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Unusual Traces on Operator Ideals

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1. Introduction

In this note we answer a question of A. PIETSCH by constructing an example of a quasi-normed operator ideal on a HILBERT space which admits more than one continuous trace. We also characterize the class of uniquely traceable operators as described below.

Let H be a separable HILBERT space and let $\mathcal{D}$ be an ideal in $\mathcal{K}(H)$ (the compact operators on H). Then a trace ([2]) on $\mathcal{D}$ is a linear functional $\tau: J \to C$ so that

(T 1) $\tau(P) = 1$ if P is a projection of rank one.

(T 2) $\tau(AB) = \tau(BA)$ if $A \in \mathcal{D}$ and $B \in \mathcal{L}(H)$.

In addition τ is called separately continuous or $(\mathcal{L}, \mathcal{L})$-continuous ([2]) if

(T 3) For every $A \in \mathcal{D}$, the linear functional $B \to \tau(AB)$ is bounded on $\mathcal{L}(H)$.

Let C_1 be the trace-class and let $\operatorname{tr}: C_1 \to C$ denote the standard trace. If $A \in \mathcal{D}$ and rank $(A) < \infty$ then

$$\tau(A) = \operatorname{tr}(A).$$

If τ verifies (T 3) then for every $A \in \mathcal{D}$ we have

$$\sup_{\substack{\|B\| \leq 1 \\ \operatorname{rank} B < \infty}} |\operatorname{tr}(AB)| < \infty$$

and so $A \in C_1$. Hence if $\mathcal{D}$ supports a separately continuous trace then $\mathcal{D} \subset C_1$.

We shall say that a positive $T \in C_1$ is *uniquely traceable* if the ideal $\mathcal{D}$ it generates supports exactly one separately continuous trace, namely the standard trace tr.

An ideal $\mathcal{D}$ is said to be *quasi-normed* if there is a quasi-norm $|\cdot|$ on $\mathcal{D}$ verifying

(Q 1) $(\mathcal{D}, |\cdot|)$ is complete

(Q 2) $|A| \geq \beta \|A\| \qquad A \in \mathcal{D}$

for some $\beta > 0$, and

(Q 3) $|SAT| \leq \|S\| \, |A| \, \|T\| \qquad S, T \in \mathcal{L}(H) \quad A \in \mathcal{D}$

Proposition 1. *If $\mathcal{D}$ is a quasi-normed ideal and τ is a separately continuous trace on $\mathcal{D}$ then τ is continuous on $(\mathcal{D}, |\cdot|)$.*

Remark. The converse is clear ([2] p. 68).

Proof. Suppose first that $A_n \in \mathcal{D}$ is a sequence of normal operators verifying $|A_n| \to 0$. We show $\tau(A_n) \to 0$. Indeed if not we can find normal B_n with $|B_n| \leqq 2^{-n}$ and $\tau(B_n) \geqq n$. Now there exists isometries $U_n\colon H \to H$ (not necessarily surjective) so that the sequence $C_n = U_n B_n U_n^*$ commutes. Let $P_n = |C_n| = (C_n^* C_n)^{1/2}$. Then $\sum P_n$ converges in to an operator P and each C_n can be written $C = PT_n$ where $\|T_n\| \leqq 1$. Thus

$$\sup_n \tau(C_n) < \infty .$$

However $\tau(C_n) = \tau(U_n B_n U_n^*) = \tau(B_n U_n^* U_n) = n$.

For the general case, if A_n is any sequence in $\mathcal{D}$ with $|A_n| \to 0$, then $|A_n + A_n^*| \to 0$ and $|A_n - A_n^*| \to 0$ and hence $\tau(A_n) \to 0$.

2. Some preparatory lemmas

For $A \in \mathcal{K}(H)$ denote by $s_n(A)$ the sequence of singular values of A so that $\|A\| = s_1(A) \geqq s_2(A) \geqq \cdots \geqq s_n(A) \to 0$. Define

$$\varphi_A(t) = \sup \{n : s_n(A) > t\} \qquad t > 0 .$$

Then φ_A is a monotone decreasing function continuous on the right.

