \le m$

$$\xi_j - \xi_{j-1} = 2\sum_{k=1}^{2^{m-j}}\sum_{h\in A_{j-1,2k-1}}\sum_{i\in A_{j-1,2k}}\epsilon_h\epsilon_i x_{hi}.$$

Hence

$$\|\xi_j - \xi_{j-1}\|_{L_2(\mathbb{P};X)} \le 2C\left(\sum_{k=1}^{2^{m-j}} \sum_{h,i\in A_{jk}} \lambda_{hi}\right)^{\frac{1}{s}} \left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon'_k x_{jk}\right\|^2\right)^{\frac12}.$$

Thus

$$\|\xi_m\|^r \le (2C)^r \sum_{j=1}^m \left(\sum_{k=1}^{2^{m-j}} \sum_{h,i\in A_{jk}} \lambda_{hi}\right)^{\frac{r}{s}} \left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon'_k x_{jk}\right\|^2\right)^{\frac{r}{2}}.$$

This estimate remains valid if we permute the indices. Let $\Pi([n])$ be the set of all permutations of $[n]$. We have

$$\|\xi_m\|^r \le (2C)^r M^r \left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon'_k x_{jk}\right\|^2\right)^{\frac{r}{2}}$$

where

$$M^r = \sum_{j=1}^m \operatorname*{Ave}_{\pi\in\Pi([n])} \left(\sum_{k=1}^{2^{m-j}} \sum_{\pi(h),\pi(i)\in A_{jk}} \lambda_{hi}\right)^{\frac{r}{s}}.$$

Now

$$\begin{aligned}\operatorname*{Ave}_{\pi\in\Pi([n])} \left(\sum_{k=1}^{2^{m-j}} \sum_{\pi(h),\pi(i)\in A_{jk}} \lambda_{hi}\right)^{\frac{r}{s}} &\le \left(\operatorname*{Ave}_{\pi\in\Pi([n])} \sum_{k=1}^{2^{m-j}} \sum_{\pi(h),\pi(i)\in A_{jk}} \lambda_{hi}\right)^{\frac{r}{s}}\\ &\le 2^{-\frac{(m-j)r}{s}}.\end{aligned}$$

Thus

$$M^r \le \sum_{j=1}^m 2^{-\frac{(m-j)r}{s}} \le \sum_{j=0}^\infty 2^{-\frac{jr}{s}}.$$

It follows that we have an estimate of the form

$$\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k x_{jk}\right\|^2\right)^{\frac12} \le C\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon'_k x_{jk}\right\|^2\right)^{\frac12}$$

as required which together with (4.2) gives the conclusion. □

We consider $(\mathbb{T},(2\pi)^{-1}d\theta)$. Then H_p is identified as the closed linear span $[e^{in\theta}]_{n\ge0}$ in $L_p(\mathbb{T})$. We will next show that the space L_p/H_p for $0<p<1$ has property (α) and (hence) the decoupling property. We note that this space was shown by Pisier [35] to have cotype 2.

It is possible to deduce the result from the ideas of Pisier [35] and Kislyakov [24] on K-closed couples. However we shall give a self-contained argument for a more general result, which also relies the same underlying ideas. Suppose $0<q<\infty$ and that V is a closed subspace of $L_q(\Omega,\mathbb{P})$. We define $V\otimes_{L_q} X$ to be the closed linear span in $L_q(\Omega,\mathbb{P};X)$ of functions of the time $\xi\otimes x(\omega)=\xi(\omega)x$ for $x\in X$ and $\xi\in V$. Now suppose E is a closed subspace of X; we will say that the quotient map $Q: X\to X/E$ is $(V;L_q)$-*exact* (or V-*exact* if the choice of q is unambiguous) if the naturally induced map $V\otimes_{L_q} X\to V\otimes_{L_q}(X/E)$ is a quotient map. In the special case when $V=R$ the span of a sequence of independent Rademachers, the space $R\otimes_{L_p} X$ is independent of p (up to equivalence of quasi-norm) and we denote by

Rad X this common space. We will say Q is *Rademacher exact* if it is $(R; L_p)$-exact for any and hence for every $0 < p < \infty$. The concept of Rademacher exactness has appeared (without the name) several times before, particularly in the work of Pisier (e.g., [32] and [34]).

We will need the following lemma. Let us say that a subspace V of $L_q(\Omega, \mathbb{P})$ is strongly embedded if for some $\epsilon > 0$ and C we have $\|f\|_{q+\epsilon} \le C\|f\|_q$ for $f \in V$.

Lemma 4.2. *Suppose $0 < p, s \le q < \infty$ and that V is a strongly embedded subspace of $L_q(\Omega, \mathbb{P})$. Then there exists $C = C(p, q, V)$ so that for $\xi \in L_q(\Omega, \mathbb{P}; L_p(\mu))$ we have*

$$C^{-1}\|(\mathbb{E}|\xi|^s)^{\frac{1}{s}}\|_p \le (\mathbb{E}\|\xi\|_p^q)^{\frac{1}{q}} \le C\|(\mathbb{E}|\xi|^s)^{\frac{1}{s}}\|_p. \tag{4.4}$$

Proof. Since V is strongly embedded in L_q it suffices to prove (4.4) in the case $s = q$. First notice that

$$\begin{aligned}\|(\mathbb{E}|\xi|^q)^{\frac{1}{q}}\|_p &\le C\|(\mathbb{E}|\xi|^p)^{\frac{1}{p}}\|_p \\ &= C(\mathbb{E}\|\xi\|_p^p)^{\frac{1}{p}} \\ &\le C(\mathbb{E}\|\xi\|_p^q)^{\frac{1}{q}}.\end{aligned}$$

Conversely if $F = (\mathbb{E}|\xi|^q)^{\frac{1}{q}} + \nu$, where $\nu > 0$, then

$$\begin{aligned}\|\xi(\omega)\|_p &= \left(\int |F^{\frac{p}{q}-1}\xi(\omega)|^p F^{\frac{p}{q}(q-p)} d\mu\right)^{\frac{1}{p}} \\ &\le \|F^{\frac{p}{q}-1}\xi(\omega)\|_q \|F\|_p^{1-\frac{p}{q}}.\end{aligned}$$

Thus

$$\begin{aligned}(\mathbb{E}\|\xi\|_p^q)^{\frac{1}{q}} &\le \|F\|_p^{1-\frac{p}{q}}(\mathbb{E}\|F^{\frac{p}{q}-1}\xi\|_q^q)^{\frac{1}{q}} \\ &= \|F\|_p^{1-\frac{p}{q}}\|(|F|^{p-q}\mathbb{E}|\xi|^q)^{\frac{1}{q}}\|_q \\ &\le \|F\|_p\end{aligned}$$

and the result follows letting $\nu \to 0$. $\square$

Theorem 4.3. *Suppose $0 < p < 1 < q < \infty$ and that V is a strongly embedded closed subspace of $L_q(\Omega, \mathbb{P})$. Then the quotient map $Q : L_p \to L_p/H_p$ is $(V; L_s)$-exact for $0 < s \le q$.*

Proof. As usual, C will denote a constant which may vary from line to line but depends only on V, p, q and s. We will use $\|\cdot\|_p$ to denote the norm in $L_p(\mathbb{T})$ and Q will be the quotient map $Q : L_p \to L_p/H_p$.

