Locally Complemented Subspaces and \( L_p \)-Spaces for \( 0 < p < 1 \)

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Abstract. We develop a theory of \( \mathcal{L}_p \)-spaces for \( 0 < p < 1 \), basing our definition on the concept of a locally complemented subspace of a quasi-Banach space. Among the topics we consider are the existence of basis in \( \mathcal{L}_p \)-spaces, and lifting and extension properties for operators. We also give a simple construction of uncountably many separable \( \mathcal{L}_p \)-spaces of the form \( \mathcal{L}_p(X) \) where \( X \) is not a \( \mathcal{L}_p \)-space. We also give some applications of our theory to the spaces \( H_p, 0 < p < 1 \).

1. Introduction

\( \mathcal{L}_p \)-spaces (\( 1 \leq p \leq \infty \)) were introduced by Lindenstrauss and Pełczyński [15] as Banach spaces whose local structure resembles that of the spaces \( L_p \). Thus a Banach space \( X \) is an \( \mathcal{L}_p \)-space if there is a constant \( \lambda \) such that for every finite dimensional subspace \( F \) of \( X \) there is a finite-dimensional subspace \( G \supseteq F \) and a linear isomorphism \( T : G \to F^{(n)} \) with \( ||T|| \cdot ||T^{-1}|| \leq \lambda \). The study of \( \mathcal{L}_p \)-spaces has proved to be rich and rewarding.

There has been little effort at a systematic treatment of \( \mathcal{L}_p \)-spaces for \( 0 < p < 1 \). There is however, in the author’s opinion, some interest in giving such a treatment. For example in [12], it is shown that the quotient \( \mathcal{L}_p/1 \) of \( \mathcal{L}_p \) by a one-dimensional subspace is not an \( \mathcal{L}_p \)-space if \( 0 < p < 1 \) and hence it cannot be isomorphic to \( L_p \).

Suppose now \( \Sigma_0 \) is a sub-\( \sigma \)-algebra of the Borel sets of \( (0, 1) \) and let \( L_p(\Sigma_0) \) be the closed subspace of all \( \Sigma_0 \)-measurable functions in \( L_p \). We denote by \( \Lambda(\Sigma_0) \) the quotient space \( L_p/L_p(\Sigma_0) \). In [9] it is shown that, “usually”, \( L_p(\Sigma_0) \) is uncomplemented in \( L_p \) if \( 0 < p < 1 \). Thus N. T. Peck raised the question whether \( \Lambda(\Sigma_0) \) can be isomorphic to \( L_p \) if \( L_p(\Sigma_0) \) is uncomplemented, and equally whether \( \Lambda(\Sigma_0) \) could be an \( \mathcal{L}_p \)-space.

The definition of an \( \mathcal{L}_p \)-space used in [12] is slightly different from the definition given above for \( 1 \leq p \leq \infty \). It is merely required that \( X \) contains an increasing net of finite-dimensional subspaces uniformly isomorphic to finite-dimensional \( L_p \)-spaces, whose union is dense. This distinction is unimportant for \( p = 1 \), but for \( 0 < p < 1 \) it is significant, for, as W. J. Stiles pointed out to the author it is not

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clear that even $L_p(0 < p < 1)$ would satisfy the 
Lindenstrauss-Pelczyński definition. This led us to consider whether there is an alternative indirect definition of $\ell_p$-spaces more suitable for $0 < p < 1$.

The crucial notion we introduce in this paper is that of a locally complemented subspace of a quasi-Banach space. This idea is entirely natural we believe and leads to an attractive definition of $\ell_p$-spaces. Thus a quasi-Banach $X$ is an $\ell_p$-space if and only if it is isomorphic to a locally complemented subspace of a space $L_p(\Omega, \Sigma, \mu)$. There is a local version of this definition (see Theorem 6.1 below); $X$ is an $\ell_p$-space if there is a uniform constant $\lambda$, such that whenever $F$ is a finite-dimensional subspaces of $X$ and $\varepsilon > 0$ there are operators $S : F \to I_p$, $T : I_p \to X$ with $\|T\| \cdot \|S\| \geq \lambda$ and $\|TSf - f\| \leq \varepsilon \|f\|$ for $f \in F$. For $p = 1$ or $\infty$, this simply reduces to the standard definition, but for $1 < p < \infty$ ($p + 2)$ it gives a very slightly wider class (Hilbert spaces are $\ell_p$-spaces for $1 < p < 2$).

We now discuss the layout and main results of the paper. Section 2 is purely preparatory and in Section 3 we introduce the notion of a locally complemented subspace. In a Banach space this has several equivalent attractive formulations; for example $N$ is a locally complemented subspace of $X$ if and only if $N^{**}$ is complemented in $X^{**}$. The Principle of Local Reflexivity plays an important role here, as it states that $X$ is locally complemented in $X^{**}$.

The absence of a bidual for non-locally convex quasi-Banach spaces leads us to consider ultraproducts in Section 4, and we give a number of connections between these ideas. Section 5 contains our first main result that a locally complemented subspace of a quasi-Banach space with a basis, under certain conditions, also has a basis; these conditions include the case of a weakly dense subspace. This result is similar in spirit to some results of Johnson, Rosenthal and Zippin [7].

In Section 6, we introduce $\ell_p$-spaces and give some of their properties. We also show that if $0 < p < 1$, it is convenient to separate separable $L_p$-spaces into three categories — discrete, continuous and hybrid $\ell_p$-spaces. A separable $\ell_p$-space has a basis if and only if it is discrete, i.e., a locally complemented subspace of $I_p$. Separable $\ell_p$-spaces with trivial dual are called continuous and correspond to the locally complemented subspaces of $L_p$. We point out (Theorem 6.7) that the kernel of any operator from a $p$-Banach space with a basis onto a continuous $\ell_p$-space (including $L_p$ itself) will again have a basis. We also produce a simple explicit example of a weakly dense subspace of $I_p$ ($0 < p < 1$) failing to have a basis (or even the Bounded Approximation Property). In view of the results of Davie and Enflo [3], [5] and recently Szankowski [22] the existence of such a subspace is hardly surprising; however the construction is very easy and the subspace has the additional property that every compact operator defined on it may be extended to $I_p$.

In Section 7 we show that the subspace $L_p(\Sigma_0)$ is locally complemented but not complemented (see [9]). A deduction is that in this case $\Lambda(\Sigma_0)$ is an $\ell_p$-space; however we have shown in [11] that, in the case where $(\Omega, \Sigma, \mu)$ is separable, that $\Lambda(\Sigma_0) \cong L_p$ implies that $L_p(\Sigma_0)$ is complemented. If we take the special example
where \( \Omega = (0, 1) \times (0, 1) \), \( \Sigma \) is the Borel sets of \((0, 1)^2 \) and \( \Sigma_0 \) is \( \sigma \)-algebra of sets of the form \( B \times (0, 1) \) for \( B \) a Borel subset of \((0, 1) \), then \( \Lambda(\Sigma_0) \cong L_p(L_p/1) \) where \( L_p/1 \) ([12]) is the quotient of \( L_p \) by a single line. However \( L_p/1 \) is not an \( \mathcal{L}_p \)-space. This shows also that for \( 0 < p < 1 \) it is possible to have \( L_p(X) \) an \( L_p \)-space without having \( X \) as \( \mathcal{L}_p \)-space, in contrast to the situation for \( p = 1 \). We go on to construct an uncountable collection of separable \( \mathcal{L}_p \)-spaces of this type.

In Section 8, we prove a number of lifting theorems (similar to those of [12]) and extension theorems for operators. For example if \( X \) is a \( p \)-Banach space and \( N \) is a closed subspace such that \( X/N \) is a continuous \( \mathcal{L}_p \)-space then an operator \( T : N \to Z \) can be extended to an operator \( T_1 : X \to Z \) under any one of three hypotheses: (1) \( T \) is compact, (2) \( Z \) is a \( q \)-Banach space for some \( q > p \) or (3) \( Z \) is a pseudo-dual space. In each the extension is unique.

In Section 9, we give an application of these ideas to an example involving \( H_p \) for \( 0 < p < 1 \). Let \( J_p \) for \( 0 < p < 1 \) be the closed subspace of \( H_p \) (regarded as a subspace of \( L_p(\mathbb{S}) \), where \( \mathbb{S} \) is the unit circle with Lebesgue measure) of all \( f \) such that \( f \in H_p \). Exploiting a recent result of Aleksandrov [1] we show \( J_p \) is isomorphic to a locally complemented subspace of \( H_p \oplus H_p^* \) (where \( H_p = \{ f \in L_p(\mathbb{S}) : f \in L_p \} \)). We deduce that \( J_p \) has \((BAP)\) and that as \( H_p \) has a basis then so does \( J_p \). We also quickly obtain the dual space of \( J_p \); every continuous linear functional \( \varphi \in J_p^* \) is of the form
\[
\varphi(f) = \psi_1(f) + \psi_2(f) \quad f \in J_p
\]
where \( \psi_1 \in H_p^* \) and \( \psi_2 \in H_p^* \). We show that \( J_p \) is non-isomorphic to \( H_p \), but \( L_p/J_p \cong L_p/H_p \). Finally we characterize translation-invariant operators \( T : J_p \to J_p \) using the extension theorems of Section 8. We show that every translation-invariant operator \( T : J_p \to J_p \) takes the form
\[
Tf(z) = \sum_{n=1}^{\infty} c_n \omega_n(z) + \alpha_1 \theta_0(f) + \alpha_2 \theta_\omega(f)
\]
where \( \omega_n \in \mathbb{S} \), \( \sum |c_n|^p < \infty \) and, \( \theta_\omega(f) = f(\omega) \) regarding \( f \) as a member of \( H_p \) and \( \theta_\omega(f) = f(0) \) regarding \( f \) as a member of \( H_p \).