We note first some easy inequalities. First

$$s_{m+n-1}(A + B) \leqq s_m(A) + s_n(B) \tag{1}$$

for $A, B \in \mathcal{K}(H)$, $m, n \in \boldsymbol{N}$ (cf. [1]). It follows that

$$\varphi_{A+B}(t) \leqq \varphi_A\left(\frac{t}{2}\right) + \varphi_B\left(\frac{t}{2}\right). \tag{2}$$

Further note that

$$\sum_{n=1}^{\infty} s_n(A) = \int_0^{\infty} \varphi_A(t)\, dt . \tag{3}$$

Now suppose $A \geqq 0$ and $A \in \mathcal{K}(H)$. Then the sequence $\{s_n(A)\}$ consists of the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of A arranged in decreasing order. Let us define

$$\operatorname{tr}_m (A) = \sum_{j=1}^{m} s_n(A) . \tag{4}$$

Then

$$\operatorname{tr}_m (A) = \max \big(\operatorname{tr} (PAP)\big) \tag{5}$$

where P ranges over all self-adjoint projections of rank m.

Lemma 2. *For $A, B \geqq 0$ and $m, n \in \boldsymbol{N}$ we have*

(i) $\operatorname{tr}_m (A + B) \leqq \operatorname{tr}_m (A) + \operatorname{tr}_m (B)$

(ii) $\operatorname{tr}_{m+n} (A + B) \geqq \operatorname{tr}_m (A) + \operatorname{tr}_n (B)$.

Proof. (i) is immediate. For (ii) note that there exists self-adjoint projections P, Q so that rank $P = m$, rank $Q = n$ and

$$\mathrm{tr}\,(PAP) = \mathrm{tr}_m\,(A)$$

$$\mathrm{tr}\,(QBQ) = \mathrm{tr}_n\,(B).$$

Let R be a self-adjoint projection onto a subspace of dimension $m + n$ containing $P(H)$ and $Q(H)$. Then

$$\mathrm{tr}\,(RAR) \geqq \mathrm{tr}_m\,(A)$$

$$\mathrm{tr}\,(RBR) \geqq \mathrm{tr}_n\,(B)$$

but

$$\mathrm{tr}\,\big(R(A+B)\,R\big) \leqq \mathrm{tr}_{m+n}\,(A+B).$$

Lemma 3. *If $A, B \in \mathcal{K}(H)$ and $A, B \geqq 0$, then*

(i) $\quad \varphi_{A+B}(t) \geqq \max\big(\varphi_A(t), \varphi_B(t)\big) \qquad t > 0$

(ii) $\quad \varphi_{A+B}(4t) \leqq 3\max\big(\varphi_A(t), \varphi_B(t)\big) \qquad t > 0.$

Proof. (i) is immediate, for if $\varphi_A(t) = m$ then $\langle (A - tI)\,x, x\rangle > 0$ on a subspace of dimension m. Hence $\langle (A + B - tI)\,x, x\rangle > 0$ on the same subspace and so $\varphi_{A+B}(t) \geqq m$.

(ii) Let $\varphi_A(t) = m$, $\varphi_B(t) = n$ and suppose $\varphi_{A+B}(4t) \geqq 2\max(m, n)$. Let $\varphi_{A+B}(4t) = k$; then

$$\begin{aligned}\mathrm{tr}_k\,(A+B) &> \mathrm{tr}_{m+n}\,(A+B) + 4(k-m-n)\,t \\ &\geqq \mathrm{tr}_m\,(A) + \mathrm{tr}_n\,(B) + 4(k-m-n)\,t.\end{aligned}$$

However

$$\mathrm{tr}_k\,(A) \leqq \mathrm{tr}_m\,(A) + (k-m)\,t$$

$$\mathrm{tr}_k\,(B) \leqq \mathrm{tr}_n\,(B) + (k-n)\,t.$$

Hence

$$\mathrm{tr}_k\,(A+B) \leqq \mathrm{tr}_m\,(A) + \mathrm{tr}_n\,(B) + (2k-m-n)\,t.$$