The key fact which we use is that L_p is BMO-regular ([17], [24]). This means that there is a constant $C = C(p)$ so that if $f \in L_p$ then there exists $w_0 \ge |f|$ with $\|w_0\|_{L_p(\mathbb{T})} \le C\|f\|_{L_p(\mathbb{T})}$ and $\|\log w_0\|_{\mathrm{BMO}} \le C$. In fact w can be obtained as $\mathcal{M}(|f|^u)^{\frac{1}{u}}$ where $0 < u < p$ and $\mathcal{M}$ is the Hardy–Littlewood maximal function [5]; in particular if f is bounded then w can be chosen bounded. Let us note that since $\log w_0 \in BMO$ then we can write $\log w_0 = \varphi_1 + \mathcal{H}(\varphi_2) + \alpha$ where α is constant, $\mathcal{H}$ is the Hilbert transform and $\|\varphi_1\|_\infty, \|\varphi_2\|_\infty \le C\|\log w_0\|_{\mathrm{BMO}} \le C$. If we then write $\log w = \log w_0 - \varphi_1 + \|\varphi_1\|_\infty$ then we have $\|\mathcal{H}(\log w)\|_\infty \le C$, $w \ge |f|$ and $\|w\|_p \le C\|f\|_p$.

If we then define the outer function

$$F(z) = \exp\left(\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log w(e^{i\theta})\frac{d\theta}{2\pi}\right) \qquad |z|<1$$

then $F \in H_p$ and on the boundary we have $|F| = |w| \geq |f|$ a.e., $\|F\|_p \leq C\|f\|_p$. Furthermore we have an estimate (from the bound on $\mathcal{H}(\log w)$) that $\Im \log F$ is uniformly bounded on the boundary. Hence F^{it} is in H_∞ and $\|F^{it}\|_\infty \leq Ce^{Ct}$ where $C = C(p)$.

Suppose $\eta \in V \otimes L_p/H_p$ is of the form $\sum_{j=1}^n v_j(\omega)Qg_j$ where $v_1,\dots,v_n \in V$ and g_j are bounded functions. Suppose $\mathbb{E}\|\eta\|^s \leq 1$. Let M be defined by

$$M = \inf\left\{(\mathbb{E}\|\xi\|_p^s)^{\frac{1}{s}} :\ \xi \in L_s(\Omega,\mathbb{P};L_p(\mathbb{T})),\ Q\xi = \eta\right\}.$$

It clearly suffices to prove an estimate on M which is independent of η. We may pick a nearly optimal choice of ξ such that $(\mathbb{E}\|\xi\|_p^s)^{\frac{1}{s}} \leq 2M$ with the additional constraint that $\xi(\omega) \in L_\infty(\mathbb{T})$ for all ω. At the same time we may choose for each ω, $f_\omega \in L_\infty$ such that $Qf_\omega = \eta(\omega)$ and $\|f_\omega\|_p \leq 2\|\eta(\omega)\|$.

By Lemma 4.4 we have an estimate $\|(\mathbb{E}|\xi|^s)^{\frac{1}{s}}\|_p \leq CM$. Next we use the above remarks to find a bounded outer function G with $|G| \geq (\mathbb{E}|\xi|^s)^{\frac{1}{s}} + 1$, $\|G\|_p \leq C(M+1)$ and $\|G^{it}\|_\infty \leq Ce^{Ct}$. Similarly for each ω we find a bounded outer function F_ω so that $|F_\omega| \geq |f_\omega|$, $\|F\|_p \leq C(\|\eta(\omega)\|+1)$ and $\|F^{it}\|_\infty \leq Ce^{Ct}$.

Now fix $1 < r < q$. Suppose φ is any polynomial of the form $\varphi(e^{i\theta}) = \sum_{k=1}^m a_k e^{ik\theta}$ such that $\|\varphi\|_{r'} \leq 1$. For each fixed ω we consider the entire function

$$\Phi(z) = e^{z^2}\int_0^{2\pi} G^{\frac{p}{r}-1-z}F_\omega^z \xi(\omega)\varphi\,\frac{d\theta}{2\pi}.$$

We claim that Φ is bounded on vertical strips. In fact if $z = a+it$ then

$$|\Phi(z)| \leq Ce^{a^2-t^2}e^{Ct}\int_0^{2\pi} |G|^{\frac{p}{r}-1-a}|F_\omega|^a|\xi(\omega)||\varphi|\,\frac{d\theta}{2\pi}.$$

Let us consider the line $z = \frac{p}{r}-1+it$. Then

$$\Phi(z) = e^{z^2}\int_0^{2\pi} G^{-it}F_\omega^{it}F_\omega^{\frac{p}{r}-1}f_\omega\varphi\,\frac{d\theta}{2\pi}$$

so that

$$\begin{aligned}|\Phi(z)| &\leq C\int_0^{2\pi} |f_\omega|^{\frac{p}{r}}|\varphi|\frac{d\theta}{2\pi}\\ &\leq C(\|\eta(\omega)\|+1)^{\frac{p}{r}}.\end{aligned}$$

Now suppose $z = \frac{p}{r}-\frac{p}{q}+it$. Then

$$\Phi(z) = e^{z^2}\int_0^{2\pi} G^{-it}F_\omega^{it}G^{\frac{p}{q}-1}F_\omega^{\frac{p}{r}-\frac{p}{q}}\xi(\omega)\varphi\,\frac{d\theta}{2\pi}$$

so that

$$|\Phi(z)| \leq C\|G^{\frac{p}{q}-1}\xi(\omega)\|_q\|F_\omega^{\frac{p}{r}-\frac{p}{q}}\|_u$$

where $\frac{1}{q}+\frac{1}{u}+\frac{1}{r'} = 1$, i.e., $\frac{1}{u} = \frac{1}{r}-\frac{1}{q}$. Thus

$$\|F_\omega^{\frac{p}{r}-\frac{p}{q}}\|_u = \|F_\omega\|_p^{\frac{p}{u}} \leq C(\|\eta(\omega)\|+1)^{\frac{p}{r}-\frac{p}{q}}.$$

Now by applying the Maximum Modulus Principle to a function of the type $e^{cz}\Phi(z)$ where c is real, we obtain an estimate

$$|\Phi(0)| \le C(\|\eta(\omega)+1\|)^{1-\theta}\|G^{\frac{p}{q}-1}\xi(\omega)\|_q^\theta$$

where $\theta=(1-\frac{p}{r})(1-\frac{p}{q})^{-1}$. As this estimate holds for all such φ we obtain

$$\inf_{h\in H_r}\|G^{\frac{p}{r}-1}\xi(\omega)-h\|_r \le C(\|\eta(\omega)+1\|)^{1-\theta}\|G^{\frac{p}{q}-1}\xi(\omega)\|_q^\theta.$$

Since the Riesz projection $\mathcal{R}$ is bounded on L_r we thus have

$$\|G^{\frac{p}{r}-1}\xi(\omega)-\mathcal{R}(G^{\frac{p}{r}-1}\xi(\omega))\|_r \le C(\|\eta(\omega)+1\|)^{1-\theta}\|G^{\frac{p}{q}-1}\xi(\omega)\|_q^\theta.$$

Hence

$$\|\xi(\omega)-G^{1-\frac{p}{r}}\mathcal{R}(G^{\frac{p}{r}-1}\xi(\omega))\|_p \le C\|G\|_p^{1-\frac{p}{r}}(\|\eta(\omega)+1\|)^{1-\theta}\|G^{\frac{p}{q}-1}\xi(\omega)\|_q^\theta.$$

It follows that

$$M\le CM^{1-\frac{p}{r}}((\mathbb{E}\|\eta\|^s)^{\frac1s}+1)^{1-\theta}(\mathbb{E}\|G^{\frac{p}{q}-1}\xi\|_q^s)^{\frac{\theta}{s}}. \tag{4.5}$$

Now by Lemma 4.4 we obtain

$$\begin{aligned}E\|G^{\frac{p}{q}-1}\xi\|_q^s)^{\frac1s} &\le C\|G^{\frac{p}{q}-1}(\mathbb{E}|\xi|^s)^{\frac1s}\|_q\\ &\le C\|G^{\frac{p}{q}}\|_q\\ &\le C\|G\|_p^{\frac{p}{q}}\\ &\le CM^{\frac{p}{q}}.\end{aligned}$$

Now by (4.5) we have

$$M\le CM^\theta$$

which gives the required bound on M and completes the proof. □

Now by taking the special cases of $V=[\epsilon_j]_{j\in\mathbb{N}}$ or $V=[\epsilon_j'\epsilon_k]_{j,k\in\mathbb{N}}$ or $V=[\epsilon_j\epsilon_k]_{j<k}$ in any L_s-space we can deduce the following. (We have already noted that the fact that L_p/H_p has cotype two is due to Pisier [35].)