2. Preliminaries

As usual a quasi-norm on real (or complex) vector space \( X \) is a map \( x \mapsto \|x\| \) \((x \in \mathbb{R})\) satisfying

\[
\begin{align*}
(2.0.1) \quad & |x| > 0 \quad x \neq 0 \\
(2.0.2) \quad & |ax| = |a| \cdot |x|, \quad x \in \mathbb{R} \quad \text{(or } \mathbb{C}) \quad x \in X \\
(2.0.3) \quad & \|x + y\| \leq k (\|x\| + \|y\|) \quad x, y \in X,
\end{align*}
\]
where \( k \) is a constant independent of \( x \) and \( y \). A quasi-norm defines a locally bounded vector topology on \( X \). A complete quasi-normed space is called a quasi-Banach space. If, in addition the quasi-norm satisfies for some \( p \), \( 0 < p \leq 1 \),
\[
(2.0.4) \quad \|x + y\|^p \leq \|x\|^p + \|y\|^p \quad x, y \in X
\]
then we say $X$ is a $p$-Banach space. A basic theorem due to Aoki and Rolewicz asserts that every quasi-Banach space may be equivalently renormed as a $p$-Banach space for some $p$, $0 < p \leq 1$. We shall therefore assume without losing any generality that every quasi-Banach space considered is a $p$-Banach space for some suitable $p$ where $0 < p \leq 1$ (i.e. that (2.0.4) is satisfied).

If $(\Omega, \Sigma, \mu)$ is a measure space then by $L_p(\Omega, \Sigma, \mu)$ we denote the space of all real (or complex) $\Sigma$-measurable functions $f$ satisfying:

$$
\|f\|_p = \left\{ \int \|f(x)\|^p \, d\mu(x) \right\}^{1/p} < \infty
$$

$L_p(\Omega, \Sigma, \mu)$ is a $p$-Banach space, after the standard identification of functions agreeing $\mu$-almost everywhere. If $\Sigma$ is the power set of $\Omega$ and $\mu$ is counting measure on $\Sigma$ (i.e. $\mu(A)$ is the cardinality of $A$ if $A$ is a finite subset of $\Omega$ and $\infty$ otherwise), then $L_p(\Omega, \Sigma, \mu)$ is written $l_p(\Omega)$. If $\Omega$ is countable this reduces to the standard sequence space $l_p$.

On the other hand if $(\Omega, \Sigma, \mu)$ is separable non-atomic probability space then $L_p(\Omega, \Sigma, \mu)$ can be identified isometrically with the function space $L_p(0, 1)$ and will be written $L_p$.

If $X$ is a quasi-Banach space the $L_p(\Omega, \Sigma, \mu; X)$ will denote the space of $\Sigma$-measurable maps $f : \Omega \to X$ with separable range satisfying:

$$
\|f\|_p = \left\{ \int \|f(\omega)\|^p \, d\mu(\omega) \right\}^{1/p} < \infty
$$

Again $L_p(\Omega, \Sigma, \mu; X)$ is a quasi-Banach space; if $X$ is a $p$-Banach space, then it is also a $p$-Banach space. If $\Omega = \mathbb{N}$ and $\mu$ is counting measure we write this space as $l_p(X)$, while if $(\Omega, \Sigma, \mu)$ is separable non-atomic probability space we write it as $L_p$.

If $X$ is a $p$-Banach space, then for any index set $I$, the space $l_\infty(I; X)$ is the space of "generalized sequences", $\{x_i\}_{i \in I}$ satisfying

$$
\|\{x_i\}_{i \in I} \| = \sup_{i \in I} \|x_i\| < \infty
$$

$l_\infty(I; X)$ is also a $p$-Banach space. If $\mathfrak{U}$ is a non-principal ultrafilter on $I$, then the ultraproduct $X_\mathfrak{U}$ of $X$ is the space $l_\infty(I; X)/C_{0,\mathfrak{U}}(I; X)$ where $C_{0,\mathfrak{U}}(I; X)$ is the closed subspace of $l_\infty(I; X)$ of all $\{x_i\}$ such that

$$
\lim_{\mathfrak{U}} \|x_i\| = 0
$$

It is often convenient to think of $X_\mathfrak{U}$ as the Hausdorff quotient of the space $l_\infty(I; X)$ with the "semi-quasi-norm"

$$
\|\{x_i\}\|_\mathfrak{U} = \lim_{\mathfrak{U}} \|x_i\|
$$

We also shall identify $X$ as a subspace of $X_\mathfrak{U}$ by identifying each $x \in X$ with the constant sequence $x_i = x$ for $i \in I$.

The main theorem we shall require here is due to Schreiber [20] (the case $p = 1$ is due to Dacunha-Castelle and Krivine [2]).
Theorem 2.1. Any ultraproduct of a space $L_p(\Omega, \Sigma, \mu)$ is isometrically isomorphic to $L_p(\Omega, \Sigma, \mu_1)$ for a suitably chosen measure space $(\Omega, \Sigma, \mu_1)$.

Any separable $p$-Banach space $X$ is a quotient of the space $l_p$. In the case $p = 1$, Lindenstrauss and Rosenthal [16] showed that there is a form of uniqueness of the quotient map of $l_1$ onto $X$. Precisely if $T_1: l_1 \to X$ are any two quotient maps and $X$ is not isomorphic to $l_1$ then there is an automorphism $\tau: l_1 \to l_1$ such that $T_1 = T_2\tau$. Stiles [21] asked whether this can be generalized to $l_p$ when $p < 1$. In the stated form this is impossible, since as shown by Stiles, $l_p$ contains a subspace $M$ which contains no copy of $l_p$ complemented in the whole space; then $l_p/M \not\cong l_p \oplus l_p/M$ and there can be no isomorphism of $l_p$ onto $l_p \oplus M$. However, excepting this case, the argument of Lindenstrauss and Rosenthal can be extended. We therefore state for $0 < p < 1$:

Theorem 2.2. Suppose $X$ is a separable $p$-Banach space and suppose $T_1: l_p \to X$ and $T_2: l_p \to X$ are open mappings. Provided the kernels of $T_1$ and $T_2$ both contain copies of $l_p$ which are complemented in $l_p$, there is an automorphism $\tau: l_p \to l_p$ with $T_1 = T_2\tau$.

The proof given in Lindenstrauss-Tzafriri [18] p. 108 goes through undisturbed, once one observes that the operator $S$ defined therein is subjective for purely algebraic reasons (the proof in [18] appeals to duality) indeed given $x \in l_1$, $x - \hat{T}_1T_2x$ is clearly in $S(U)$ while $\hat{T}_1\hat{T}_2x \in S(V)$.

A closed subspace $M$ of a quasi-Banach space $X$ is said to have the Hahn-Banach Extension Property (HBEP) if every continuous linear functional $\tilde{g} \in M^*$ can be extended to a continuous linear functional $\tilde{g} \in X^*$. 

Corollary 2.3. Suppose $X$ is a separable $p$-Banach space non-isomorphic is $l_p$. Suppose $T_1: l_p \to X$ and $T_2: l_p \to X$ are two open mappings and suppose the kernel of $T_1$ has (HBEP). Then there is an automorphism $\tau$ of $l_p$ such that $T_1 = T_2\tau$.

Proof. If ker $T_1$ has HBEP then $X$ is a $\mathcal{S}_p$-space as defined in [12] and so ker $T_2$ also has HBEP. But this means by results of Stiles that both ker $T_1$ and ker $T_2$ contain copies of $l_p$ complemented in $l_p$.

We conclude by recalling some definitions. A quasi-Banach space $X$ is a pseudo-dual space if there is Hausdorff vector topology on $X$ for which the unit ball is relatively compact. $X$ has the Bounded Approximation Property (BAP) if there is a sequence of finite-rank operation $T_n: X \to X$ such that $T_nx \to x$ for $x \in X$.