Thus

$$2k - m - n > 4(k-m-n)$$

and so

$$k < \frac{3}{2}\,(m+n) \leqq 3\max(m, n).$$

Now for $a > 0$, $A \in C_1$ with $A \geqq 0$ we shall define

$$F_A(a) = \int_0^a \varphi_A(t)\,dt.$$

Lemma 4. *If $A, B \in C_1$ with $A, B \geqq 0$ then*

$$|F_{A+B}(a) - F_A(a) - F_B(a)| \leqq 9a\varphi_{A+B}(a).$$

Proof. First note that

$$F_A(a) = \sum_{j=1}^{\infty} \min\big(s_j(A), a\big).$$

Let $\varphi_A(a) = m$, $\varphi_B(a) = n$, $\varphi_{A+B}(a) = p$ and $\varphi_{A+B}(4a) = q$. Then

$$F_A(a) = \operatorname{tr}(A) - \sum_{j=1}^{m} \bigl(s_j(A) - a\bigr)$$
$$= \operatorname{tr}(A) + ma - \operatorname{tr}_m(A).$$

Now since $p \geqq \max(m, n)$ (Lemma 3)

$$\operatorname{tr}_{m+n}(A + B) \leqq a \min(m, n) + \operatorname{tr}_p(A + B)$$

and hence

$$\operatorname{tr}_m(A) + \operatorname{tr}_n(B) \leqq \operatorname{tr}_p(A + B) + a \min(m, n).$$

Thus

$$F_{A+B}(a) \leqq \operatorname{tr}(A + B) + a\bigl(p + \min(m, n)\bigr) - \operatorname{tr}_m(A) - \operatorname{tr}_n(B)$$
$$\leqq F_A(a) + F_B(a) + a\bigl(p - \max(m, n)\bigr)$$
$$\leqq F_A(a) + F_B(a) + a\varphi_{A+B}(a).$$

Conversely $q \leqq 3 \max(m, n)$ and hence

$$\operatorname{tr}_q(A + B) \leqq \operatorname{tr}_{3N}(A + B)$$

where $N = \max(m, n)$. Thus

$$\operatorname{tr}_q(A + B) \leqq \operatorname{tr}_{3N}(A) + \operatorname{tr}_{3N}(B)$$
$$\leqq \operatorname{tr}_m(A) + \operatorname{tr}_n(B) + a(6N - m - n).$$

Hence

$$\operatorname{tr}_p(A + B) \leqq \operatorname{tr}_q(A + B) + 4a(p - q)$$
$$\leqq \operatorname{tr}_m(A) + \operatorname{tr}_n(B) + a(4p + 6N - 4q - m - n).$$

We conclude

$$F_{A+B}(a) \geqq F_A(a) + F_B(a) + a(p - m - n) - a(4p + 6N - 4q - m - n)$$
$$= F_A(a) + F_B(a) - a(3p + 6N - 4q)$$
$$\geqq F_A(a) + F_B(a) - 9pa.$$

Lemma 5. *Suppose $\psi\colon (0, \infty) \to (0, \infty)$ is a right-continuous integer-valued monotone decreasing function. Suppose*

(a) $\int_0^\infty \psi(u)\, du < \infty$.

(b) *There exist constants $C < \infty$ and $0 < \gamma < 1$ so that*

$$\int_0^a \psi(u)\, du \leqq C\gamma a \psi(\gamma a) \qquad 0 < a \leqq 1.$$

Then there exists constants $K < \infty$ and $\alpha > 0$ so that

$$\psi(st) \leqq K s^{\alpha-1} \psi(t)$$

for $0 < s, t \leqq 1$.

Proof. First we observe if $0 < a < \gamma$ and $0 < b < a$ then

$$b\psi(b) \leqq \int_0^{a/\gamma} \psi(u)\,du \leqq Ca\psi(a).$$

In particular if $b = \gamma a$

$$\gamma a \psi(\gamma a) \leqq Ca\psi(a)$$

and hence

$$\int_0 \psi(u)\,du \leqq C^2 a\psi(a) \qquad 0 < a \leqq \gamma^{-1}.$$

Next we observe that we may suppose $\gamma = 2^{-p}$ where $p \in N$. Let

$$d_k = 2^{-k}\psi(2^{-k}) \qquad k \in N.$$

For $n \in N$ we define $v_n \in \omega$, the space of all real sequences, by

$$v_n(k) = \frac{d_{k+n}}{d_n} \qquad k \in N.$$

If $n \geqq p$ then

$$0 \leqq v_n(k) \leqq C$$

so that $\{v_n : n \geqq p\}$ is bounded in ω. Let Γ be the closed convex null of $\{v_n : n \geqq p\}$. Then Γ is compact.