Theorem 4.4. *For $0<p<1$, the space L_p/H_p has has cotype two, property (α) and the decoupling property.*

Let us remark that L_p/H_p is an example of a non-A-convex space (first proved by Aleksandrov [1]); it is therefore of some interest that it nevertheless has the decoupling property.

Our next result is an extension of a result of Pisier [32] to the context of quasi-Banach spaces. Note that Pisier's argument uses the convexity of the norm in an essential way so that a new proof is required.

Theorem 4.5. *Let X be a quasi-Banach space and suppose E is a closed subspace with type greater than one. If either X or X/E has nontrivial cotype then the quotient map $X\to X/E$ is Rademacher exact.*

Proof. Assume X has nontrivial cotype. We argue that X/E must also have nontrivial cotype. Indeed if not by passing to an ultraproduct we can find a similar example where c_0 embeds in X/E; then by passing to a subspace we have an example where X/E is isomorphic to c_0. Now by the results of [20] and [10] X is locally convex, i.e., a Banach space and this contradicts Pisier's result [32].

Conversely if X/E has nontrivial cotype we claim X has nontrivial cotype. Indeed by a similar ultraproduct argument we need only show that c_0 cannot be embedded into X. Let $T : c_0 \to X$ be an embedding. If $Q : X \to X/E$ is the quotient map we have $\lim_{n\to\infty} \|QTe_n\| = 0$ and so there is a sequence (y_n) in E with $\lim_{n\to\infty} \|y_n - Te_n\| = 0$. Simple perturbation arguments then yield that some subsequence of (y_n) is a basic sequence equivalent to the c_0-basis, contradicting the fact that E has type.

Now assume X/E has nontrivial cotype and let $\sum_{n=1}^{\infty} \epsilon_n v_n$ be a converging series in Rad $(X/E) := R \otimes_{L_2} (X/E)$. Then there is a bounded operator $T : c_0 \to L_2(X/E)$ defined by $T((a_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} a_n \epsilon_n v_n$. By the results of [19] for some $q < \infty$, T may be factored in the form $T = UV$ where $V : c_0 \to \ell_q$ and $U : \ell_q \to L_2(X/E)$ are bounded operators.

We now construct the pullback of the short exact sequence $0 \to L_2(E) \to L_2(X) \to L_2(X/E) \to 0$ by the operator $U : \ell_q \to L_2(X/E)$, so that we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & L_2(E) \arrow[r] & L_2(X) \arrow[r] & L_2(X/E) \arrow[r] & 0 \\
0 \arrow[r] & L_2(E) \arrow[r] \arrow[u, equal] & Y \arrow[r] \arrow[u, "S"'] & \ell_q \arrow[r] \arrow[u, "U"'] & 0.
\end{tikzcd}$$

Since $L_2(E)$ and ℓ_q both have nontrivial type, the space Y is a Banach space with nontrivial type [10]. Hence the quotient map $Q_Y : Y \to \ell_q$ is Rademacher exact. In particular we can find $y_n \in Y$ so that $\sum_{n=1}^{\infty} \epsilon'_n y_n \in L_2(Y)$ and $Q_Y y_n = Ve_n$ where e_n is the canonical basis of c_0. Let $Sy_n = f_n \in L_2(X)$. Then $\sum_{n=1}^{\infty} \epsilon'_n f_n \in L_2(L_2(X))$ and $Qf_n(\omega) = \epsilon_n(\omega) v_n$. Note that by a symmetry argument if X is r-normed we have

$$\left(\mathbb{E}\left\|\sum_{k=1}^{m} \epsilon'_k f_k(\omega)\right\|^2\right)^{\frac{1}{2}} \le 2^{\frac{1}{r}-1}\left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon'_k f_k(\omega)\right\|^2\right)^{\frac{1}{2}} \qquad m < n.$$

Hence

$$\int_{\Omega} \sup_m \mathbb{E}\left\|\sum_{k=1}^{m} \epsilon'_k f_k(\omega)\right\|^2 d\mathbb{P}(\omega) < \infty$$

and hence we may choose ω so that

$$\sup_m \mathbb{E}\left\|\sum_{k=1}^{m} \epsilon'_k f_k(\omega)\right\|^2 < \infty.$$

Let $x_n = \epsilon_n(\omega) f_n(\omega)$. Then since X has cotype it is easy to see that $\sum \epsilon_n x_n \in$ Rad (X) and that $Qx_n = v_n$. $\square$

Theorem 4.6. *Let X be a quasi-Banach space with property (α) and suppose E is a closed subspace with type greater than one. Then X/E also has property (α) (and hence the decoupling property).*

Proof. Note that X must have some nontrivial cotype because it has (α). Hence the quotient map $Q : X \to X/E$ is Rademacher-exact by Theorem 4.5; we then also obtain that the quotient Rad $(X) \to$ Rad (X/E) is Rademacher-exact. This implies that $X \to X/E$ is $(V; L_2)$-exact where $V = [\epsilon_j \epsilon'_k]_{j,k}$. Hence if $(v_{jk})_{j,k=1}^{n}$ is an $n\times n$-matrix with values in X/E it follows that we can find $(x_{jk})_{j,k=1}^n \in X$ with $Qx_{jk}=v_{jk}$ and

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' x_{jk}\Big\|^2\Big)^{\frac12} \le C\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' v_{jk}\Big\|^2\Big)^{\frac12}$$

where $C=C(X,E)$. Now

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n a_{jk}\epsilon_j\epsilon_k' x_{jk}\Big\|^2\Big)^{\frac12} \le C\max_{j,k}|a_{jk}|\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' x_{jk}\Big\|^2\Big)^{\frac12}$$

and so

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n a_{jk}\epsilon_j\epsilon_k' v_{jk}\Big\|^2\Big)^{\frac12} \le C\max_{j,k}|a_{jk}|\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' v_{jk}\Big\|^2\Big)^{\frac12}. \qquad \square$$

Note that this implies that any quotient of L_p, $(0<p<1)$, by a subspace with nontrivial type has the decoupling property. Let us remark that we do not know whether such a space is in fact already natural. We also remark that we do not know if the theorem above can be improved to show that if X has some cotype and the decoupling property and E has nontrivial type then X/E has the decoupling property.

Theorem 4.7. *Suppose X is a quasi-Banach space and E is a closed subspace.*

(a) *Suppose* $\dim E<\infty$. *If X/E has cotype two then X has cotype two. If X/E has additionally property* (α) *then X has property* (α) *and hence the decoupling property.*
(b) *Suppose E is Hilbertian and X has cotype two. Then if X/E has property* (α) *then X also has property* (α) *and hence the decoupling property.*

Proof. We will start with the following claim. We denote the canonical basis of ℓ_2^n by $(e_j)_{j=1}^n$ and the Hilbert–Schmidt norm by $\|\cdot\|_{\mathrm{HS}}$. Let $Q:X\to X/E$ be the quotient map.

