Unusual Traces on Operator Ideals

By N. J. Kalton of Missouri — Columbia

(Received October 25, 1986)

1. Introduction

In this note we answer a question of A. Pietsch by constructing an example of a quasi-normed operator ideal on a Hilbert space which admits more than one continuous trace. We also characterize the class of uniquely traceable operators as described below.

Let $H$ be a separable Hilbert space and let $\mathcal{D}$ be an ideal in $\mathcal{K}(H)$ (the compact operators on $H$). Then a trace ([2]) on $\mathcal{D}$ is a linear functional $\tau: \mathcal{D} \to \mathbb{C}$ so that

(T1) $\tau(P) = 1$ if $P$ is a projection of rank one.
(T2) $\tau(AB) = \tau(BA)$ if $A \in \mathcal{D}$ and $B \in \mathcal{L}(H)$.

In addition $\tau$ is called separately continuous or $(\mathcal{J}, \mathcal{L})$-continuous ([2]) if

(T3) For every $A \in \mathcal{D}$, the linear functional $B \mapsto \tau(AB)$ is bounded on $\mathcal{L}(H)$.

Let $C_1$ be the trace-class and let $\text{tr}: C_1 \to \mathbb{C}$ denote the standard trace. If $A \in \mathcal{D}$ and rank $(A) < \infty$ then

$$\tau(A) = \text{tr}(A).$$

If $\tau$ verifies (T3) then for every $A \in \mathcal{D}$ we have

$$\sup_{\text{rank} B < \infty} |\text{tr}(AB)| < \infty$$

and so $A \in C_1$. Hence if $\mathcal{D}$ supports a separately continuous trace then $\mathcal{D} \subset C_1$.

We shall say that a positive $T \in C_1$ is uniquely traceable if the ideal $\mathcal{D}$ it generates supports exactly one separately continuous trace, namely the standard trace $\text{tr}$.

An ideal $\mathcal{D}$ is said to be quasi-normed if there is a quasi-norm $|.|$ on $\mathcal{D}$ verifying

(Q1) $(\mathcal{D}, |.|)$ is complete
(Q2) $|A| \geq \beta \|A\| \quad A \in \mathcal{D}$

for some $\beta > 0$, and

(Q3) $|SAT| \leq \|S\| \|A\| \|T\| \quad S, T \in \mathcal{L}(H) \quad A \in \mathcal{D}$
Proposition 1. If $D$ is a quasi-normed ideal and $\tau$ is a separately continuous trace on $D$ then $\tau$ is continuous on $(D, \|\cdot\|)$.

Remark. The converse is clear ([2] p. 68).

Proof. Suppose first that $A_n \in D$ is a sequence of normal operators verifying $|A_n| \to 0$. We show $\tau(A_n) \to 0$. Indeed if not we can find normal $B_n$ with $|B_n| \leq 2^{-n}$ and $\tau(B_n) \geq n$. Now there exists isometries $U_n : H \to H$ (not necessarily surjective) so that the sequence $C_n = U_n B_n U_n^*$ commutes. Let $P_n = |C_n| = (C_n^* C_n)^{1/2}$. Then $\sum P_n$ converges in to an operator $P$ and each $C_n$ can be written $C = PT_n$ where $\|T_n\| \leq 1$. Thus

$$\sup_n \tau(C_n) < \infty.$$  

However $\tau(C_n) = \tau(U_n B_n U_n^*) = \tau(B_n U_n^* U_n) = n$.

For the general case, if $A_n$ is any sequence in $D$ with $|A_n| \to 0$, then $|A_n + A_n^*| \to 0$ and $|A_n - A_n^*| \to 0$ and hence $\tau(A_n) \to 0$.

2. Some preparatory lemmas

For $A \in \mathcal{K}(H)$ denote by $s_n(A)$ the sequence of singular values of $A$ so that $\|A\| = s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A) \to 0$. Define

$$\varphi_A(t) = \sup \{n : s_n(A) > t\} \quad t > 0.$$  

Then $\varphi_A$ is a monotone decreasing function continuous on the right.

We note first some easy inequalities. First

(1) $s_{m+n-1}(A + B) \leq s_m(A) + s_n(B)$

for $A, B \in \mathcal{K}(H)$, $m, n \in N$ (cf. [1]). It follows that

(2) $\varphi_{A+B}(t) \leq \varphi_A \left( \frac{t}{2} \right) + \varphi_B \left( \frac{t}{2} \right)$.