3. Locally complemented subspaces

We shall say that a closed subspace $E$ of a quasi-Banach space $X$ is locally complemented in $X$ if there is a constant $\lambda$ such that whenever $F$ is a finite-dimensional subspace of $X$ and $\epsilon > 0$ there is a linear operator $T = T_p: F \to E$ such that $\|T\| \leq \lambda$ and $\|Tf - f\| \leq \epsilon \|f\|$ for $f \in E \cap F$.
By way of motivation let us observe that the Principle of Local Reflexivity for Banach spaces (Lindenstrauss and Rosentha [17]) states that every Banach space $X$ is locally complemented in its bidual $X^{**}$ (with $\lambda = 1$).

We shall start with two rather technical lemmas which will be needed to identify locally complemented subspaces.

**Lemma 3.1.** Suppose $X$ is a quasi-Banach space and that $E$ is closed subspace of $X$. Suppose there is an increasing net $X_n$ of subspaces of $X$ so that $\bigcup (X_n \cap E)$ is dense in $E$ and $\bigcup X_n$ is dense in $X$. Suppose there are operators $Q_n : X_n \to E$ such that $\sup \|Q_n\| < \infty$ and $Q_n e - e$ for $e \in \bigcup (X_n \cap E)$. Then $E$ is locally complemented in $X$.

**Proof.** Suppose $F \subset X$ is a finite-dimensional subspace and $\{f_1, \ldots, f_n\}$ is a normalized basis of $F$ such that for some $m \leq n$, $\{f_1, \ldots, f_n\}$ is a basis of $E \cap F$. Then there is a constant $c > 0$ such that for any $(a_1, \ldots, a_n)$

$$\left\| \sum_{i=1}^{n} a_i f_i \right\| \leq c \left( \sum_{i=1}^{n} \left| a_i \right|^2 \right)^{1/2}.$$ 

For fixed $0 < \varepsilon < 1$ select $x$ and $g_1, \ldots, g_m \in X_n$ so that $\|g_i - f_i\|_p \leq \frac{1}{4} c^p \varepsilon^p$ for $1 \leq i \leq n$ and $g_i \in E$ for $1 \leq i \leq m$. Choose $\beta \equiv \alpha$ that $\|Q_\beta e - e\|_p \leq \frac{1}{4} c^p \|e\|_p$ for $e \in [g_1, \ldots, g_m]$. Then define $T : F \to E$ by

$$T \left( \sum_{i=1}^{n} a_i f_i \right) = Q_\beta \left( \sum_{i=1}^{n} a_i g_i \right).$$

Then

$$\left\| \sum_{i=1}^{n} a_i (g_i - f_i) \right\|_p \leq \frac{1}{4} c^p \varepsilon^p \sum_{i=1}^{n} \|a_i\|^p \leq \frac{1}{4} \varepsilon^p \|\sum_{i=1}^{n} a_i f_i\|^p.$$ 

Thus $\|T\|_p \leq 2\|Q_\beta\|_p$ and if $e \in F \cap E$,

$$\|Te - e\| \leq \varepsilon \|e\|.$$ 

**Lemma 3.2.** Let $X$ be a quasi-Banach space and suppose $E$ is a locally complemented subspace of $X$. Thus there is a constant $\lambda$ such that whenever $Y$ is a closed subspace of $X$ containing $E$ with $\dim Y/E < \infty$ there is a projection $P : Y \to E$ with $\|P\| \leq \lambda$.

**Remark.** Clearly the converse of Lemma 3.2 is immediate.

**Proof.** There is a constant $\lambda_0$ so that for every $\varepsilon > 0$ and finite-dimensional subspace $F$ of $X$ there is a linear map $T : F \to E$ with $\|Tf - f\| \leq \varepsilon \|f\|$ for $f \in E \cap F$ and $\|T\| \leq \lambda_0$. We can suppose that $X$ is a $p$-Banach space where $0 < p \leq 1$. 

Suppose $Q : Y \to E$ is any bounded projection (there is a bounded projection since $\dim Y|E| < \infty$). Suppose $q = \|I - Q\|$ and choose $\varepsilon > 0$ so that
$$\varepsilon < (3\lambda_0^p + 4)^{-1} q^{-1}.$$ 
Let $G = Q^{-1}(0)$ and let $\{g_1, \ldots, g_n\}$ is an $\varepsilon$-net for the unit ball of $G$. Let
$$\delta_i = d(g_i, E) = \inf_{e \in E} \|g_i - e\| \quad 1 \leq i \leq n$$
and choose $e_i \in E$ so that
$$\|g_i - e_i\|^p \leq 2\delta_i^p \quad 1 \leq i \leq n.$$ 
Let $H$ be the linear span of $G$ and $\{e_1, \ldots, e_n\}$. Since $\|e_i\|^p \leq 3$ there is a linear map $T : H \to E$ so that $\|T\| \leq \lambda_0$ and
$$\|Te_i - e_i\| \leq \varepsilon \quad 1 \leq i \leq n.$$ 
Suppose $g \in G$ and let $\|g\| = 0$. For some $i$, $1 \leq i \leq n$

(3.2.1) $\|g - 0g_i\| \leq \varepsilon\|g\|$ 
and so

(3.2.2) $\|g - 0e_i\|^p \leq (2\delta^p + \varepsilon^p) \|g\|^p$
while

(3.2.3) $\delta_i^p \|g\|^p \leq d(g, E)^p + \varepsilon^p \|g\|^p.$

Combining (3.2.2) and (3.2.3) we obtain:

(3.2.4) $\|g - 0e_i\|^p \leq 2d(g, E)^p + 3\varepsilon^p \|g\|^p$

Now define $P : Y \to E$ by $P = Q + T (I - Q)$. Then $P$ is a projection. Suppose $y \in Y$ and $\|y\| = 1$; let $g = y - Qy$ and $\theta = \|g\|$. Choose $i$ so that (3.2.1) holds. Then by (3.2.4)

$$\|Qy + 0e_i\|^p \leq \|y\|^p + \|g - 0e_i\|^p$$
$$\leq 1 + 2d(g, E)^p + 3\varepsilon^p \|g\|^p$$
$$\leq 3 + 3\varepsilon^p \|g\|^p.$$ 

On the other hand

$$\|Tg - 0e_i\|^p \leq \lambda_0^p \|g - 0e_i\|^p + \|g\|^p \|Te_i - e_i\|^p$$
$$\leq 2\lambda_0^p d(g, E)^p + (3\lambda_0^p + 1) \varepsilon^p \|g\|^p.$$ 

Thus we have

$$\|Py\|^p \leq 2\lambda_0^p + 3 + \varepsilon^p \|g\|^p (3\lambda_0^p + 4) \leq 2\lambda_0^p + 4.$$ 

Setting $\lambda^p = 2\lambda_0^p + 4$ we have the desired conclusion.

**Lemma 3.3.** implies the following proposition whose proof we omit:

**Proposition 3.3.** Suppose $X$ is a quasi-Banach space and $E \subset F$ are closed subspaces of $X$. If $F$ is locally complemented in $X$ and $E$ is locally complemented in $F$, then $E$ is locally complemented in $X$. 

We shall say a closed subspace \( E \) of a quasi-Banach space \( X \) has the Compact Extension Property (CEP) in \( X \) if whenever \( Z \) is a quasi-Banach space and \( K : E \rightarrow Z \) is a compact operator then there is a compact operator \( K_1 : X \rightarrow Z \) with \( K_1 e = K e \) for \( e \in E \). An argument exactly as in Theorem 2.2 of [14] shows that if \( E \) has (CEP) then for any fixed \( r > 0 \) there is a constant \( \lambda \) so that whenever \( Z \) is an \( r \)-Banach space and \( K : E \rightarrow Z \) is compact then we can determine \( K_1 \) so that \( ||K_1|| \leq \lambda ||K|| \).

**Theorem 3.4.** If \( E \) is a locally complemented subspace of \( X \) then \( E \) has (CEP).

**Proof.** We shall not give full details here as this is a straightforward "Lindenstrauss compactness argument". If \( K : E \rightarrow Z \) is compact consider the net \( \{KP_Y\} \) where \( Y \) ranges over all subspaces of \( X \) with \( Y \supset E \) and \( \dim Y/E < \infty \) and \( P_Y : Y \rightarrow E \) is a uniformly bounded set of projections as in Lemma 3.2.

The next result is essentially known, but helps to clarify the situation for Banach spaces.

**Theorem 3.5.** Let \( X \) be a Banach space and let \( E \) be a closed subspace of \( X \). The following conditions on \( E \) are equivalent:

1. \( E \) has (CEP) in \( X \).
2. \( E \) is locally complemented in \( X \).
3. \( E^{**} \) is complemented in \( X^{**} \) under its natural embedding.
4. There is a linear extension operator \( L : E^* \rightarrow X^* \) such that \( L(e^*)(e) = e^*(e) \) for \( e \in E \) and \( e^* \in E^* \).

**Proof.** (2) \( \Rightarrow \) (1): Theorem 3.4.

(1) \( \Rightarrow \) (4): There is a constant \( \lambda \) so that wherever \( K : E \rightarrow Y \) is a compact operator into a Banach space \( Y \) then \( K \) has extension \( K_1 : X \rightarrow Y \) with \( ||K_1|| \leq \lambda ||K|| \).