By hypothesis if $k \geqq p$

$$\int_0^{2^{-k}} \psi(u)\,du \leqq C^2 d_k$$

and hence

$$\sum_{j=k}^{\infty} d_j \leqq 2C^2 d_k.$$

Hence if $m > k \geqq p$

$$\sum_{j=k+n}^{m+n} d_j \leqq 2C^2 d_{k+n}$$

and so

$$\sum_{j=k}^{m} v_n(j) \leqq 2C^2 v_n(k).$$

Now if $w \in \Gamma$

$$\sum_{j=k}^{m} w(j) \leqq 2C^2 w(k).$$

Since w is bounded this implies that $w \in c_0$. Let P be the closed positive cone of ω; then $\Gamma - P$ is closed and the constant sequence $e = (1, 1, \ldots)$ is not in $\Gamma - P$. Thus there exist $\beta_n \geqq 0$ finitely non-zero such that

$$\sum \beta_k = 1$$

and

$$\sum \beta_k v_n(k) \leqq 1 - \theta$$

for all $n \geqq p$. If $\beta_k = 0$ for $k > N$ we conclude that

$$\min_{k \leqq N} v_n(k) \leqq 1 - \theta$$

for $n \geqq p$. Thus for every $n \geqq p$ there exists $k \leqq N$ with

$$d_{n+k} \leqq (1 - \theta)\, d_n .$$

Fix $\alpha > 0$ by $2^{-N\alpha} = 1 - \theta$. We deduce that if $n \geqq p$, $\sigma \in N$ there exists k with $\sigma N < k \leqq (\sigma + 1)\, N$ and

$$d_{n+k} \leqq (1 - \theta)^{\sigma+1}\, d_n = 2^{-(\sigma+1)\alpha N} d_n .$$

Hence if $(\sigma - 1)\, N < l \leqq \sigma N$, since ψ is monotone decreasing,

$$d_{n+l} \leqq 2^{k-l} d_{n+k} \leqq 2^{2N} 2^{-(\sigma+1)\alpha N} d_n = 2^{l-(\sigma-1)N} 2^{-l\alpha} d_n \leqq 2^N 2^{-l\alpha} d_n .$$

We easily deduce that for some constant K we have

$$st\psi(st) \leqq K s^{\alpha} t\psi(t)$$

for every $0 < s, t < 1$ and the result follows.

3. The main results

Let T be a positive compact operator and let $\psi(t) = \varphi_T(t)$. Then the two-sided ideal $\mathcal{D}_T$ generated by T is determined solely by ψ. In fact $A \in \mathcal{D}_T$ if and only if for some γ, $0 < \gamma < 1$ and some $C < \infty$ we have

$$\varphi_A(t) \leqq C\psi(\gamma t) \qquad t > 0 . \tag{6}$$

We denote the set of such A by $\mathcal{D}(\psi)$. To see that $\mathcal{D}(\psi)$ is an ideal one must use equation (2).

Suppose in addition we have that for some λ, $0 < \lambda < 1$,

$$\psi(\lambda t) \geqq 2\psi(t) \qquad t > 0 .$$

Then $\mathcal{D}(\psi)$ is a quasi-normed ideal if we define $|A|$ to be the infimum of all $c > 0$ so that

$$\varphi_A(ct) \leqq \psi(t) \qquad t > 0 .$$

Note here that

$$\varphi_{A+B}(t) < \varphi_A\left(\frac{t}{2}\right) + \varphi_B\left(\frac{t}{2}\right) \leqq \max\left(\varphi_A\left(\frac{\lambda}{2}\, t,\right), \varphi_B\left(\frac{\lambda}{2}\, t\right)\right)$$

so that

$$|A + B| \leqq \frac{2}{\lambda} \max\,(|A|, |B|) .$$

We also remark that if ψ is any right-continuous monotone decreasing integer valued function with $\lim_{t\to\infty} \psi(t) = 0$ then there is a positive compact operator T with $\varphi_T = \psi$. Clearly $\mathcal{D}(\psi) \subset C_1$ if and only if $T \in C_1$, i.e.

$$\int_0^\infty \psi(u)\,du < \infty.$$

Theorem 6. *Suppose ψ is a nonnegative monotone-decreasing, integer-valued left-continuous function on $(0, \infty)$ with*

$$\int_0^\infty \psi(u)\,du < \infty.$$

Then the following are equivalent:

(i) *If τ is a separately continuous trace on $\mathcal{D}(\psi)$ then*

$$\tau(A) = \operatorname{tr}(A) \qquad A \in \mathcal{D}(\psi).$$

(ii) *There exists $K < \infty$, $\alpha > 0$ so that if $0 \leqq s, t \leqq 1$*

$$\psi(st) \leqq K s^{\alpha-1} \psi(t).$$

Proof. (i) $\Rightarrow$ (ii): Let us suppose (ii) fails. Then, according to Lemma 5 we can find a sequence a_n with $0 < a_n \leqq 1$ and

$$\int_0^{a_n} \psi(u)\,du > n a_n \psi\left(\frac{a_n}{n}\right).$$

For $A \geqq 0$ in $\mathcal{D}(\psi)$ we set

$$\Lambda_n(A) = F_A(a_n) \Big/ \int_0^{a_n} \psi(u)\,du.$$

Suppose $\varphi_A(t) \leqq C\psi(\gamma t)$. Then

$$F_A(a_n) \leqq C \int_0^{a_n} \psi(\gamma u)\,du = C\gamma^{-1} \int_0^{\gamma a_n} \psi(u)\,du$$

so that

$$0 \leqq \Lambda_n(A) \leqq C\gamma^{-1}.$$

Define

$$\Lambda(A) = \lim_{n\in\mathcal{U}} \Lambda_n(A)$$

where $\mathcal{U}$ is some non-principal ultrafilter on N.
We observe that if $0 \leqq \lambda \leqq 1$,

$$F_{\lambda A}(a_n) = \int_0^{a_n} \varphi_{\lambda A}(u)\,du = \int_0^{a_n} \varphi_A(\lambda^{-1}u)\,du = \lambda \int_0^{\lambda^{-1}a_n} \varphi_A(u)\,du$$

so that

$$|F_{\lambda A}(a_n) - \lambda F_A(a_n)| \leqq \lambda \left| \int_{a_n}^{\lambda^{-1}a_n} \varphi_A(u)\, du \right|$$

$$\leqq \lambda(\lambda^{-1} - 1)\, \varphi_A(a_n) \leqq C(1 - \lambda)\, \psi(\gamma a_n)\, a_n .$$

Thus if $n > \gamma^{-1}$

$$|F_{\lambda A}(a_n) - \lambda F_A(a_n)| \leqq \frac{C}{n} (1 - \lambda) \int_0^{a_n} \psi(u)\, du$$

and

$$|A_n(\lambda A) - A_n(A)| \leqq \frac{C}{n} (1 - \lambda).$$

Thus

$$A(\lambda A) = \lambda A(A) \qquad \lambda \geqq 0, \quad A \geqq 0.$$

Now by Lemma 4 if $A, B \geqq 0$

$$|F_{A+B}(a_n) - F_A(a_n) - F_B(a_n)| \leqq 9 a_n \varphi_{A+B}(a_n)$$

If $\varphi_{A+B}(t) \leqq C_1 \psi(\gamma_1 t)$ we have

$$|F_{A+B}(a_n) - F_A(a_n) - F_B(a_n)| \leqq 9 C_1 a_n \psi(\gamma, a_n) \leqq \frac{9C_1}{n} \int_0^{a_n} \psi(u)\, du$$

if $n > \dfrac{1}{\gamma_1}$. It follows that

$$A(A + B) = A(A) + A(B).$$

Note further that

$$0 \leqq A(A) \leqq \frac{C}{\gamma}$$

where $\varphi_A(t) \leqq C\psi(\gamma t)$.

We also note that if F has finite rank then $\varphi_A(t)$ is bounded and then

$$F_A(a_n) \leqq K a_n$$

for some n. Hence $A(A) = 0$ if rank $A < \infty$. Now extend A to a linear functional still denoted by A, defined on $\mathcal{D}(\psi)$. We note that

$$A(U^{-1} A U) = A(A)$$

if U is unitary.