Further note that

(3) $\sum_{n=1}^{\infty} s_n(A) = \int_0^\infty \varphi_A(t) \, dt$.

Now suppose $A \geq 0$ and $A \in \mathcal{K}(H)$. Then the sequence $(s_n(A))$ consists of the eigenvalues $(\lambda_n)_{n=1}^\infty$ of $A$ arranged in decreasing order. Let us define

(4) $\text{tr}_m(A) = \sum_{j=1}^{m} s_n(A)$.

Then

(5) $\text{tr}_m(A) = \max \{ \text{tr} (PAPA) \}$

where $P$ ranges over all self-adjoint projections of rank $m$.

Lemma 2. For $A, B \geq 0$ and $m, n \in N$ we have

(i) $\text{tr}_m(A + B) \leq \text{tr}_m(A) + \text{tr}_m(B)$

(ii) $\text{tr}_{m+n}(A + B) \geq \text{tr}_m(A) + \text{tr}_n(B)$. 

Proof. (i) is immediate. For (ii) note that there exist self-adjoint projections $P, Q$ so that $\text{rank } P = m$, $\text{rank } Q = n$ and
\[
\text{tr } (PAP) = \text{tr}_m (A)
\]
\[
\text{tr } (QBQ) = \text{tr}_n (B).
\]
Let $R$ be a self-adjoint projection onto a subspace of dimension $m + n$ containing $P(H)$ and $Q(H)$. Then
\[
\text{tr } (RAR) \geq \text{tr}_m (A)
\]
\[
\text{tr } (RBR) \geq \text{tr}_n (B)
\]
but
\[
\text{tr } (R(A + B)R) \leq \text{tr}_{m+n} (A + B).
\]

Lemma 3. If $A, B \in \mathcal{X}(H)$ and $A, B \geq 0$, then

(i) $\varphi_{A+B}(t) \geq \max \{\varphi_A(t), \varphi_B(t)\}$ for $t > 0$

(ii) $\varphi_{A+B}(4t) \leq 3 \max \{\varphi_A(t), \varphi_B(t)\}$ for $t > 0$.

Proof. (i) is immediate, for if $\varphi_A(t) = m$ then $\langle (A - it) x, x \rangle > 0$ on a subspace of dimension $m$. Hence $\langle (A + B - it) x, x \rangle > 0$ on the same subspace and so $\varphi_{A+B}(t) \geq m$.

(ii) Let $\varphi_A(t) = m$, $\varphi_B(t) = n$ and suppose $\varphi_{A+B}(4t) \geq 2 \max (m, n)$. Let $\varphi_{A+B}(4t) = k$; then
\[
\text{tr}_k (A + B) > \text{tr}_{m+n} (A + B) + 4(k - m - n) t
\]
\[
\geq \text{tr}_m (A) + \text{tr}_n (B) + 4(k - m - n) t.
\]

However
\[
\text{tr}_k (A) \leq \text{tr}_m (A) + (k - m) t
\]
\[
\text{tr}_k (B) \leq \text{tr}_n (B) + (k - n) t.
\]

Hence
\[
\text{tr}_k (A + B) \leq \text{tr}_m (A) + \text{tr}_n (B) + (2k - m - n) t.
\]

Thus
\[
2k - m - n > 4(k - m - n)
\]
and so
\[
k < \frac{3}{2} (m + n) \leq 3 \max (m, n).
\]

Now for $a > 0$, $A \in C_1$ with $A \geq 0$ we shall define
\[
F_A(a) = \int_0^a \varphi_A(t) dt.
\]

Lemma 4. If $A, B \in C_1$ with $A, B \geq 0$ then
\[
|F_{A+B}(a) - F_A(a) - F_B(a)| \leq 2a \varphi_{A+B}(a).
\]

Proof. First note that
\[
F_A(a) = \sum_{j=1}^{\infty} \min \{s_j(A), a\}.
\]
Let \( \varphi_A(a) = m, \varphi_B(a) = n, \varphi_{A+B}(a) = p \) and \( \varphi_{A+B}(4a) = q \). Then

\[
F_A(a) = \text{tr} (A) - \sum_{j=1}^{m} (a_j(A) - a)
= \text{tr} (A) + ma - \text{tr}_m (A).
\]