Let \( G \) be a finite-dimensional subspace of \( E^* \) and let \( G^1 = \{ e \in E : g(e) = 0 \text{ for } g \in G \} \). Let \( Y \) be the quotient space \( E/G^1 \) and \( q : E \rightarrow Y \) be the quotient map. Then there exists a linear operator \( K : X \rightarrow Y \) with \( Ke = ge \) for \( e \in E \) and \( ||K|| \leq \lambda \). Now \( K^* : E^* \rightarrow X^* \), \( ||K^*|| \leq \lambda \) and \( K^* g = g(e) \) for \( g \in G \) and \( e \in E \). The conclusion of (4) can then be obtained by a standard compactness argument.

(4) \( \Rightarrow \) (3) The adjoint \( L^* : X^{**} \rightarrow E^{**} \) is a projection.

(3) \( \Rightarrow \) (2) This follows from Proposition 3.3 and the Principle of Local Reflexivity.

**Remark.** In general, so we shall see, the property (CEP) is strictly weaker than local complementation for a subspace.

### 4. Ultraproducts

The first part of the following theorem serves as a replacement in the non-locally convex setting for the Principle of Local Reflexivity.

**Theorem 4.1.** Suppose \( X \) is a quasi-Banach space, \( I \) is an index set and \( \mathcal{U} \) is a non-principal ultrafilter on \( I \).

1. \( X \) is locally complemented in \( X_{\mathcal{U}} \).
(2) If $Y$ is a locally complemented subspace of $X$ then $Y_\mathcal{U}$ is locally complemented in $X_\mathcal{U}$.

Proof. (1): Let $F$ be a finite-dimensional subspace of $X_\mathcal{U}$ and let $\{f^{(1)}, \ldots, f^{(n)}\}$ be a basis of $F$. We shall regard $f^{(i)}$ as members of $L_\infty(I; X)$ by selecting representatives. For each $i \in I$, define $T_i : F \to X$ by

$$T_i \left\{ \sum_{k=1}^{n} a_k f^{(k)} \right\} = \sum_{k=1}^{n} a_k f_i^{(k)}.$$ 

Clearly we have

$$\sup_{i \in I} ||T_i f|| < \infty$$

and

$$\lim_{\mathcal{U}} ||T_i f|| = ||f|| \quad f \in F.$$ 

By an elementary compactness argument $\lim_{\mathcal{U}} ||T_i|| = 1$. If $f \in F \cap X$ then

$$\lim_{\mathcal{U}} T_i f = f.$$ 

Again by a compactness argument we may select $i$ so that for any $\varepsilon > 0$, $||T_i f - f||^p \leq \varepsilon^p / 2 ||f||^p$ ($f \in F \cap X$) and $||T_i ||^p \leq 1 + \varepsilon^p / 2$. Letting $S = (1 + \varepsilon^p / 2)^{-1/p} T_i$ we have $||S|| \leq 1$ and $||S f - f|| \leq \varepsilon ||f||$ for $f \in F \cap X$.

(2): Here we may suppose that for some $\lambda$, we have, for every subspace $W$ of $X$ containing $Y$ with $\dim W/Y < \infty$, a projection $P : W \to Y$ with $||P|| \leq \lambda$. Again let $F$ be a finite-dimensional subspace of $X_\mathcal{U}$ and select a basis $\{f^{(1)}, \ldots, f^{(n)}\}$ for $F$. For each $i \in I$ let $W_i = [Y, f^{(1)}_i, \ldots, f^{(n)}_i]$ be the linear span of $Y$ and $f^{(1)}_i, \ldots, f^{(n)}_i$. Let $P_i : W_i \to Y$ be a projection with $||P_i|| \leq \lambda$. Define $T : F \to Y_\mathcal{U}$ by $T f = \{P_i f\}_{i \in I}$ for $f \in F$. Then $||T|| \leq \lambda$ and if $f \in Y_\mathcal{U}$ then $T f = f$.

Let us define a quasi-BANACH space $X$ to be an ultra-summand if $X$ is complemented in $X_\mathcal{U}$ for every ultraproduct $X_\mathcal{U}$ of $X$. Then we have:

**Theorem 4.2.** Let $X$ be a quasi-BANACH space and $E$ be a locally complemented subspace of $X$. Suppose $Y$ is an ultra-summand. Then any bounded linear operator $T_0 : E \to Y$ can be extended to a bounded linear operator $T : X \to Y$.

Proof. For an index set $\mathfrak{A}$ we take the collection of subspaces $W$ of $X$ with $W \supseteq E$ and $\dim W/E < \infty$. We let $\mathcal{U}$ be any ultrafilter on $\mathfrak{A}$ containing all subsets of $\mathfrak{A}$ of the form $\{ W \in \mathfrak{A} : W \supseteq W_0 \}$ for $W_0 \in \mathfrak{A}$. For each $W \in \mathfrak{A}$ there is a projection $P_W : W \to E$ so that $\sup ||P_W|| = \lambda < \infty$.

Define $T : X \to Y_\mathcal{U}$ by

$$(T x)_{W} = 0 \quad x \notin W$$

$$= T_0 P_W x \quad x \in W.$$ 

Then $T$ factors to a linear map into $Y_\mathcal{U}$ and $||T|| \leq \lambda ||T_0||$. If $Q : Y_\mathcal{U} \to Y$ is any projection then $T = QT$ provides the desired extension.

**Proposition 4.3.** A complemented subspace of a pseudo-dual space is an ultra-summand.
Proof. Suppose \( Y \) is a pseudo-dual space and \( P : Y \rightarrow X \) is a projection onto a closed subspace \( X \) of \( Y \). We may assume the unit ball of \( Y \) is compact in a Hausdorff vector topology \( \gamma \). If \( X_\mu \) is any ultraproduct of \( X \) then we can define \( Q : X_\mu \rightarrow X \) by

\[ Q(\{x_i\}) = P (\gamma - \lim \mu_i x_i). \]

Then \( Q \) is a projection of \( X_\mu \) onto \( X \).

**Theorem 4.4.** Consider the following properties of a quasi-Banach space \( X \):

1. \( X \) is an ultra-summand.
2. Whenever \( X \) is a locally complemented subspace of a quasi-Banach space \( Z \) then \( X \) is complemented in \( Z \).
3. \( X \) is isomorphic to a complemented subspace of a pseudo-dual space.

Then (1) and (2) are equivalent in general. If \( X \) has (BAP) then (1), (2) and (3) are equivalent.

Proof. (1) \Rightarrow (2): This follows directly from Theorems 4.1 and 4.2.

(2) \Rightarrow (3) when \( X \) has (BAP): Suppose \( T_n : X \rightarrow X \) is a sequence of finite-rank operators with \( T_n x \rightarrow x \) for \( x \in X \). Then \( \sup \|T_n\| = \lambda < \infty \). Form the space \( Z \) of all sequence \( \xi = (\xi_n)_{n=1}^{\infty} \) where \( \xi_n \in T_n(X) \) such that \( \|\xi\| = \sup \|\xi_n\| \leq \infty \). Then \( Z \) is a pseudo-dual space since its unit ball is compact for coordinatewise convergence. Define \( J : X \rightarrow Z \) by \( Jx = (T_n x)_{n=1}^{\infty} \). Then \( J \) is an isomorphic embedding of \( X \) into \( Z \). Define \( Q_k : Z \rightarrow J(X) \) by \( Q_k(\xi) = J\xi_k \); then \( \|Q_k\| \leq \|J\| \) and \( Q_k u \rightarrow u \) for \( u \in J(X) \). By Lemma 3.1, \( J(X) \) is locally complemented in \( Z \) and hence is complemented in \( Z \).

**Theorem 4.5.** Suppose \( E \) is a locally complemented subspace of \( X \). Then \( X/E \) is isomorphic to a locally complemented subspace of an ultraproduct \( X_\mu \) of \( X \).

Proof. Again let \( \mathcal{F} \) be the collection of all subspaces \( W \) of \( X \) with \( W \supseteq E \) and \( \dim W/E < \infty \). Let \( \mathcal{U} \) be an ultrafilter on \( \mathcal{F} \) containing all subsets of the form \( \{W : W \subseteq W_\mu\} \) for \( W_\mu \in \mathcal{F} \). There exist projections \( P_\mu : W \rightarrow E \) so that \( \sup \|P_\mu\| = \lambda < \infty \). Define \( Q : X \rightarrow X_\mu \) by

\[ (Qx)_\mu = 0 \quad x \notin W \]

\[ = x - P_\mu x \quad x \in W. \]

Again \( Q \) is linear into \( X_\mu \) (after factoring out sequences tending to zero through \( \mu \)) and \( \|Q\| \equiv (1 + \lambda^p)^{1/p} \) (where we assume \( X \) to be a \( p \)-Banach space). If \( x \in E \) then \( Qx = 0 \) and clearly in general,

\[ \|Qx\| \equiv d(x, E). \]

Thus \( Q \) factors to an embedding of \( X/E \) into \( X_\mu \). It remains to show that \( Q(X) \) is locally complemented in \( X_\mu \).