If rank $A < \infty$ then $A(A) = 0$. Furthermore if H is hermitian then we can write $H = P_1 - P_2$ where P_1, P_2 are positive and

$$\varphi_{P_1+P_2}(t) = \varphi_H(t).$$

Hence if $\varphi_H(t) \leqq C\psi(\gamma t)$ then

$$|A(H)| \leqq C\gamma^{-1}.$$

In general if $A \in \mathcal{D}(\psi)$ then

$$\varphi_{A+A^*}(t) \leqq 2\varphi_A\left(\frac{t}{2}\right)$$

$$\varphi_{i(A-A^*)}(t) \leqq 2\varphi_A\left(\frac{t}{2}\right)$$

and hence if $\varphi_A(t) \leqq C\psi(\gamma t)$ then

$$\varphi_{A+A^*}(t) \leqq 2C\psi\left(\frac{1}{2}\gamma t\right)$$

$$\varphi_{i(A-A^*)}(t) \leqq 2C\varphi\left(\frac{1}{2}\gamma t\right).$$

Hence

$$|\Lambda(A)| \leqq 8C\gamma^{-1}.$$

It follows easily that Λ is separately continuous, i.e. for each $B \in \mathcal{D}(\psi)$ the linear functional $A \to \Lambda(AB)$ is bounded on $\mathcal{L}(H)$.

Now define $\tau(A) = \operatorname{tr}(A) + \Lambda(A)$ for $A \in \mathcal{D}(\psi)$. Then τ is separately continuous and $\tau(P) = 1$ for every rank one projection. Furthermore

$$\tau(U^{-1}AU) = \tau(A)$$

for every unitary U and hence

$$\tau(AB) = \tau(BA)$$

for every $B \in \mathcal{L}(H)$, $A \in \mathcal{D}(\psi)$ (cf. [2] p. 63).

To see that τ is a trace distinct from tr we need only produce a positive operator T with

$$\varphi_T(t) = \psi(t).$$

Then $\Lambda(T) = 1$ and hence $\tau(T) = \operatorname{tr}(T) + 1$. This shows that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i): Let us assume τ is a separately continuous trace on $\mathcal{D}(\psi)$. Let us write $\Lambda(A) = \tau(A) - \operatorname{tr}(A)$, so that Λ is separately continuous and $\Lambda(F) = 0$ if rank $F < \infty$. Suppose $C < \infty$ and $0 < \gamma < 1$; then there exists $M < \infty$ so that if

(7) $$\varphi_A(t) \leqq C\psi(\gamma t)$$

then $|\Lambda(A)| \leqq M$.

Fix A satisfying (7). Then for $n \in N$ using standard representation theorems we can write

$$A = F + A_1 + \cdots + A_n$$

where rank $F < \infty$ and

$$\varphi_{A_j}(t) \leqq \frac{1}{n}\varphi_A(t) \qquad 0 \leqq t \leqq \lambda$$

and

$$\varphi_{A_j}(t) = 0 \qquad t > \lambda.$$

Consider for $0 < \lambda < 1$,

$$\begin{aligned}
\varphi_{\lambda^{-1}A_j}(t) &= \varphi_{A_j}(\lambda t) \\
&\leqq \frac{1}{n}\,\varphi_{A_j}(\lambda t) \qquad 0 < t \leqq 1 \\
&\leqq \frac{C}{n}\,\psi(\lambda\gamma t) \qquad 0 < t \leqq 1 \\
&\leqq \frac{C}{n}\,K\lambda^{\alpha-1}\psi(\gamma t) \quad 0 < t \leqq 1 \\
&\leqq C\psi(\gamma t) \qquad 0 < t \leqq 1
\end{aligned}$$

provided $K\lambda^{\alpha-1} = n$. Similarly if $t > 1$

$$\varphi_{\lambda^{-1}A_j}(t) = 0 \leqq C\psi(\gamma t)$$

and hence

$$|\Lambda(\lambda^{-1}A_j)| \leqq M$$

$$|\Lambda(A_j)| \leqq M\lambda = M\left(\frac{K}{n}\right)^{1/(1-\alpha)}$$

Hence

$$\Lambda(A) \leqq Mn\left(\frac{K}{n}\right)^{1/(1-\alpha)}$$

and hence $\Lambda(A) = 0$. It follows that $\operatorname{tr}(A) = \tau(A)$ for $A \in \mathcal{D}(\psi)$.