Now since \( p \geq \max (m, n) \) (Lemma 3)

\[
\text{tr}_{m+n} (A + B) \leq a \min (m, n) + \text{tr}_p (A + B)
\]

and hence

\[
\text{tr}_m (A) + \text{tr}_n (B) \leq \text{tr}_p (A + B) + a \min (m, n).
\]

Thus

\[
F_{A+B}(a) \leq \text{tr} (A + B) + a(p + \min (m, n)) - \text{tr}_m (A) - \text{tr}_n (B)
\leq F_A(a) + F_B(a) + a(p - \max (m, n))
\leq F_A(a) + F_B(a) + a\varphi_{A+B}(a).
\]

Conversely \( q \leq 3 \max (m, n) \) and hence

\[
\text{tr}_q (A + B) \leq \text{tr}_{3N} (A + B)
\]

where \( N = \max (m, n) \). Thus

\[
\text{tr}_q (A + B) \leq \text{tr}_{3N} (A) + \text{tr}_{3N} (B)
\leq \text{tr}_m (A) + \text{tr}_n (B) + a(6N - m - n).
\]

Hence

\[
\text{tr}_p (A + B) \leq \text{tr}_q (A + B) + 4a(p - q)
\leq \text{tr}_m (A) + \text{tr}_n (B) + a(4p + 6N - 4q - m - n).
\]

We conclude

\[
F_{A+B}(a) \geq F_A(a) + F_B(a) + a(p - m - n) - a(4p + 6N - 4q - m - n)
= F_A(a) + F_B(a) - a(3p + 6N - 4q)
\geq F_A(a) + F_B(a) - 3pa.
\]

Lemma 5. Suppose \( \psi: (0, \infty) \to (0, \infty) \) is a right-continuous integer-valued monotone decreasing function. Suppose

(a) \( \int_{0}^{\infty} \psi(u) \, du < \infty \).

(b) There exist constants \( C < \infty \) and \( 0 < \gamma < 1 \) so that

\[
\int_{0}^{a} \psi(u) \, du \leq C \gamma^a \psi(a) \quad 0 < a \leq 1.
\]

Then there exists constants \( K < \infty \) and \( \alpha > 0 \) so that

\[
\psi(st) \leq K s^{-\alpha} \psi(t)
\]

for \( 0 < s, t \leq 1 \).
Proof. First we observe if \( 0 < a < \gamma \) and \( 0 < b < a \) then

\[
\psi(b) \leq \int_0^{a/\gamma} \psi(u) \, du \leq C \psi(a).
\]

In particular if \( b = \gamma a \)

\[
\gamma \psi(\gamma a) \leq C \psi(a)
\]

and hence

\[
\int_0^a \psi(u) \, du \leq C^2 \psi(a) \quad 0 < a \leq \gamma^{-1}.
\]

Next we observe that we may suppose \( \gamma = 2^{-p} \) where \( p \in \mathbb{N} \). Let

\[
d_k = 2^{-k} \psi(2^{-k}) \quad k \in \mathbb{N}.
\]

For \( n \in \mathbb{N} \) we define \( v_n \in \omega \), the space of all real sequences, by

\[
v_n(k) = \frac{d_{k+n}}{d_n} \quad k \in \mathbb{N}.
\]

If \( n \geq p \) then

\[
0 \leq v_n(k) \leq C
\]

so that \( \{v_n: n \geq p\} \) is bounded in \( \omega \). Let \( \Gamma \) be the closed convex null of \( \{v_n: n \geq p\} \).

Then \( \Gamma \) is compact.

By hypothesis if \( k \geq p \)

\[
\int_0^{2^{-k}} \psi(u) \, du \leq C \psi(k)
\]

and hence

\[
\sum_{j=k}^\infty d_j \leq 2C^2 d_k.
\]

Hence if \( m > k \geq p \)

\[
\sum_{j=k}^{m+n} d_j \leq 2C^2 d_{k+n}
\]

and so

\[
\sum_{j=k}^m v_n(j) \leq 2C^2 v_n(k).
\]

Now if \( w \in \Gamma \)

\[
\sum_{j=k}^m w(j) \leq 2C^2 w(k).
\]