Let \( F \) be a finite-dimensional subspace of \( X_\mu \) with a basis \( \{f^{(1)}, \ldots, f^{(n)}\} \). For each \( W \in \mathcal{F} \) define \( T_\mu : F \rightarrow Q(X) \) by

\[ T_\mu \left( \sum_{j=1}^{n} a_j f^{(j)} \right) = \sum_{j=1}^{n} a_j Qf^{(j)} \mu. \]
Now \( \sup_{w} \| T_{w} \| < \infty \) and \( \lim_{w} \| T_{w} \| = \| Q \| \) as in the proof of Theorem 4.1. If \( f \in Q(X) \cap F \) then \( f = Qx \) for some \( x \in X \). Hence
\[
T_{w}f = Q(x - P_{n}x)
\]
eventually (as \( W \to \infty \) through \( \mathfrak{N} \)). Thus \( T_{w}f = Qx = f \) eventually.

Now we can clearly choose \( W \in \mathfrak{N} \) so that \( T_{w}f = f \) for \( f \in Q(X) \cap F \) and \( \| T_{w} \| \leq 2\| Q \| \), thus showing \( Q(X) \) is locally complemented in \( X_{\mathfrak{N}} \).

5. Bases

If a quasi-Banach space \( X \) has (BAP) then it is possible to give a generalization of Theorem 3.5.

Theorem 5.1. Suppose \( X \) is a quasi-Banach space with (BAP): Then a closed subspace \( E \) of \( X \) is locally complemented if and only if \( E \) has both (BAP) and (CEP).

Proof. Suppose first that \( E \) has (BAP) and (CEP). Then where is a sequence \( T_{n} : E \to E \) of finite-rank operators with \( T_{n}e = e \) for \( e \in E \) and \( \sup_{n} \| T_{n} \| < \infty \). Now by (CEP) (and remarks following the definition) there is a uniformly bounded sequence of operators \( Q_{n} : X \to T_{n}(E) \) such that \( Q_{n}e = T_{n}e \) for \( e \in E \). Now by Lemma 3.1, \( E \) is locally complemented.

Conversely supposed \( T_{n} : X \to X \) are finite-rank operators satisfying \( T_{n}x = x \) for \( x \in X \) and \( \sup_{n} \| T_{n} \| < \infty \). If \( E \) is locally complemented there are uniformly bounded projections \( P_{n} : E + T_{n}(X) \to E \). Define \( Q_{n} = P_{n}T_{n} \); then \( \sup_{n} \| Q_{n} \| < \infty \), \( Q_{n}(X) \subseteq E \) and \( Q_{n}e = e \) for \( e \in E \). Thus \( E \) has (BAP); it has (CEP) by Theorem 3.4.

Remark. See below Example 6.7.

Corollary 5.2. If \( X \) has (BAP) and \( E \) is locally complemented in \( X \) there is a sequence of operators \( S_{n} : X \to E \) such that \( \sup_{n} \| S_{n} \| < \infty \) and \( S_{n}e = e \) for \( e \in E \).

Now suppose \( X \) has a basis. It is unlikely that in general every complemented subspace of \( X \) has a basis. This would require for Banach spaces the equivalence of (BAP) and the existence of a basis; see Lindenstrauss and Tzafriri [18] p. 38 and p. 92. However under certain circumstances we shall show that a locally complemented subspace does have a basis.

Suppose \( X \) has a basis \( (b_{n}) \) and \( E \) is a closed subspace of \( X \). Let \( \Gamma \) be the linear span in \( X^{*} \) of the biorthogonal functionals \( (b_{n}^{*}) \). We shall say that \( E \) is residual in \( X \) if there is a uniformly bounded sequence of operators \( T_{n} : X \to E \) such that \( T_{n}^{*} \gamma = \gamma \) for \( \gamma \in \Gamma \) in the weak*-topology (i.e. \( \gamma(T_{n}x) \to \gamma(x) \) for \( x \in X \)).

We shall denote by \( P_{m} \) the partial summation operators with respect to the basis i.e.
\[
P_{m}x = \sum_{k=1}^{m} b_{k}^{*}(x) b_{k}.
\]
Let \( X_{0} \) be the algebraic linear span of \( (b_{k})_{k=1}^{\infty} \).
Our main theorem will be that every residual locally complemented subspace of $X$ has a basis. This theorem is similar in spirit to result of **Johnson, Rosenthal and Zippin** [7] on the existence of bases in Banach spaces. Our proof will be achieved in several steps; the first is:

**Lemma 5.3.** Suppose $E_0$ is a residual locally complemented subspace of $X$. Then there is a residual locally complemented subspace $E$ of $X$ isomorphic to $E_0$ and uniformly bounded sequences of finite-rank operators $S_n : X \to E \cap X_0 T_n : X \to E \cap X_0$ such that

1. $S_n e - e \in E$
2. $T_n^* \gamma \to \gamma$ weak* for $\gamma \in \Gamma$.

**Proof.** Since $E_0$ is residual and locally complemented there are uniformly bounded operators $\tilde{S}_n : X \to E_0$, $\tilde{T}_n : X \to E_0$ so that $\tilde{S}_n e_0 - e_0$ for $e_0 \in E_0$ and $\tilde{T}_n^* \gamma \to \gamma$ weak* for $\gamma \in \Gamma$.

Choose a countable dimensional dense subspace of $E_0$, $E_{00}$ say, such that $\tilde{S}_n(X_0) \subset E_{00}$ for $n \in \mathbb{N}$ and $\tilde{T}_n(X_0) \subset E_{00}$ for $n \in \mathbb{N}$. Since $\Gamma$ separates the points of $E_{00}$ it is possible to chose a Hamel basis $(w_n : n \in \mathbb{N})$ of $E_{00}$ such that the biorthogonal functionals $\varphi_n \in \Gamma$. Now for each $n \in \mathbb{N}$ choose $m(n) \in \mathbb{N}$ so that

$$\|w_n - P_{m(n)} w_n\|_p \leq 2^{-m(n)} \|\varphi_n\|_p.$$

Let $v_n = w_n - P_{m(n)} w_n$ and define $K : X \to X$ by

$$Kx = \sum_{n=1}^\infty \varphi_n(x) v_n.$$

Then $\|K\| < 1$ and so $A = I - K$ is invertible. Now let $E = A(E_0)$.

Clearly $\{A \tilde{S}_n A^{-1} : n \in \mathbb{N}\}$ is uniformly bounded and $A \tilde{S}_n A^{-1} e - e$ for $e \in E$. Let $S_n = A \tilde{S}_n A^{-1} T_n$, then $\{S_n : n \in \mathbb{N}\}$ is a uniformly bounded sequence of finite-rank operators and $S_n e - e$ for $e \in E$.

If $\gamma \in \Gamma$ and $x \in X$

$$\gamma( A \tilde{T}_n A^{-1} x) = \gamma( \tilde{T}_n A^{-1} x) - \gamma( K \tilde{T}_n A^{-1} x)$$

and

$$\gamma( \tilde{T}_n A^{-1} x) \to \gamma( A^{-1} x)$$

as $n \to \infty$.

On the other hand

$$\gamma( K \tilde{T}_n A^{-1} x) = \sum_{j=1}^\infty \varphi_j( T_n A^{-1} x) \gamma( v_j).$$

Now

$$|\varphi_j( \tilde{T}_n A^{-1} x)| |\gamma(v_j)| \leq C \|\varphi_j\| \|v_j\|.$$

where

$$C = (\sup \|\tilde{T}_n\|) \|A^{-1}\| \|\gamma\| \|x\|.$$
Hence, by a form of the Dominated Convergence Theorem,

$$\lim_{n \to \infty} \gamma(K \check{T}_n A^{-1}x) = \sum_{j=1}^{\infty} \varphi_j(A^{-1}x) \gamma(v_j)$$

noting that

$$\lim_{n \to \infty} \varphi_j(\check{T}_n A^{-1}x) = \varphi_j(A^{-1}x) \quad \text{since} \quad \varphi_j \in \Gamma.$$ 

Thus

$$\lim_{n \to \infty} \gamma(A \check{T}_n A^{-1}x) = \gamma(A^{-1}x) - \gamma(KA^{-1}x) = \gamma(x).$$

Now let $T_n = T_n P_n$; then (5.3.2) follows immediately.