Corollary 7. *Let T be a positive trace-class operator. In order that T is uniquely traceable it is necessary and sufficient that there exists $p > 1$ and $C < \infty$ so that the singular values (λ_n) of T satisfy*

$$\lambda_m \leqq C\left(\frac{m}{n}\right)^{-p}\lambda_n \qquad m > n.$$

Proof. It suffices to show that φ_T satisfies (ii) of Theorem 6 if and only if it satisfies the criterion of the Corollary. In fact if

$$\varphi_T(st) \leqq Ks^{\alpha-1}\varphi_T(t) \qquad 0 < s,\ t < 1,$$

$\lambda_n < 1$ and $m > n$ then

$$\varphi_T(\lambda_n) < n$$

$$\varphi_T(\lambda_n -) \leqq n$$

$$\varphi_T(\lambda_m -) \geqq m$$

and hence

$$m \leqq K\left(\frac{\lambda_m}{\lambda_n}\right)^{\alpha-1} n$$

and the Corollary follows with $p = \dfrac{1}{1-\alpha}$.

If the condition of the Corollary fails we may for every $C > 0$ $p > 1$ find $m > n$ with $\lambda_n < 1$ so that

$$\lambda_m > C\left(\frac{m}{n}\right)^{-p} \lambda_n.$$

Then $\varphi(\lambda_n) < n$ and

$$\varphi\left(C\left(\frac{m}{n}\right)^{-p} \lambda_n\right) > m.$$

Thus

$$KC^{\alpha-1}\left(\frac{m}{n}\right)^{p(1-\alpha)} n > m$$

so that

$$\left(\frac{m}{n}\right)^{1-p(1-\alpha)} < KC^{\alpha-1}$$

Clearly this is a contradiction if $p = (1-\alpha)^{-1}$ and $KC^{\alpha-1} < 1$.

Example. We now construct an explicit example of a function ψ so that $\mathcal{E}(\psi)$ is a quasi-normed ideal supporting at least two distinct continuous traces.

We define only $\psi(2^{-2m})$ for $m \in \boldsymbol{N} \cup \{0\}$. We do this by induction. Let $\psi(2^{-2m}) = \psi_m$. Set $\psi_0 = 0$, $\psi_1 = 1$. For $n = 1, 2, \ldots$, let

$$\psi_m = 4\psi_{m-1} \quad \text{if} \quad n! < m \leqq n! + n$$

$$\psi_m = 2\psi_{m-1} \quad \text{if} \quad n! + n < m \leqq (n+1)!$$

Let ψ be a monotone-decreasing right-continuous integer-valued function with $\psi(2^{-2m}) = \psi_m$. Clearly $\mathcal{D}(\psi)$ is a quasi-normed ideal since

$$\psi\left(\frac{1}{16}\,t\right) \geqq 2\psi(t) \qquad t > 0.$$

Furthermore

$$\psi_{(n!+n)} = 2^{2n}\psi_{n!}$$

so that ψ fails Theorem 6(ii).

We must check that

$$\int_0^\infty \psi(t)\,dt < \infty.$$

To do this note we need

$$\sum_{m=1}^\infty \frac{1}{2^{2m}}\,\psi_m < \infty.$$

Now

$$\sum_{n!+1}^{n!+n} \frac{1}{2^{2m}}\,\psi_m = n\psi_{n!}$$

while

$$\sum_{n!+n+1}^{(n+1)!} \frac{1}{2^{2m}} \psi_m \leqq \psi_{n!}$$

Now

$$\psi_{(n+1)!} \leqq 2^{n!+n+1-(n+1)!}\psi_{n!} = 2^{(1-n)n!}\psi_{n!} \leqq \frac{1}{2} \psi_{n!}$$

if $n \geqq 2$. Hence

$$\sum (n+1) \psi_{n!} < \infty$$

and so

$$\int_0^\infty \psi(t)\, dt < \infty .$$

By Theorem 6, $\mathcal{D}(\psi)$ gives the promised example.

Added in proof. For an alternative treatment see [3] pp. 312—321.

References

[1] A. Pietsch, Operator Ideals, North-Holland, Amsterdam 1979
[2] —, Operator Ideals with a trace, Math. Nachr. 100 (1981) 61—91
[3] —, Eigenvalues and s-numbers, Mathematik und ihre Anwendungen in Physik und Technik, Band 43, Leipzig 1987

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