Since \( w \) is bounded this implies that \( w \in \mathcal{C} \). Let \( P \) be the closed positive cone of \( \omega \); then \( \Gamma - P \) is closed and the constant sequence \( \epsilon = (1, 1, \ldots) \) is not in \( \Gamma - P \). Thus
there exist \( \beta_n \geq 0 \) finitely non-zero such that
\[
\sum \beta_k = 1
\]
and
\[
\sum \beta_k v_n(k) \leq 1 - \theta
\]
for all \( n \geq p \). If \( \beta_k = 0 \) for \( k > N \) we conclude that
\[
\min_{k \leq N} v_n(k) \leq 1 - \theta
\]
for \( n \geq p \). Thus for every \( n \geq p \) there exists \( k \leq N \) with
\[
d_{n+k} \leq (1 - \theta) d_n.
\]
Fix \( \alpha > 0 \) by \( 2^{-N\alpha} = 1 - \theta \). We deduce that if \( n \geq p, \sigma \in N \) there exists \( k \) with \( \sigma N \leq k \leq (\sigma + 1) N \) and
\[
d_{n+k} \leq (1 - \theta)^{\sigma+1} d_n = 2^{-(\sigma+1)N} d_n.
\]
Hence if \( (\sigma - 1) N < l \leq \sigma N \), since \( \psi \) is monotone decreasing,
\[
d_{n+l} \leq 2^{-l} d_{n+k} \leq 2^{-(\sigma+1)N} d_n = 2^{-(\sigma-1)N} 2^{-l} d_n \leq 2^{N} 2^{-l} d_n.
\]
We easily deduce that for some constant \( K \) we have
\[
\text{st}_{\psi}(st) \leq K \text{st}_{\psi}(t)
\]
for every \( 0 < s, t < 1 \) and the result follows.

3. The main results

Let \( T \) be a positive compact operator and let \( \psi(t) = \varphi_T(t) \). Then the two-sided ideal \( \mathcal{D}_T \) generated by \( T \) is determined solely by \( \psi \). In fact \( A \in \mathcal{D}_T \) if and only if for some \( \gamma, 0 < \gamma < 1 \) and some \( C < \infty \) we have
\[
(6) \quad \varphi_{\lambda}(t) \leq C \psi(\gamma t) \quad t > 0.
\]
We denote the set of such \( A \) by \( \mathcal{D}(\psi) \). To see that \( \mathcal{D}(\psi) \) is an ideal one must use equation (2).

Suppose in addition we have that for some \( \lambda, 0 < \lambda < 1, \)
\[
\varphi_{\lambda}(t) \geq 2 \varphi(t) \quad t > 0.
\]

Then \( \mathcal{D}(\psi) \) is a quasi-normed ideal if we define \( |A| \) to be the infimum of all \( c > 0 \) so that
\[
\varphi_{\lambda}(ct) \leq \psi(t) \quad t > 0.
\]
Note here that
\[
\varphi_{\lambda+\beta}(t) \leq \varphi_{\lambda+\beta} \left( \frac{t}{2} \right) + \varphi_{\beta} \left( \frac{t}{2} \right) \leq \max \left( \varphi_{\lambda} \left( \frac{1}{2} t \right), \varphi_{\beta} \left( \frac{1}{2} t \right) \right)
\]
so that
\[
|A + B| \leq \frac{2}{\lambda} \max (|A|, |B|).
\]
We also remark that if \( \psi \) is any right-continuous monotone decreasing integer valued function with \( \lim_{t \to \infty} \psi(t) = 0 \) then there is a positive compact operator \( T \) with \( \varphi_T = \psi \).

Clearly \( \mathcal{D}(\psi) \subset C_1 \) if and only if \( T \in C_1 \), i.e.
\[
\int_0^\infty \psi(u) \, du < \infty.
\]

**Theorem 6.** Suppose \( \psi \) is a nonnegative monotone-decreasing, integer-valued left-continuous function on \((0, \infty)\) with
\[
\int_0^\infty \psi(u) \, du < \infty.
\]

Then the following are equivalent:

(i) If \( \tau \) is a separately continuous trace on \( \mathcal{D}(\psi) \) then
\[
\tau(A) = \text{tr}(A) \quad A \in \mathcal{D}(\psi).
\]

(ii) There exists \( K < \infty \), \( \alpha > 0 \) so that if \( 0 \leq s, t \leq 1 \)
\[
\psi(st) \leq K \psi^{-1}(t).
\]

**Proof.** (i) \( \Rightarrow \) (ii): Let us suppose (ii) fails. Then, according to Lemma 5 we can find a sequence \( a_n \) with \( 0 < a_n \leq 1 \) and
\[
\int_0^{a_n} \psi(u) \, du > na_n \psi \left( \frac{a_n}{n} \right).
\]