Claim. There is a constant C so that if $v_1,\dots,v_n\in X/E$ then there are bounded operators $U:\ell_2^n\to\ell_2^n$ and $V:\ell_2^n\to X$ with $\|V\|\le 1$, $QVUe_j=v_j$ for $1\le j\le n$, and

$$\|U\|_{\mathrm{HS}} \le C\Big(\mathbb{E}\Big\|\sum_{k=1}^n \epsilon_k v_k\Big\|^2\Big)^{\frac12}. \tag{4.6}$$

To verify the claim it is sufficient to show that if $(v_k)_{k=1}^\infty$ is an infinite sequence with $\sum_{k=1}^\infty \epsilon_k v_k$ converging in $L_2(X/E)$ then we can find operators $U:\ell_2\to\ell_2$, $V:\ell_2\to X$ with $\pi_2(U)<\infty$, $\|V\|<\infty$ and $QVUe_k=v_k$ for $k\in\mathbb{N}$.

Consider the operator $T:\ell_2\to X/E$ defined by $Te_j=v_j$. We will show that this operator is 2-absolutely summing. Indeed suppose $u_1,\dots,u_m\in[e_j]_{j=1}^N$ and $\sum_{j=1}^m|(u_j,x)|^2\le\|x\|^2$ for all $x\in\ell_2$. Then we may find $u_1',\dots,u_l'\in[e_j]_{j=1}^N$ so that

$$\sum_{j=1}^m|(u_j,x)|^2+\sum_{j=1}^l|(u_j',x)|^2+\sum_{j=N+1}^\infty|(e_j,x)|^2=\|x\|^2 \qquad x\in\ell_2.$$

Hence if $(\gamma_j)_{j=1}^\infty, (\gamma_j')_{j=1}^l, (\gamma_j'')_{j=N+1}^\infty$ denote independent normalized Gaussians

$$\begin{aligned}\Big(\sum_{j=1}^m \|Tu_j\|^2\Big)^{\frac12} &\le C\Big(\mathbb{E}\Big\|\sum_{j=1}^m \gamma_j Tu_j\Big\|^2\Big)^{\frac12}\\ &\le 2^{\frac1r-1}C\Big(\mathbb{E}\Big\|\sum_{j=1}^m \gamma_j Tu_j + \sum_{j=1}^l \gamma_j' Tu_j' + \sum_{j=N+1}^\infty \gamma_j'' Te_j\Big\|^2\Big)^{\frac12}\\ &= 2^{\frac1r-1}C\Big(\mathbb{E}\Big\|\sum_{j=1}^\infty \gamma_j v_j\Big\|^2\Big)^{\frac12}\\ &\le C\Big(\mathbb{E}\Big\|\sum_{j=1}^\infty \epsilon_j v_j\Big\|^2\Big)^{\frac12}.\end{aligned}$$

It follows that $\pi_2(T) \le C(\mathbb{E}\|\sum_{j=1}^\infty \epsilon_j v_j\|^2)^{\frac12}$. Now by Proposition 2.4 we have a factorization of T given by $T = T_0 jL$ where $L : \ell_2 \to C(K)$, $j : C(K) \to L_2(K,\mu)$ is the inclusion and μ is a probability measure on K and $T_0 : L_2(K,\mu) \to X$ is bounded. Identifying $L_2(K,\mu)$ with ℓ_2 we let $U = jL$ which is therefore Hilbert–Schmidt. Now consider the pull back using T_0,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & X & \longrightarrow & X/E & \longrightarrow & 0\\ & & \| & & \uparrow{\scriptstyle V_0} & & \uparrow{\scriptstyle T_0} & & \\ 0 & \longrightarrow & E & \longrightarrow & Y & \longrightarrow & \ell_2 & \longrightarrow & 0.\end{array}$$

Now in case (a) we note that Y is necessarily locally convex [10] and hence the second row splits. In case (b) Y is again locally convex and is also a subspace of $X \oplus \ell_2$ which implies Y has cotype 2. Since E is Hilbertian, we certainly have that Y has some nontrivial type ([9]) and so Y^* has type two. By the Maurey extension theorem [27] the dual short exact sequence $0 \to \ell_2 \to Y^* \to E^* \to 0$ splits; by the reflexivity of the spaces this is enough again to show that the second row splits. Let $S : \ell_2 \to Y$ be a lifting and set $V = V_0 S$. Then the claim (4.6) is established.

Now let us show in case (a) that X has cotype two. Suppose $x_1, \ldots, x_n \in X$. Use (4.6) to find U, V with $QVUe_j = Qx_j$ and

$$\|U\|_{\mathrm{HS}} \le C\Big(\mathbb{E}\Big\|\sum_{j=1}^n \epsilon_j Qx_j\Big\|^2\Big)^{\frac12}.$$

Let $y_j = VUe_j$. Then

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n \epsilon_j y_j\Big\|^2\Big)^{\frac12} \le \Big(\mathbb{E}\Big\|\sum_{j=1}^n \epsilon_j Ue_j\Big\|^2\Big)^{\frac12}$$

and so

$$\Big(\sum_{j=1}^n \|y_j\|^2\Big)^{\frac12} \le C\Big(\mathbb{E}\Big\|\sum_{j=1}^n \epsilon_j x_j\Big\|^2\Big)^{\frac12}.$$

Now

$$\Big(\sum_{j=1}^n \|x_j - y_j\|^2\Big)^{\frac12} \le C\Big(\mathbb{E}\Big\|\sum_{j=1}^n \epsilon_j (x_j - y_j)\Big\|^2\Big)^{\frac12}$$

since E is isomorphically Hilbertian (in fact, finite-dimensional). Combining we have

$$\left(\sum_{j=1}^n \|x_j\|^2\right)^{\frac12} \le C\left(\mathbb{E}\left\|\sum_{j=1}^n \epsilon_j x_j\right\|^2\right)^{\frac12}$$

, i.e., X is cotype 2.

Next in either case (a) or (b) we assume that X/E has property (α). Now suppose $(x_{jk})_{j,k=1}^n$ is an $n\times n$ array in X. Then since X/E has property (α) we can find a map $U:\ell_2(n^2)\to\ell_2^{n^2}$ and a map $V:\ell_2^{n^2}\to X$ with $QVUe_{jk}=Qx_{jk}$ (labelling the canonical basis of $\ell_2^{n^2}$ as (e_{jk}) for $1\le j,k\le n$) such that $\|V\|\le 1$ and

$$\|U\|_{\mathrm{HS}} \le C\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' Qx_{jk}\right\|^2\right)^{\frac12}.$$

Let $y_{jk}=VUe_{jk}$. Then if $|a_{jk}|\le 1$,

$$\begin{aligned}\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n a_{jk}\epsilon_j\epsilon_k' y_{jk}\right\|^2\right)^{\frac12} &\le \left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n a_{jk}\epsilon_j\epsilon_k' Ue_{jk}\right\|^2\right)^{\frac12}\\ &\le \|U\|_{\mathrm{HS}}\\ &\le C\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' x_{jk}\right\|^2\right)^{\frac12}.\end{aligned}$$

Note in particular

$$\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' y_{jk}\right\|^2\right)^{\frac12} \le C\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' x_{jk}\right\|^2\right)^{\frac12}.$$

Again since E is Hilbertian

$$\begin{aligned}\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n a_{jk}\epsilon_j\epsilon_k'(x_{jk}-y_{jk})\right\|^2\right)^{\frac12} &\le C\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k'(x_{jk}-y_{jk})\right\|^2\right)^{\frac12}\\ &\le C\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' x_{jk}\right\|^2\right)^{\frac12}.\end{aligned}$$

Combining we see that X has property (α). □

A minimal extension of a quasi-Banach space X is a space Y so that there is a one-dimensional subspace E of Y with Y/E isomorphic to X. We have the following immediate corollary:

Corollary 4.8. *Every minimal extension of ℓ_1 or L_1 has the decoupling property.*

We remark that there is a minimal extension of ℓ_1 [18] with the property that it contains no basic sequence. In particular the complex version of this space is not A-convex [38], and hence not natural. Let us remark that in [14] an example of a non-A-convex minimal extension of L_1/H_1 is created and it follows also from our arguments that this space also has decoupling.