**Lemma 5.4.** If $E$ satisfies the conclusions of Lemma 5.3, then there is a uniformly bounded sequence of finite-rank operators $V_n : X \to E \cap X_0$ such that

(5.4.1) $V_n e \to e, \quad e \in E$

(5.4.2) $P_n V_n = P_n, \quad n \in N.$

**Proof.** Let $W_n = S_n + T_n - T_n S_n$. Then for fixed $k$,

$$P_k T_n x = \sum_{i=1}^{k} T_n b_i^{*}(x) b_i$$

and $\|P_k T_n - P_k\| \to 0$, as $n \to \infty$. Hence $P_k W_n \to P_k$ as $n \to \infty$. Choose $m(k)$ an increasing sequence so that

$$\|P_k W_{m(k)} - P_k\|^p \leq \frac{1}{2} k^{-p} \quad k = 1, 2, \ldots$$

Then on $[b_1, \ldots, b_k]$, $P_k W_{m(k)}$ is invertible with inverse $A_k$ with

$$\|A_k - I\|^p \leq \frac{1}{2} k^{-p} (1 - \frac{1}{2} k^{-p})^{-1} \leq k^{-p}.$$ 

Let $V_k = W_{m(k)} A_k P_k$. Then $P_k V_k = P_k$ and $\{V_k\}$ is uniformly bounded. If $e \in E$

$$V_k e = P_k e = (W_{m(k)} A_k - I) P_k e$$

so that

$$\|V_k e - P_k e\|^p \leq \|W_{m(k)}\|^p k^{-p} \|e\| + \|(W_{m(k)} - I) P_k e\|^p$$

and

$$W_{m(k)} - I = (T_{m(k)} - I) (I - S_{m(k)}).$$

Hence

$$\|(W_{m(k)} - I) e\| \to 0$$

and

$$\|(W_{m(k)} - I) (e - P_k e)\| \to 0.$$

Thus $V_k e \to e$ for $e \in E.$
Lemma 5.5. If $E$ satisfies the conclusions of Lemma 5.4., then we can find a constant $\lambda$, an increasing sequence of positive integers $(h_n : n = 0, 1, 2, \ldots)$ with $h_0 = 0$ and $h_1 = 1$, and a (not necessarily continuous) linear operator $T : X_0 \to X_0$ such that

\begin{align}
(5.5.1) & \quad \text{If } G_n = \{ b_i : h_{i-1} < i \leq h_n \} \text{ for } n \geq 1, \text{ then } T(G_n) \subset G_{n+1}.
(5.5.2) & \quad \text{If } g \in G_n \text{ then } \| Tg \| \leq \lambda \| g \| \text{ and } \| T^p g \| \leq \lambda^p (d(g, E)^p + \lambda^{-2np}\| g \|^p)
(5.5.3) & \quad \text{If } x \in X_0, \text{ then } x - Tx \in E.
\end{align}

Proof. Choose $\lambda$ sufficiently large so that $\lambda^p > 2$,

$$\| P_m - P_n \| \leq \lambda, \quad m, n \geq 0$$

(where $P_0 = 0$) and

\begin{align}
\| V_n \| & \leq \lambda, \quad n \in \mathbb{N}.
\| I - V_n \| & \leq \lambda, \quad n \in \mathbb{N}.
\end{align}

Next observe that if $x \in X$ and $\epsilon > 0$ then we can find $\epsilon \in E$ so that

$$\| x - \epsilon \| < d(x, E)^p + \epsilon \| x \|^p$$

and as $(I - V_n) \epsilon \to 0$, for large enough $n$ we have

$$\| (I - V_n) x \|^p \leq \lambda^p (d(x, E)^p + \| x \|^p).$$

By obvious compactness argument if $F$ is a finite-dimensional subspace of $X$ we can choose $n \in N$ so that (5.5.4) holds for any $x \in F$.

Using this remark it is possible to construct two increasing sequences of positive integers $\{ h_n : n = 0, 1, 2, \ldots \}$ and $\{ m_n : n = 1, 2, 3, \ldots \}$ so that $h_0 = 0$, $h_1 = 1$, $m_n \geq h_n$ and

\begin{align}
(5.5.5) & \quad (I - V_{m(n)}) (G_n) \subset G_{n+1}
(5.5.6) & \quad \| (I - V_{m(n)}) g \|^p \leq \lambda^p (d(g, E)^p + \lambda^{-2np}\| g \|^p), \quad g \in G_n.
\end{align}

Here we have used the fact that $P_{m(n)} (I - V_{m(n)}) = 0$.

Let $T : X_0 \to X_0$ be the linear map defined by $Tg = (I - V_{m(n)}) g$ for $g \in G_n$. Then the lemma follows.

Theorem 5.6. If $E$ is a residual locally complemented subspace of a space $X$ with a basis; then $E$ also has a basis.

Proof. We may assume that $E$ satisfies the conclusions of Lemma 5.5. We start with some observations where we let $T_0 = I$.

\begin{align}
(5.6.1) & \quad P_n T^j = 0 \quad h_j = n
(5.6.2) & \quad P_n T^j = P_n T^j P_n \quad h_j < n, \quad j \geq 0
(5.6.3) & \quad T P_{h_j} - P_{h_j} T = T Q_j \quad j = 1, 2, 3, \ldots
\end{align}

where

$$Q_j = P_{h_j} - P_{h_j-1} \quad (j = 1, 2, 3, \ldots).$$

Note that $(I - T) (X_0) \subset E$ and define $w_n = (I - T) b_n$. Clearly $b_n^*(w_n) = 1$ so that
$w_n \neq 0$. We shall show that $(w_n)$ is a basis for $E$. To this end we define a sequence of operators $U_n : X \to E$ by

$$U_n = (I - T) P_n \left( \sum_{j=0}^{k-1} T^j \right) P_n$$

where $h_{k-1} < n \leq h_k$. If $x \in X_0$,

$$U_n x = (I - T) P_n \sum_{j=0}^{k-1} T^j x .$$

by (5.5.1). If $1 \leq l \leq n$

$$U_n w_l = (I - T) P_n \left( \sum_{j=0}^{k-1} T^j (I - T) b_l \right)$$

$$= (I - T) P_n (I - T^k) b_l = (I - T) b_l = w_l .$$

Similarly if $l > n$, $U_n w_l = 0$. Similar calculations show that $U_m U_n = U_n U_m = U_n$ whenever $m \geq n$. Thus to show $(w_n)$ is a basis it will suffice to show $U_n e = e$ for $e \in E$.

First we make a preliminary calculation; suppose $x \in X_0$ and $k \geq 1$. Let

$$y = \sum_{j=0}^{k-1} T^j Q_{k-j} x .$$

Then $y \in G_k$ and $\sum_{j=0}^{k-1} (T^j - I) Q_{k-j} x \in E$. Hence

$$d(y, E) = d(P_{h_k} x, E) .$$

Also

$$\| T^j Q_{k-j} x \| \leq \lambda^j \| Q_{k-j} x \|$$

by (5.5.2). Thus

$$\| y \|_p \leq \left( \sum_{j=0}^{k-1} \lambda^{(j+1)p} \right) \| x \|_p \leq \lambda^{(k+1)p} \| x \|_p$$

since $\lambda^p > 2$. Returning to 5.5.2, we have

$$(5.6.4) \quad \left\| T \left( \sum_{j=0}^{k-1} T^j Q_{k-j} x \right) \right\|_p \leq \lambda^p \left( d(P_{h_k} x, E) \right)^p + \lambda^{(k-1)p} \| x \|_p$$

and in particular,

$$(5.6.5) \quad \left\| T \left( \sum_{j=0}^{k-1} T^j Q_{k-j} \right) \right\|_p \leq \lambda^{2p} + \lambda^{2(k-1)p} \leq \lambda^{3p}$$

Now for any $k$

$$P_{h_k} - U_{h_k} = TQ_k \sum_{j=0}^{k-1} T^j P_{h_k} .$$
by (5.6.3). Thus
\[
P_{k} - U_{k} = T \left( \sum_{j=0}^{k-1} T'^{j} Q_{k-j} \right) P_{k}
\]
so that
\[
\|P_{k} - U_{k}\| \leq \lambda^{k}. 
\]
Finally \(\|U_{k}\| \leq \lambda^{k}\) for all \(k\). If \(e \in E\)
\[
\|P_{k} e - U_{k} e\| \leq \lambda^{p} \left[ (d(P_{k} e, E))^{p} + \lambda^{(1-k)p} \|P_{k} e\|^{p} \right]
\]
by (5.6.4) and so \(P_{k} e - U_{k} e \to 0\) i.e. \(U_{k} e \to e\).

If \(h_{k+1} < n \equiv h_{k}\), then
\[
U_{k} e - U_{n} = (I - T) (P_{k} - P_{n}) Q_{k} \left( \sum_{j=0}^{k-1} T'^{j} \right) P_{k}
\]
\[
= (I - T) (P_{k} - P_{n}) + (I - T) (P_{k} - P_{n}) Q_{k} \left( \sum_{j=1}^{k-1} T'^{j} \right) P_{k}. 
\]

Now
\[
Q_{k} \left( \sum_{j=1}^{k-1} T'^{j} \right) P_{k} = \sum_{j=1}^{k-1} T'^{j} Q_{k-j} P_{k} = T \left( \sum_{j=0}^{k-2} T'^{j} Q_{k-1-j} \right) P_{k}.
\]

Hence by (5.6.5)
\[
\|U_{k} - U_{n}\|^{p} \leq (\lambda^{p} + 1) \lambda^{p} [1 + \lambda^{3p}] \leq \lambda^{7p}.
\]
We conclude \(\|U_{n}\| \leq \lambda^{n}\) for all \(n \in \mathbb{N}\). If \(h_{k+1} < n \equiv h_{k}\) then for \(e \in E\)
\[
eq U_{n} e = (I - U_{n}) (e - U_{k+1} e)
\]
\[
\to 0 \quad \text{and} \quad n \to \infty.
\]
Thus \((w_{n})\) is a basis for \(E\).