For \( A \geq 0 \) in \( \mathcal{D}(\psi) \) we set
\[
A_n(A) = F_A(a_n) \int_0^{a_n} \psi(u) \, du.
\]

Suppose \( \varphi_A(t) \leq C\psi(\gamma t) \). Then
\[
F_A(a_n) \leq C \int_0^{a_n} \psi(\gamma u) \, du = C \gamma^{-1} \int_0^{a_n} \psi(u) \, du
\]

so that
\[
0 \leq A_n(A) \leq C \gamma^{-1}.
\]

Define
\[
A(A) = \lim_{n \in \mathcal{U}} A_n(A)
\]

where \( \mathcal{U} \) is some non-principal ultrafilter on \( N \).

We observe that if \( 0 \leq \lambda \leq 1 \),
\[
F_{\lambda}A(a_n) = \int_0^{a_n} \varphi_{\lambda}A(u) \, du = \int_0^{a_n} \varphi_{\lambda}(\lambda^{-1}u) \, du = \lambda \int_0^{\lambda^{-1}a_n} \varphi_A(u) \, du
\]
so that
\[
|F_{\lambda A}(a_n) - \lambda F_A(a_n)| \leq \lambda \left| \int_{a_n}^{x_n} \varphi_A(u) \, du \right| \\
\leq \lambda (\lambda^{-1} - 1) \varphi_A(a_n) \leq C(1 - \lambda) \psi(ya_n) a_n.
\]
Thus if \( n > \gamma^{-1} \)
\[
|F_{\lambda A}(a_n) - \lambda F_A(a_n)| \leq \frac{C}{n} (1 - \lambda) \int_{0}^{a_n} \psi(u) \, du
\]
and
\[
|A_n(\lambda A) - A_n(A)| \leq \frac{C}{n} (1 - \lambda).
\]
Thus
\[
A(\lambda A) = \lambda A(A) \quad \lambda \geq 0, \ A \geq 0.
\]
Now by Lemma 4 if \( A, B \geq 0 \)
\[
|F_{\lambda A+B}(a_n) - F_{\lambda A}(a_n) - F_{\lambda B}(a_n)| \leq 9a_n \varphi_{\lambda A+B}(a_n)
\]
If \( \varphi_{\lambda A+B}(t) \leq C \psi(\gamma_1 t) \) we have
\[
|F_{\lambda A+B}(a_n) - F_{\lambda A}(a_n) - F_{\lambda B}(a_n)| \leq 9C_1 a_n \psi(\gamma, a_n) \leq \frac{9C_1}{n} \int_{0}^{a_n} \psi(u) \, du
\]
if \( n > \frac{1}{\gamma_1} \). It follows that
\[
A(A + B) = A(A) + A(B).
\]
Note further that
\[
0 \leq A(A) \leq \frac{C}{\gamma}
\]
where \( \varphi_A(t) \leq C \psi(\gamma t) \).
We also note that if \( F \) has finite rank then \( \varphi_A(t) \) is bounded and then
\[
F_{\lambda A}(a_n) \leq K a_n
\]
for some \( n \). Hence \( A(A) = 0 \) if \( \text{rank } A < \infty \). Now extend \( A \) to a linear functional still denoted by \( A \), defined on \( D(\psi) \). We note that
\[
A(U^{-1}AU) = A(A)
\]
if \( U \) is unitary.
If \( \text{rank } A < \infty \) then \( A(A) = 0 \). Furthermore if \( H \) is hermitian then we can write
\( H = P_1 - P_2 \) where \( P_1, P_2 \) are positive and
\[
\varphi_{P_1 + P_2}(t) = \varphi_H(t).
\]
Hence if \( \varphi_H(t) \leq C \psi(\gamma t) \) then
\[
|A(H)| \leq C \gamma^{-1}.
\]
In general if \( A \in \mathcal{D}(\psi) \) then

\[
\varphi_{\text{d} \mathcal{A} \mathcal{A}}(t) \leq 2 \varphi_{\mathcal{A}} \left( \frac{t}{2} \right)
\]

\[
\varphi_{\mathcal{A} \mathcal{A} \mathcal{A} \mathcal{A}}(t) \leq 2 \varphi_{\mathcal{A}} \left( \frac{t}{2} \right)
\]

and hence if \( \varphi_{\mathcal{A}}(t) \leq C \psi(\gamma t) \) then

\[
\varphi_{\mathcal{A} \mathcal{A} \mathcal{A} \mathcal{A}}(t) \leq 2C \psi \left( \frac{1}{2} \gamma t \right)
\]

\[
\varphi_{\mathcal{A} \mathcal{A} \mathcal{A} \mathcal{A}}(t) \leq 2C \psi \left( \frac{1}{2} \gamma t \right).
\]