In the next section we will give an example of a minimal extension of the trace class $\mathcal{S}_1$ (which has cotype two but fails (α)) which fails to have decoupling.

5. Further remarks on decoupling

In this section we will examine some problems related to Theorem 3.1. By (iii) we see that a quasi-Banach space X has the decoupling property if (3.3) holds for some $0 < p < \infty$, which we may split as two distinct inequalities. Precisely, we require the existence of a constant C so that for every X-valued symmetric matrix $(x_{jk})_{j,k=1}^n$ with $x_{jj} = 0$ for $1 \le j \le n$

$$\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k x_{jk}\right\|^2\right)^{\frac12} \le C\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' x_{jk}\right\|^2\right)^{\frac12} \tag{5.1}$$

and

$$\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' x_{jk}\right\|^2\right)^{\frac12} \le C\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k x_{jk}\right\|^2\right)^{\frac12}. \tag{5.2}$$

In this section we show that there are quasi-Banach spaces where (5.1) fails and spaces where (5.2) fails.

Proposition 5.1. *Let X be a quasi-Banach space. Then for $0 < p < \infty$ there exists a constant $C = C(p, X)$ with the following property: suppose $(x_{jk})_{j,k=1}^n$ is any X-valued matrix. Then*

$$V_p\left(\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k x_{jk}\right) \le C\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' x_{jk}\right\|^p\right)^{\frac1p}. \tag{5.3}$$

Proof. Suppose X is r-normed and $s = \min(p, r)$. Suppose $u, v \in \ell_\infty^n$. Let

$$\zeta(u,v) = \sum_{j=1}^n\sum_{k=1}^n u_j v_k \epsilon_j\epsilon_k x_{jk},$$
$$\zeta'(u,v) = \sum_{j=1}^n\sum_{k=1}^n u_j v_k \epsilon_j\epsilon_k' x_{jk}.$$

Let $\chi = (1, 1, \dots, 1)$ and χ_A be the indicator of a subset A of $[n] = \{1, 2, \dots, n\}$. Let $\zeta = \zeta(\chi, \chi)$ and $\zeta' = \zeta'(\chi, \chi)$. Then for any A, B and $\delta_1, \delta_2 = \pm 1$, we have

$$\zeta'(\chi_A + \delta_1\chi_{\tilde A}, \chi_B + \delta_2\chi_{\tilde B}) \approx_1 \zeta'$$

and so, by averaging, for any subset A of $[n]$ we have

$$(\mathbb{E}\|\zeta'(\chi_A, \chi_B)\|^p)^{\frac1p} \le 4^{\frac1s - 1}(\mathbb{E}\|\zeta'\|^p)^{\frac1p}.$$

If $A \cap B = \emptyset$ then $\zeta(\chi_A, \chi_B) \approx_1 \zeta'(\chi_A, \chi_B)$ and so

$$(\mathbb{E}\|\zeta(\chi_A, \chi_B)\|^p)^{\frac1p} \le 4^{\frac1s - 1}(\mathbb{E}\|\zeta'\|^p)^{\frac1p} \qquad A \cap B = \emptyset. \tag{5.4}$$

Note also that if $\delta = \pm 1$

$$\zeta(\chi_A + \delta\chi_{\tilde A}, \chi_A + \delta\chi_{\tilde A}) \approx \zeta.$$

Hence

$$V_p(\zeta(\chi_A, \chi_A) + \zeta(\chi_{\tilde A}, \chi_{\tilde A})) \le 2^{\frac1s - 1}V_p(\zeta), \tag{5.5}$$

$$V_p(\zeta(\chi_A, \chi_{\tilde A}) + \zeta(\chi_{\tilde A}, \chi_A)) \le 2^{\frac1s - 1}V_p(\zeta). \tag{5.6}$$

We now prove (5.3). In fact by rewriting $\delta = \chi_A - \chi_{\widetilde{A}}$ and $\delta' = \chi_B - \chi_{\widetilde{B}}$ we need only estimate $V_p(\zeta(\chi_A, \chi_B))$ for arbitrary A, B. Then we write

$$\zeta(\chi_A, \chi_B) = \zeta(\chi_{A\cap B}, \chi_{A\cap B}) + \zeta(\chi_{A\cap B}, \chi_{B\setminus A}) + \zeta(\chi_{A\setminus B}, \chi_{A\cap B}).$$

Combining (5.4) and (5.5) gives an estimate

$$V_p(\zeta(\chi_A, \chi_B)) \leq C(\mathbb{E}\|\zeta'\|^p)^{\frac{1}{p}} + V_p(\zeta))$$

and hence

$$V_p\Big(\sum_{j=1}^n \sum_{k=1}^n \delta_j \delta'_k \epsilon_j \epsilon_k x_{jk}\Big) \leq C(\mathbb{E}\|\zeta'\|^p)^{\frac{1}{p}} + V_p(\zeta)).$$

It follows that if $\eta_j, \eta'_j = \pm 1$ we also have

$$V_p\Big(\sum_{j=1}^n \sum_{k=1}^n \epsilon_j \epsilon_k x_{jk}\Big) \leq C(\mathbb{E}\|\zeta'(\eta,\eta')\|^p + \mathbb{E}\|\zeta(\eta,\eta')\|^p)^{\frac{1}{p}}.$$

Treating η_j, η'_k are independent Rademachers, also independent of ϵ_j we note that $\mathbb{E}\|\zeta(\eta,\eta')\|^p = \mathbb{E}\|\zeta'(\eta,\eta')\|^p = \mathbb{E}\|\zeta'\|^p$ and so (5.3) follows. □

We would also like to introduce the notion of Gaussian decoupling, replacing the Rademachers with normalized Gaussians. We say that X has *Gaussian decoupling* if there is a constant such that for every symmetric X-valued matrix $(x_{jk})_{j,k=1}^n$ with $x_{jj} = 0$ we have

$$C^{-1}\Big(\mathbb{E}\Big\|\sum_{j=1}^n \sum_{k=1}^n x_{jk}\gamma_j\gamma'_k\Big\|^2\Big)^{\frac{1}{2}} \leq \Big(\mathbb{E}\Big\|\sum_{j=1}^n \sum_{k=1}^n x_{jk}\gamma_j\gamma_k\Big\|^2\Big)^{\frac{1}{2}} \leq C\Big(\mathbb{E}\Big\|\sum_{j=1}^n \sum_{k=1}^n x_{jk}\gamma_j\gamma'_k\Big\|^2\Big)^{\frac{1}{2}}, \tag{5.7}$$

where, as usual $(\gamma_j)_{j=1}^n$ and $(\gamma'_j)_{j=1}^n$ denote two mutually independent sequences of independent normalized Gaussians.