**Theorem 5.7.** If \(X\) is a quasi-BANACH space with a basis and \(E\) is a weakly dense locally complemented subspace of \(X\) then \(E\) also has a basis.

**Proof.** There is a uniformly bounded sequence of operators \(S_{n} : X \to E\) with \(S_{n} e \to e\) for \(e \in E\). Then if \(\gamma \in \Gamma\), consider the map \(A : X \to l_{\infty}\) defined by \(Ax = (\gamma (x - S_{n} x))_{n=1}^{\infty}\). Since \(l_{\infty}\) is locally convex then \(A^{-1}(c_{0})\) is weakly closed. However \(A^{-1}(c_{0}) \supset E\) is weakly dense so that \(A(X) \subset c_{0}\) i.e. \(S_{n}^{*} \gamma \to \gamma\). Thus \(E\) is also residual with \(T_{n} = S_{n}\).

6. \(l_{p}\)-spaces when \(0 < p < 1\)

We shall say that a quasi-BANACH space \(X\) is an \(l_{p}\)-space for \(0 < p < 1\) if it is isomorphic to a locally complemented subspace of a space \(L_{p}(\Omega, \Sigma, \mu)\) where \((\Omega, \Sigma, \mu)\) is measure space.
Let us note that the standard definition of an \( \ell_p \)-space for \( 1 \leq p \leq \infty \) due to Lindenstrauss and Pełczyński [15] is local in character. \( X \) is an \( \ell_p \)-space if for some constant \( \lambda \) and for every finite-dimensional subspace \( F \) of \( X \) there is a finite-dimensional subspace \( G \) containing \( F \) an isomorphism \( S : G \to l_p^n \) (where \( n = \text{dim} \ G \)) with \( ||S|| \cdot ||S^{-1}|| \leq \lambda \). The problem with this definition for \( 0 < p < 1 \) (pointed out to us by W. J. Stiles) is that it is by no means clear that even \( L_p(0,1) \) satisfies this condition. A possible alternative would be to define \( X \) to be an \( \ell_p \)-space if there is a constant \( A \) and an increasing net of finite-dimensional subspaces \( (B, : x \in A) \) with \( U E_x \) dense in \( X \) and isomorphisms \( S_x : E_x \to l_p^n \) with \( ||S_x|| \cdot ||S_x^{-1}|| \leq \lambda \). This definition was adopted in [12]. It is a consequence of Theorem 6.1 below that every such space is an \( \ell_p \)-space in our sense here, but we do not know whether the converse holds.

In our opinion, the definition given above would serve as a natural definition for all \( p, 0 < p \leq \infty \). However for \( 1 < p < \infty \), it would make a Banach space \( X \) an \( \ell_p \)-space if and only if it is a complemented subspace of a space \( L_p(\Omega, \Sigma, \mu) \). The standard definition makes \( X \) an \( L_p \)-space if it is a complemented non-Hilbertian subspace of a space \( L_p(\Omega, \Sigma, \mu) \) [17]. For \( p = 1 \) or \( p = \infty \) our definition is the same as the standard one. The equivalence follows easily from Theorem 3.5 and results in [17] (Corollary to Theorem 3.2, and Theorem 3.3 (a)).

Note that every \( \ell_p \)-space is (isomorphic to) a \( p \)-Banach space when \( 0 < p \leq 1 \).

The following theorem lists several equivalent formulations of the statement that \( X \) is an \( \ell_p \)-space.

**Theorem 6.1.** Let \( X \) be a \( p \)-Banach space where \( 0 < p \leq 1 \). The following conditions on \( X \) are equivalent:

1. \( X \) is a \( \ell_p \)-space
2. \( X \) is isomorphic to a locally complemented subspace of some \( \ell_p \)-space
3. \( X \) is isomorphic to the quotient of a \( \ell_p \)-space by a locally complemented subspace
4. \( X \) is isomorphic to the quotient of a space \( l_p(I) \) by a locally complemented subspace.
5. Whenever \( Z \) is a \( p \)-Banach space and \( Q : Z \to X \) is an open map then ker \( Q \) is locally complemented in \( Z \).
6. There is a constant \( \lambda \) such that whenever \( F \) is a finite-dimensional subspace of \( X \) and \( \epsilon > 0 \) there are linear operators \( S : F \to l_p, T : l_p \to X \) with \( ||S|| \cdot ||T|| \leq \lambda \) and \( ||TSf - f|| \leq \epsilon ||f|| \) for \( f \in F \).

**Proof.** (1) \( \Rightarrow \) (2) follows from Proposition 3.3. Since every \( p \)-Banach space is a quotient of \( l_p(I) \) for some index set \( I \), we have (5) \( \Rightarrow \) (4) \( \Rightarrow \) (3). To conclude the proof we shall show (1) \( \Rightarrow \) (6), (6) \( \Rightarrow \) (5) and (3) \( \Rightarrow \) (1).

(1) \( \Rightarrow \) (6): We suppose \( X \) is a locally complemented subspace of \( L_p(\Omega, \Sigma, \mu) \). Let \( \lambda \) be a constant so that whenever \( Y \supseteq X \) there is a projection \( P_Y : Y \to X \) with \( ||P_Y|| \leq \lambda \). Suppose \( F \subseteq X \) is a finite-dimensional subspace and \( \epsilon > 0 \). By a
routine approximation argument there is a finite subalgebra $\Sigma_0$ of $\Sigma$ and a linear map $S : F \to L_p(\Omega, \Sigma_0, \mu)$ with $\|S\| \leq 1$ and $\|Sf - f\| \leq \lambda^\ell \|f\|$. Let $Y = X + L_p(\Omega, \Sigma_0, \mu)$ and let $T = P_X|L_p(\Omega, \Sigma_0, \mu)$. Then $\|T\| \leq \lambda$ and $\|TSf - f\| \leq \varepsilon \|f\|$ for $f \in F$. Since $L_p(\Omega, \Sigma_0, \mu)$ is isometric to a subspace of $L_p$ which is the range of a norm-one projection, (1) follows.

(6) $\Rightarrow$ (5). For convenience we may suppose $Q$ is a quotient map. Let $F$ be a subspace of $Z$ of dimension $n$ with a basis $f_1, \ldots, f_n$ where $\|f_i\| = 1$ for $1 \leq i \leq n$. Suppose $0 < \varepsilon < 1$ and let $\alpha > 0$ be a constant so that

$$\left| \sum_{i=1}^n a_i f_i \right| \leq \alpha (\sum_{i=1}^n |a_i|^p)^{1/p}$$

for all $a_1, \ldots, a_n$. Choose operators $S : Q(F) \to L_p$ and $T : L_p \to X$ with $\|T\| \cdot \|S\| \leq \lambda$ and

$$\|TSQf - Qf\| \leq \frac{1}{2} \alpha \|Qf\| \quad f \in F.$$ 

Since $L_p$ is projective for $p$-BANACH spaces there is an operator $T_1 : l_p \to Z$ with $|T_1| = \|T\|$ and $QT_1 = T$. Define $R : F \to Z$ by $R = I - T_1S$. Then $QR = (I - TS)Q$ and $\|QR\| \leq \frac{1}{2} \alpha \varepsilon$. Thus we can find $g_1, \ldots, g_n \in Z$ with $Qg_i = QRf_i$ and $\|g_i\| < \alpha \varepsilon$.

Define $L : F \to Z$ by $Lf_i = g_i$. Then

$$\left| \sum_{i=1}^n a_i f_i \right| \leq \alpha \left( \sum_{i=1}^n |a_i|^p \right)^{1/p} \leq \alpha \left( \sum_{i=1}^n a_i f_i \right).$$

Hence $\|L\| \leq \alpha \varepsilon$. Let $V = R - L$; then $V(F) \subset \ker Q$ and $\|V\| \leq (\lambda^p + 2)^{1/p}$. If $f \in F \cap \ker Q$, we have $Rf = f$ and $\|f - Vf\| \leq \alpha \varepsilon \|f\|$. Hence $\ker Q$ is locally complemented in $Z$.

(3) $\Rightarrow$ (1). We may suppose $X$ is the quotient of a space $Y$ by a locally complemented subspace, where $Y$ is itself a locally complemented subspace of $L_p(\Omega, \Sigma, \mu)$.