Hence

\[
|\Lambda(\mathcal{A})| \leq 8C \gamma^{-1}.
\]

It follows easily that \( \Lambda \) is separately continuous, i.e. for each \( B \in \mathcal{D}(\psi) \) the linear functional \( \mathcal{A} \rightarrow \Lambda(\mathcal{A} \mathcal{B}) \) is bounded on \( \mathcal{I}(\mathcal{H}) \).

Now define \( \tau(\mathcal{A}) = \text{tr} (\mathcal{A}) + \Lambda(\mathcal{A}) \) for \( \mathcal{A} \in \mathcal{D}(\psi) \). Then \( \tau \) is separately continuous and \( \tau(P) = 1 \) for every rank one projection. Furthermore

\[
\tau(U^{-1} \mathcal{A} \mathcal{U}) = \tau(\mathcal{A})
\]

for every unitary \( \mathcal{U} \) and hence

\[
\tau(\mathcal{A} \mathcal{B}) = \tau(\mathcal{B} \mathcal{A})
\]

for every \( \mathcal{B} \in \mathcal{I}(\mathcal{H}) \), \( \mathcal{A} \in \mathcal{D}(\psi) \) (cf. [2] p. 63).

To see that \( \tau \) is a trace distinct from \( \text{tr} \) we need only produce a positive operator \( T \) with

\[
\varphi_{\tau}(t) = \psi(t).
\]

Then \( \Lambda(T) = 1 \) and hence \( \tau(T) = \text{tr} (T) + 1 \). This shows that (i) \( \Rightarrow \) (ii).

(ii) \( \Rightarrow \) (i): Let us assume \( \tau \) is a separately continuous trace on \( \mathcal{D}(\psi) \). Let us write \( \Lambda(\mathcal{A}) = \tau(\mathcal{A}) - \text{tr} (\mathcal{A}) \), so that \( \Lambda \) is separately continuous and \( \Lambda(F) = 0 \) if rank \( F < \infty \). Suppose \( C < \infty \) and \( 0 < \gamma < 1 \); then there exists \( M < \infty \) so that if

\[
\varphi_{\lambda}(t) \leq C \psi(\gamma t)
\]

then \( |\Lambda(\mathcal{A})| \leq M \).

Fix \( \mathcal{A} \) satisfying (7). Then for \( n \in \mathbb{N} \) using standard representation theorems we can write

\[
\mathcal{A} = F + \mathcal{A}_{1} + \cdots + \mathcal{A}_{n}
\]

where rank \( F < \infty \) and

\[
\varphi_{\mathcal{A}_{i}}(t) \leq \frac{1}{n} \varphi_{\mathcal{A}}(t) \quad 0 \leq t \leq \lambda
\]

and

\[
\varphi_{\mathcal{A}}(t) = 0 \quad t > \lambda.
\]
Consider for $0 < \lambda < 1$,
\[
\varphi_{1,\lambda}(t) = \varphi_{\lambda}(\lambda t)
\]
\[
\leq \frac{1}{n} \varphi_{\lambda}(\lambda t) \quad 0 < t \leq 1
\]
\[
\leq \frac{C}{n} \psi(\lambda t) \quad 0 < t \leq 1
\]
\[
\leq \frac{C}{n} K^{1-s} \psi(\lambda t) \quad 0 < t \leq 1
\]
\[
\leq C \psi(\lambda t) \quad 0 < t \leq 1
\]
provided $K^{1-s} = n$. Similarly if $t > 1$
\[
\varphi_{1,\lambda}(t) = 0 \leq C \psi(\lambda t)
\]
and hence
\[
|A(\lambda^{-1}A)| \leq M
\]
\[
|A(A)| \leq M\lambda = M \left( \frac{K}{n} \right)^{1/(1-s)}
\]
Hence
\[
A(A) \leq Mn \left( \frac{K}{n} \right)^{1/(1-s)}
\]
and hence $A(A) = 0$. It follows that $\text{tr}(A) = \tau(A)$ for $A \in D(\psi)$.