The following proposition was pointed out by Stephen Montgomery-Smith:

Proposition 5.2. *Let X be a quasi-Banach space. Then if $(x_{jk})_{j,k=1}^n$ is any X-valued symmetric matrix then*

$$\sum_{j=1}^n \sum_{k=1}^n x_{jk}\gamma'_j\gamma_k \approx_1 \frac{1}{2}\Big(\sum_{j=1}^n \sum_{k=1}^n x_{jk}\gamma_j\gamma_k - \sum_{j=1}^n \sum_{k=1}^n x_{jk}\gamma'_j\gamma'_k\Big).$$

Hence there exists a constant $C = C(X)$ so that

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n \sum_{k=1}^n x_{jk}\gamma'_j\gamma_k\Big\|^2\Big)^{\frac{1}{2}} \leq C\Big(\mathbb{E}\Big\|\sum_{j=1}^n \sum_{k=1}^n x_{jk}\gamma_j\gamma_k\Big\|^2\Big)^{\frac{1}{2}}.$$

Thus X has Gaussian decoupling if and only if there exists a C so that for every X-valued matrix $(x_{jk})_{j,k=1}^n$ with zero diagonal,

$$\Big(E\Big\|\sum_{j=1}^n \sum_{k=1}^n x_{jk}\gamma_j\gamma_k\Big\|^2\Big)^{\frac{1}{2}} \leq C\Big(\mathbb{E}\Big\|\sum_{j=1}^n \sum_{k=1}^n x_{jk}\gamma_j\gamma'_k\Big\|^2\Big)^{\frac{1}{2}}. \tag{5.8}$$

Proof. We may replace γ_j by $2^{-\frac{1}{2}}(\gamma_j+\gamma_j')$ and γ_j' by $2^{-\frac{1}{2}}(\gamma_j-\gamma_j')$. Then the first equation is immediate and the others then follow trivially. Notice that (5.8) holds for arbitrary matrices once it holds for symmetric matrices. □

Proposition 5.3. *Suppose X is quasi-Banach space satisfying condition (5.1). Then X has Gaussian decoupling.*

Proof. It is easy to derive (5.8) from (5.1) by approximating a normalized gaussian γ by $N^{-\frac{1}{2}}(\epsilon_1+\cdots+\epsilon_N)$. We omit the details. □

We conclude the section by discussing two examples. First we recall that the Schatten ideal $\mathcal{S}_p$ fails decoupling. In fact we will see that it fails condition (5.2). For $j\neq k$ let $x_{jk}=\frac{1}{2}(e_{jk}+e_{kj})$ where $e_{jk}=e_j\otimes e_k$ and let $x_{jj}=0$. Let $\zeta_n=\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_k x_{jk}$ and let ξ_n be its decoupled version $\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_k' x_{jk}$. Observe that condition (5.2) implies that

$$(\mathbb{E}\|\xi_n\|^2)^{\frac{1}{2}}\le CV_2(\xi_n)\le CV_2(\zeta_n).$$

However

$$\left(\mathbb{E}\left\|\xi_n+\sum_{j=1}^n\epsilon_j\epsilon_j' e_{jj}\right\|^2\right)^{\frac{1}{2}}=n$$

and

$$\left(\mathbb{E}\left\|\zeta_n+\sum_{j=1}^n e_{jj}\right\|^2\right)^{\frac{1}{2}}=n.$$

Thus

$$(\mathbb{E}\|\xi_n\|^2)^{\frac{1}{2}}\ge cn^{\frac{1}{p}}$$

but

$$V_2(\zeta_n)\le Cn$$

where $0<c,C<\infty$. Notice that this shows that (5.3) cannot be replaced by an equivalence.

Let us recall that $\mathcal{S}_p$ has cotype two [40]. It follows from Proposition 2.5 (applied twice) that for some C we have

$$\begin{aligned}C^{-1}\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_k' x_{jk}\right\|^2\right)^{\frac{1}{2}}&\le\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n\gamma_j\gamma_k' x_{jk}\right\|^2\right)^{\frac{1}{2}}\\&\le C\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_k' x_{jk}\right\|^2\right)^{\frac{1}{2}}\end{aligned}$$

for every $\mathcal{S}_p$-valued matrix $(x_{jk})_{j,k=1}^n$. However it is *not* true that there exists a constant C so that for every symmetric matrix $(x_{jk})_{j,k=1}^n$ with zero diagonal we have

$$\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n\gamma_j\gamma_k x_{jk}\right\|^2\right)^{\frac{1}{2}}\le C\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n\epsilon_j\epsilon_k x_{jk}\right\|^2\right)^{\frac{1}{2}}.$$

Indeed by Proposition 5.2

$$\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k' x_{jk}\right\|^2\right)^{\frac12} \approx \left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \gamma_j\gamma_k' x_{jk}\right\|^2\right)^{\frac12}$$
$$\le C\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \gamma_j\gamma_k x_{jk}\right\|^2\right)^{\frac12}.$$

Problem. Does $\mathcal{S}_p$ fail (5.1) when $0<p<1$? Does $\mathcal{S}_p$ have Gaussian decoupling?

We conclude with an example of a minimal extension of the trace class $\mathcal{S}_1$ which fails Gaussian decoupling and hence also (5.1). Note that $\mathcal{S}_1$ has cotype two but fails to have property (α).

Let $(e_j)_{j=1}^\infty$ be an orthonormal basis of a separable Hilbert space and let $e_{jk} = e_j \otimes e_k$ as usual. Let $\mathcal{F}$ be the linear span of $\{e_{jk} : 1 \le j,k < \infty\}$. For $x \in \mathcal{F}$ let $(\lambda_j)_{j=1}^\infty$ be the sequence of eigenvalues of x repeated according to algebraic multiplicity (this sequence is finitely nonzero). Define

$$\Phi(x) = \sum_{j=1}^\infty \lambda_j \log\frac{|\lambda_j|}{\|x\|}.$$

Here we interpret the summand as zero if $\lambda_j = 0$. Then Φ is homogeneous and quasi-additive in the sense that there is a constant C so that if $x,y\in\mathcal{F}$

$$|\Phi(x+y)-\Phi(x)-\Phi(y)| \le C(\|x\|+\|y\|).$$

This is a special case of Theorem 6.8 of [16]. It follows that we can create a minimal extension $\mathbb{C}\oplus_\Phi \mathcal{S}_1$ by completing the direct sum $\mathbb{C}\oplus\mathcal{F}$ quasi-normed by

$$\|(\lambda,x)\| = |\lambda-\Phi(x)| + \|x\|_{\mathcal{S}_1}.$$

Let us denote this space by X. By Theorem 4.7 X has cotype two. We will show that this space fails Gaussian decoupling. Indeed, let us suppose it has the Gaussian decoupling property. Then using the Kahane–Khintchine inequality we can suppose that for any matrix $(x_{jk})_{j,k=1}^n$ with $x_{jj}=0$ we have an estimate using L_1-norms:

$$\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \gamma_j\gamma_k x_{jk}\right\| \le C\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \gamma_j\gamma_k' x_{jk}\right\|.$$

Let us note

$$\Phi\left(\sum_{k=1}^n a_jb_ke_{jk}\right) = \left(\sum_{j=1}^n a_jb_j\right)\log\frac{|\sum_{j=1}^n a_jb_j|}{(\sum_{j=1}^n|a_j|^2)^{\frac12}(\sum_{j=1}^n|b_j|^2)^{\frac12}}.$$

Hence

$$\left|\Phi\left(\sum_{k=1}^n a_jb_ke_{jk}\right)\right| \le \left(\sum_{j=1}^n|a_j|^2\right)^{\frac12}\left(\sum_{j=1}^n|b_j|^2\right)^{\frac12}.$$

Let us define $x_{jk} = (0,e_{jk})$ for $1\le j,k\le n$. This matrix does not have zero diagonal but we will later apply our estimate (5.8) from Proposition 5.2 to the same matrix with the diagonal removed. Then

$$\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \gamma_j\gamma_k' x_{jk}\Big\| = \mathbb{E}\Big|\Phi\Big(\sum_{j=1}^n\sum_{k=1}^n \gamma_j\gamma_k' e_{jk}\Big)\Big| + \mathbb{E}\Big(\sum_{j=1}^n |\gamma_j|^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^n |\gamma_k'|^2\Big)^{\frac{1}{2}}$$
$$\le 2n.$$