Corollary 7. Let $T$ be a positive trace-class operator. In order that $T$ is uniquely traceable it is necessary and sufficient that there exists $p > 1$ and $C < \infty$ so that the singular values $(\lambda_n)$ of $T$ satisfy
\[
\lambda_m \leq C \left( \frac{m}{n} \right)^{-p} \lambda_n \quad m > n.
\]

Proof. It suffices to show that $\varphi_T$ satisfies (ii) of Theorem 6 if and only if it satisfies the criterion of the Corollary. In fact if
\[
\varphi_T(st) \leq K n^{s-1} \varphi_T(t) \quad 0 < s, t < 1
\]
$\lambda_n < 1$ and $m > n$ then
\[
\varphi_T(\lambda_n) < n
\]
\[
\varphi_T(\lambda_n -) \leq n
\]
\[
\varphi_T(\lambda_m -) \geq m
\]
and hence
\[
m \leq K \left( \frac{\lambda_m}{\lambda_n} \right)^{-1} n
\]
and the Corollary follows with $p = \frac{1}{1 - \alpha}$.
If the condition of the Corollary fails we may for every $C > 0 \; p > 1$ find $m > n$ with $\lambda_n < 1$ so that
\[
\lambda_m > C \left( \frac{m}{n} \right)^{-p} \lambda_n.
\]
Then $\varphi(\lambda_n) < n$ and
\[
\varphi \left( C \left( \frac{m}{n} \right)^{-p} \lambda_n \right) > m.
\]
Thus
\[
 Ko^{-1} \left( \frac{m}{n} \right)^{p(1-\alpha)} n > m
\]
so that
\[
\left( \frac{m}{n} \right)^{1-p(1-\alpha)} < Ko^{-1}
\]
Clearly this is a contradiction if $p = (1 - \alpha)^{-1}$ and $Ko^{-1} < 1$.

Example. We now construct an explicit example of a function $\varphi$ so that $\mathcal{E}(\varphi)$ is a quasi-normed ideal supporting at least two distinct continuous traces.

We define only $\varphi(2^{-2m})$ for $m \in \mathbb{N} \cup \{0\}$. We do this by induction. Let $\varphi(2^{-2m}) = \varphi_m$.

Set $\varphi_0 = 0$, $\varphi_1 = 1$. For $n = 1, 2, \ldots$, let
\[
\varphi_m = 4\varphi_{m-1} \quad \text{if} \quad n! < m \leq n! + n
\]
\[
\varphi_m = 2\varphi_{m-1} \quad \text{if} \quad n! + n < m \leq (n + 1)!
\]

Let $\varphi$ be a monotone-decreasing right-continuous integer-valued function with $\varphi(2^{-2m}) = \varphi_m$. Clearly $\mathcal{E}(\varphi)$ is a quasi-normed ideal since
\[
\varphi \left( \frac{1}{16} t \right) \geq 2\varphi(t) \quad t > 0.
\]
Furthermore
\[
\varphi_{m+1} = 2^m \varphi_m
\]
so that $\varphi$ fails Theorem 6(ii).

To do this note we need
\[
\sum_{m=1}^{\infty} \frac{1}{2^{2m}} \varphi_m < \infty
\]
Now
\[
\sum_{m=1}^{n+1} \frac{1}{2^{2m}} \varphi_m = n\varphi_n
\]

9 Math. Nachr., Bd. 134
while
\begin{equation}
\sum_{n=0}^{\infty} \frac{1}{2^{2n} n!} \psi_n \leq \psi_n!
\end{equation}

Now
\begin{equation}
\psi_{(n+1)!} \leq 2^{n+1} \psi_{n+1} - 2^{n+1} \psi_{n+1} = 2^{n+1} \psi_{n+1} \leq \frac{1}{2} \psi_n!
\end{equation}

if \( n \geq 2 \). Hence
\begin{equation}
\sum (n + 1) \psi_n < \infty
\end{equation}

and so
\begin{equation}
\int_0^\infty \psi(t) \, dt < \infty.
\end{equation}

By Theorem 6, \( D(\psi) \) gives the promised example.


References


Dept. of Mathematics
University of Missouri — Columbia
Columbia, MO 65211
U.S.A.