Similarly we see that

$$\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n x_{jk}\gamma_j\gamma_k\Big\| \le 2n.$$

On the other hand

$$\mathbb{E}\Big\|\sum_{j=1}^n \gamma_j\gamma_j' x_{jj}\Big\| \le \mathbb{E}\sum_{j=1}^n |\gamma_j\gamma_j'| + \mathbb{E}\left|\sum_{j=1}^n \gamma_j\gamma_j' \log\frac{|\gamma_j||\gamma_j'|}{\sum_{k=1}^n |\gamma_k||\gamma_k'|}\right|.$$

The first term on the right is estimated by Cn. The second we split and notice first that the sequence of independent identically random variables $\gamma_j\gamma_j'\log(|\gamma_j||\gamma_j'|)$ is uniformly bounded in L_2 with mean zero. Hence

$$\left(\mathbb{E}\Big|\sum_{j=1}^n \gamma_j\gamma_j'\log(|\gamma_j\gamma_j'|)\Big|^2\right)^{\frac{1}{2}} \le C\sqrt{n}.$$

However, using the fact $t\log 1/t \le 1/e \le 1$ for $0 \le t \le 1$,

$$\left|\sum_{j=1}^n \gamma_j\gamma_j' \log\frac{|\sum_{j=1}^n \gamma_j\gamma_j'|}{\sum_{k=1}^n |\gamma_k\gamma_k'|}\right| \le \sum_{j=1}^n |\gamma_j\gamma_j'|$$

and

$$\mathbb{E}\left|\sum_{j=1}^n \gamma_j\gamma_j' \log\Big|\sum_{j=1}^n \gamma_j\gamma_j'\Big|\right| \le \mathbb{E}\left(C\left(1+\Big(\sum_{j=1}^n \gamma_j\gamma_j'\Big)^2\right)\right) \le Cn.$$

Combining all the above estimates we have

$$\mathbb{E}\Big\|\sum_{j=1}^n \gamma_j\gamma_j' x_{jj}\Big\| \le Cn.$$

It follows that

$$\mathbb{E}\Big\|\sum_{j\ne k} \gamma_j\gamma_k' x_{jk}\Big\| \le Cn.$$

Now if X has the Gaussian decoupling property we will obtain by (5.8) a similar estimate for the undecoupled version, i.e.,

$$\mathbb{E}\Big\|\sum_{j\ne k} \gamma_j\gamma_k x_{jk}\Big\| \le Cn$$

and this implies an estimate

$$\mathbb{E}\Big\|\sum_{j=1}^n \gamma_j^2 x_{jj}\Big\| \le Cn$$

which in turn gives

$$\mathbb{E}\left|\sum_{j=1}^{n}\gamma_j^2\log\frac{\gamma_j^2}{\sum_{k=1}^{n}\gamma_k^2}\right|\le Cn.$$

Now $\frac{1}{n}(\sum_{j=1}^{n}\gamma_j^2\log\gamma_j^2)$ converges in L_1-norm by the Strong Law of Large Numbers. Hence

$$\mathbb{E}\left|\frac{1}{n}\left(\sum_{j=1}^{n}\gamma_j^2\right)\log\left(\sum_{j=1}^{n}\gamma_j^2\right)\right|\le C.$$

Again by the Strong Law of Large Numbers,

$$\lim_{n\to\infty}\frac{1}{n}\left(\sum_{j=1}^{n}\gamma_j^2\right)=1 \text{ almost surely.}$$

Hence by Fatou's Lemma,

$$\mathbb{E}\liminf_{n\to\infty}\left|\log\sum_{j=1}^{n}\gamma_j^2\right|\le C$$

which is absurd since $\sum_{j=1}^{\infty}\gamma_j^2=\infty$ almost surely.

6. Bilinear maps and Grothendieck's theorem

We conclude with a discussion of applications to bilinear maps. Consider a continuous bilinear map $B : X\times Y\to Z$. If X, Y, Z are all Banach spaces then B induces a linear map $\hat{B} : X\otimes_\pi Y\to Z$ where $X\otimes_\pi Y$ is the projective tensor product. Now suppose X, Y are Banach spaces but Z is a quasi-Banach space. Then B induces a bounded linear map $\hat{B} : X\otimes_\pi Y\to Z$ if and only if the convex hull of the set $\{B(x, y) : \|x\|, \|y\|\le 1\}$ is bounded or equivalently there is a constant C so that

$$\left\|\sum_{j=1}^{n}B(x_j, y_j)\right\|\le C\sum_{j=1}^{n}\|x_j\|\|y_j\|. \tag{6.1}$$

An equivalent formulation is that B factors through some Banach space Z_0, i.e., $B = TB_0$ where $B_0 : X\times Y\to Z_0$ is a bounded bilinear form and $T : Z_0\to Z$ is a bounded linear operator.

Theorem 6.1. *Let X, Y be Banach space of type two and suppose Z is a quasi-Banach space with the decoupling property. Then any bilinear form $B : X\times Y\to Z$ factors through a Banach space.*

Proof. This is an easy modification of [15] Theorems 4.6 and 4.8. Let $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_n \in Y$. Then since Z has the decoupling property

$$\begin{aligned}\Big\|\sum_{j=1}^n B(x_j, y_j)\Big\| &\le C\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon_k B(x_j, y_k)\Big\| \\ &\le C\|B\|\,\mathbb{E}\Big\|\sum_{j=1}^n \epsilon_j x_j\Big\|\Big\|\sum_{j=1}^n \epsilon_k y_k\Big\| \\ &\le C\|B\|\Big(\mathbb{E}\Big\|\sum_{j=1}^n \epsilon_j x_j\Big\|^2\Big)^{\frac12}\Big(\mathbb{E}\Big\|\sum_{j=1}^n \epsilon_j y_j\Big\|^2\Big)^{\frac12} \\ &\le C\|B\|\Big(\sum_{j=1}^n \|x_j\|^2\Big)^{\frac12}\Big(\sum_{j=1}^n \|y_j\|^2\Big)^{\frac12}.\end{aligned}$$

Now let $a_j = \|x_j\|^{\frac12}\|y_j\|^{-\frac12}$. Then $\sum_{j=1}^n B(x_j, y_j) = \sum_{j=1}^n B(a_j^{-1}x_j, a_jy_j)$ (with appropriate modifications if $x_j = 0$ or $y_j = 0$). Thus we obtain an inequality of type (6.1). □

It is natural to consider Theorem 6.1 as a bilinear analogue of the Maurey factorization theorem which asserts that any linear map $T : X \to Y$ where X has type two and Y has cotype two factors through a Hilbert space [27]. So by analogy with Pisier's abstract Grothendieck theorem [30] (see [21] for the nonlocally convex version) we may ask whether Theorem 6.1 remains true if the type two assumption is replaced by the assumption that X and Y have the bounded approximation property and that X^* and Y^* are both of cotype two. The most important special case is when $X = Y = c_0$ and $Z = L_p$ when $0 < p < 1$, and this would be equivalent to a refinement of Grothendieck's inequality:

Problem. Suppose $0 < p < 1$. Is there a constant $K = K(p)$ so that if $(f_{jk})_{j,k=1}^n$ is a matrix with values in $L_p(0, 1)$ such that

$$\sup_{\epsilon_j=\pm1}\sup_{\epsilon'_k=\pm1}\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\epsilon'_k f_{jk}\Big\| \le 1$$

then

$$\Big\|\sum_{j=1}^n\sum_{k=1}^n (u_j, v_k) f_{jk}\Big\| \le K$$

whenever $u_1, \ldots, u_n, v_, \ldots, v_n \in \ell_2$ with $\|u_j\|, \|v_k\| \le 1$ for $1 \le j \le n,\ 1 \le k \le n$?

We will prove some partial results in this direction.

Theorem 6.2. *Suppose X and Y are quasi-Banach spaces with the bounded approximation property and such that X^* and Y^* have cotype two; such Z is a quasi-Banach space with cotype two and the decoupling property. Then there is a constant C depending only on X, Y and Z so that if $B : X \times Y \to Z$ is any bounded bilinear mapping then*

$$\mathbb{E}\Big\|\sum_{j=1}^n \epsilon_j B(x_j, y_j)\Big\| \le C\|B\|\sum_{j=1}^n \|x_j\|\|y_j\| \qquad x_1, \ldots, x_n \in X,\ y_1, \ldots, y_n \in Y. \tag{6.2}$$

Proof. Let us fix x and consider the map $T : Y \to Z$ given by $T(y) = B(x,y)$. Then by Theorem 6 of [21] (Proposition 2.3 above) we have

$$\mathbb{E}\Big\|\sum_{k=1}^n \epsilon_k T y_k\Big\| \le C\|T\|\Big(\sum_{k=1}^n \|y_k\|^2\Big)^{\frac12} \qquad y_1,\dots,y_n \in Y$$

where $C = C(Y,Z)$. Thus

$$\mathbb{E}\Big\|\sum_{k=1}^n \epsilon_k B(x,y_k)\Big\| \le C\|B\|\|x\|\Big(\sum_{k=1}^n \|y_k\|^2\Big)^{\frac12} \qquad y_1,\dots,y_n \in Y.$$

Now fix $y_1,\dots,y_n \in Y$ and consider the map $S : X \to L_1(Z)$ given by $Sx = \sum_{k=1}^n \epsilon_k B(x,y_k)$. The same reasoning yields (since $L_1(Z)$ also have cotype two) for $x_1,\dots,x_n \in X$,

$$\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon'_j\epsilon_k B(x_j,y_k)\Big\| \le C\|B\|\Big(\sum_{j=1}^n \|x_j\|^2\Big)^{\frac12}\Big(\sum_{k=1}^n \|y_k\|^2\Big)^{\frac12}$$

where $C = C(X,Y,Z)$. It remains to observe that $L_1(Z)$ also has the decoupling property and so if η_j is another sequence of independent Rademachers

$$\begin{aligned}\mathbb{E}\Big\|\sum_{j=1}^n \epsilon'_j\epsilon_j B(x_j,y_j)\Big\| &\le C\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \eta_j\epsilon'_j\eta_k\epsilon_k B(x_j,y_k)\Big\| \\ &= C\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \epsilon'_j\epsilon_k B(x_j,y_k)\Big\|.\end{aligned}$$

This establishes (6.2). ☐

We can now prove a Maurey–Nikishin style factorization for bilinear forms with values in L_p. We recall that if (Ω,μ) is any probability space then $L_{1,\infty}(\Omega,\mu)$ (weak L_1) is the space of all measurable functions f such that

$$\|f\|_{L_{1,\infty}(\mu)} = \sup_{t>0} t\mu(|f|>t) < \infty.$$

Theorem 6.3. *Suppose $0 < p < 1$ and that X and Y are quasi-Banach spaces with the bounded approximation property and such that X^* and Y^* have cotype two. Then there is a constant $C = C(X,Y,p)$ so that if $B : X\times Y \to L_p(\Omega,\mu)$ is a bounded bilinear map then there is function $w \in L_1(\mu)$ with $w \ge 0$ a.e., $\int w d\mu = 1$, $\{w = 0\} \subset \{|B(x,y)| = 0\}$ a.e. for all $x \in X$ and $y \in Y$ and*

$$\|w^{-\frac1p}B(x,y)\|_{L_{1,\infty}(w\,d\mu)} \le C\|B\|\|x\|\|y\|.$$

Remark. Theorem 6.3 states that B factors through weak L_1 via a change of density. If we take the example $X = Y = C[0,1]$ it is clear this result cannot be improved by replacing $L_{1,\infty}$ with L_1. Indeed there is a quotient map Q of $C[0,1]$ onto ℓ_2 and an isometric embedding $J : \ell_1 \to L_{1,\infty}(0,1)$ using 1-stable random variables. Let $B(f,g) = J(Qf.Qg)$ and it is easy to see that B cannot map into L_1 via any change of density. Note also that if B factored through a Banach space then Theorem 6.3 would be an immediate deduction from the classical results of Nikishin [28].

Proof of Theorem 6.3. Suppose $x_1, \dots, x_n \in X,\ y_1, \dots, y_n \in Y$ with $\max \|x_j\| \le 1,\ \max \|y_k\| \le 1$. Suppose $a_j \ge 0$ and $\sum_{j=1}^n a_j = 1$. Then if $c_j = \sqrt{a_j}$

$$\begin{aligned}\|\max |a_j B(x_j, y_j)|\,\|_p &\le \left\| \left(\sum_{j=1}^n |B(c_j x_j, c_j y_j)|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\le C\mathbb{E} \left\| \sum_{j=1}^n \epsilon_j B(c_j x_j, c_j y_j) \right\|_p \\ &\le C\|B\|\end{aligned}$$

by Theorem 6.2. Now the result follows immediately from Theorem 1.1 of [33]. □

From this result we can deduce an estimate slightly weaker than (6.1) for the special case when the range is natural.

Corollary 6.4. *Suppose that X and Y are Banach spaces with the bounded approximation property and such that X^* and Y^* have cotype two and Z is a natural quasi-Banach space. Then there is constant $C = C(X, Y, Z)$ so that if $B : X \times Y \to Z$ is a bounded bilinear map and $x_1, \dots, x_n \in X,\ y_1, \dots, y_n \in Y$ are such that $\sum_{j=1}^n \|x_j\|\|y_j\| = 1$ then*

$$\left\| \sum_{j=1}^n B(x_j, y_j) \right\| \le C\|B\| \left(1 + \sum_{j=1}^n \|x_j\|\|y_j\| \log \frac{1}{\|x_j\|\|y_j\|} \right). \tag{6.3}$$

Proof. Assume Z is r-normable and that $0 < p < r$. Let $z = \sum_{j=1}^n B(x_j, y_j)$. Then since Z is natural we can find an operator $T : Z \to L_p(\Omega, \mu)$, where (Ω, μ) is some probability space, with $\|T\| = 1$ and $\|z\| \le C\|Tz\|_p$, where $C = C(Z)$. Now by Theorem 6.3 we can find w satisfying the conclusions of the theorem for the bilinear form $T \circ B$. Then

$$\begin{aligned}\|z\| &\le C\|Tz\|_p \\ &\le C\|w^{-\frac{1}{p}} Tz\|_{L_p(w\,d\mu)} \\ &\le C \left\| w^{-\frac{1}{p}} \sum_{j=1}^n TB(x_j, y_j) \right\|_{L_{1,\infty}(w\,d\mu)}.\end{aligned}$$

Now we use the fact that $L_{1,\infty}$ is *logconvex* (this is due to Stein and Weiss [37]; see also [11]). Thus if $\theta_j = \|TB(x_j, y_j)\|_{L_{1,\infty}(w\,d\mu)}$ and $\theta = \sum_{j=1}^n \theta_j$ we have

$$\|z\| \le C \sum_{j=1}^n \theta_j \log \frac{\theta}{\theta_j}.$$

Now $\theta_j \le C\|B\|\|x_j\|\|y_j\|$ and the result follows. □

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This paper is available via http://nyjm.albany.edu:8000/j/2005/11-27.html.