By Theorem 4.4, $X$ is isomorphic to a locally complemented subspace of an ultraproduct $Y_{\mu}$ of $Y$. By Theorem 4.1, $Y_{\mu}$ is isomorphic to a locally complemented subspace of $(L_p(\Omega, \Sigma, \mu))_{\mu_0}$, which by SCHREIBER'S Theorem 2.1 is a space $L_p(\Omega_1, \Sigma_1, \mu_1)$. Hence by Proposition 3.3, $X$ is a $\ell_p$-space.

Separable infinite-dimensional spaces $L_p(\Omega, \Sigma, \mu)$ for $0 < p < 1$ are isomorphic to one of the spaces $l_p$, $L_p$ or $l_p \oplus L_p$. Based on this, we define, for $0 < p < 1$, a discrete $\ell_p$-space to be a separable $\ell_p$-space isomorphic to a locally complemented subspace of $L_p$. We also define $X$ to be a continuous $\ell_p$-space if it is isomorphic to a locally complemented subspace of $L_p$.

We shall say that a separable $\ell_p$-space is a hybrid $\ell_p$-space if it is neither discrete nor continuous.

**Theorem 6.4.** Let $X$ be a separable $\ell_p$-space where $0 < p < 1$. The following conditions on $X$ are equivalent:

1. $X$ is a discrete $\ell_p$-space
2. $X$ has (BAP)
3. $X$ has a basis.
Proof. (1) ⇒ (3) Suppose $X$ is a locally complemented subspace of $l_p$, which is non-isomorphic to $l_p$. Let $l_p/X = Y$, so that $Y$ is also an $\mathcal{L}_p$-space. Then there is a quotient map $U : l_p \rightarrow Y$ which takes the unit vector basis $(e_n)$ of $l_p$ to a sequence $Ue_n$, dense in the unit ball of $Y$. By Corollary 2.3, $\ker U \cong X$, and is locally complemented in $l_p$.

For each $n \in \mathbb{N}$ select for $1 \leq k \leq n$, $u_{n,k} \in l_p$ with $u_{n,k} \in [e_{n+1}, e_{n+2}, \ldots]$, $Uu_{n,k} = Ue_k$ and $\|u_{n,k}\| \leq 2\|Ue_n\|$. Define $T_n : l_p \rightarrow \ker U$ by

$$T_n \left( \sum_{i=1}^n a_i e_i \right) = \sum_{i=1}^n a_i (e_i - u_{n,i}).$$

Then $|T_n| \leq (1 + 2^n)^{1/p}$ and $T_n^* \gamma \rightarrow \gamma$ weak* for $\gamma$ in the linear span of the biorthogonal functionals $(e_k^*)$. Hence $\ker U$ is residual, and we can apply Theorem 5.6 to deduce that $\ker U$ (and hence $X$) has a basis.

(3) ⇒ (2): Immediate

(2) ⇒ (1): We may suppose $X$ is a locally complemented subspace of $l_p \oplus L_p$. From the proof of Theorem 5.1 it is easy to see there is a uniformly bounded sequence of finite-rank operators $S_n : l_p \oplus L_p \rightarrow X$ with $S_n x \rightarrow x$ for $x \in X$. Let $P : l_p \oplus L_p \rightarrow l_p \oplus L_p$ by defined by $P(u, v) = (u, 0)$. Clearly $S_n = S_n P$ and so $S_n P x \rightarrow x$ for $x \in X$. Thus $P$ maps $X$ isomorphically onto a space $P(X)$ of $\mathcal{L}_p \oplus \{0\}$. Now $P S_n P x \rightarrow x$ for $x \in P(X)$ and so by Lemma 3.1, $P(X)$ is also locally complemented in $l_p$. Thus $X$ is a discrete $\mathcal{L}_p$-space.

Remark. Every separable $\mathcal{L}_p$-space ($1 \leq p \leq \infty$) has a basis [7].

Theorem 6.5. Let $X$ be a separable $\mathcal{L}_p$-space where $0 < p < 1$. Then $X$ is continuous if and only if $X^* = \{0\}$.

Proof. If $X$ is locally complemented in $L_p$, then $X$ has HBEP i.e. $X^* = \{0\}$. Conversely if $X \subset L_p \oplus L_p$ and $X^* = \{0\}$ then $X \subset \{0\} \oplus L_p$.

A nice property of continuous $\mathcal{L}_p$-spaces is given by:

Theorem 6.6. Let $X$ be a $p$-Banach space and let $Y$ be a continuous $\mathcal{L}_p$-space. Suppose $Q : X \rightarrow Y$ is an open mapping. Then (a) if $X$ has (BAP), $\ker Q$ has (BAP) and (b) if $X$ has a basis, $\ker Q$ has a basis.

Proof. Since $Y^* = \{0\}$, $\ker Q$ is weakly dense in $X$. Simply apply Theorems 5.1, 5.7 and 6.1 (5).

Example 6.7 Let $C$ denote the subspace of $L_p$ of constant functions. Since $C$ fails to have HBEP, $C$ is not locally complemented. Thus $L_p/C$ is not a $\mathcal{L}_p$-space (see [12], where essentially this argument is invoked to show $L_p/C \cong L_p$). However $L_p/C$ is isomorphic to a subspace of $L_p$ by the embedding $T : L_p/C \rightarrow L_p ([0, 1) \times \times (0, 1)]$ given by [(13)]

$$Tq(s, t) = f(s) - f(t) \quad s, t \in (0, 1).$$

where $q : L_p \rightarrow L_p/C$ is the quotient map. Let $Y$ be this subspace of $L_p$. Now let $Q : l_p \rightarrow L_p$ be any quotient map and let $Z \subset l_p$ be defined by $Z = Q^{-1}(Y)$. We claim that $Z$ has (CEP). Indeed if $T : Z \rightarrow W$ is a compact operator then $T|\ker Q$. 


is compact and as ker \( Q \) is locally complemented it has an extension \( T_1 : L_p \to W \) which is also compact. Now \( T_1 - T \) factors to a compact operator on \( Y \cong L_p \vert \mathcal{C} \). Since there are no non-zero compact operators on \( L_p \), \( T_1 = T \) on \( Z \).

However \( Z \) is not locally complemented, since if it were \( L_p \vert Z \cong L_p \vert \mathcal{C} \) would be an \( \mathcal{L}_p \)-space. We conclude that \( Z \) also fails (BAP) by Theorem 5.1.

7. Example of \( \mathcal{L}_p \)-spaces

If \( 0 < p < 1 \), it is rather easy to construct numerous mutually non-isomorphic examples of separable \( \mathcal{L}_p \)-spaces. This contrasts with the case \( p=1 \) (see [6]). The construction used by Johnson and Lindenstrauss in [6] can be adapted to the case \( p<1 \) to construct examples which are in general hybrids. We shall however take another route to construct examples. The following observation is routine:

**Theorem 7.1.** Let \( X \) be a separable \( \mathcal{L}_p \)-space. Then \( L_p(X) \) is a continuous \( \mathcal{L}_p \)-space.

As we shall see, the converse of Theorem 7.1 is false, for \( 0 < p < 1 \). We can construct examples where \( X \) is not an \( \mathcal{L}_p \)-space but \( L_p(X) \) is. For \( p=1 \) this is impossible since \( L_1(X) \) contains a complemented copy of \( X \), when \( X \) is locally convex.

**Theorem 7.2.** Let \((\Omega, \Sigma, \mu)\) be a non-atomic measure space and let \( \Sigma_0 \) be a sub-\( \sigma \)-algebra of \( \Sigma \). For \( 0 < p < 1 \), let \( L_p(\Sigma_0) \) denote the closed subspace of \( L_p(\Omega, \Sigma, \mu) \) of all \( \Sigma_0 \)-measurable functions, and let \( \Lambda(\Sigma_0) \) denote the quotient \( L_p(\Omega, \Sigma, \mu)/L_p(\Sigma_0) \). Then the following statements are equivalent:

1. \( L_p(\Sigma_0) \) is locally complemented in \( L_p(\Omega, \Sigma, \mu) \)
2. \( \Lambda(\Sigma_0) \) is an \( \mathcal{L}_p \)-space
3. \( L_p(\Sigma_0)^\ast = \{0\} \)
4. \( \mu \mid \Sigma_0 \) is non-atomic.

**Proof.** It follows from Theorem 6.1 that (1) and (2) are equivalent, and the equivalence of (3) and (4) is classical (cf. [4]). Since (1) implies that \( L_p(\Sigma_0) \) has (HBEP) we have (1) \( \Rightarrow \) (3). We complete the proof by showing (4) \( \Rightarrow \) (1).

Consider the net \( \Sigma_s \) (under containment) of finite subalgebras of \( \Sigma \). For each \( \Sigma_s \), let \( A_1, \ldots, A_k \) be the atoms of \( \Sigma_s \cap \Sigma_0 \) and let \( (B_{ij} : 1 \leq j \leq m(i)) \) be the atoms of \( \Sigma_s \) contained in \( A_i \). Then there are disjoint sets \( (C_{ij} : 1 \leq j \leq m(i)) \) in \( \Sigma_0 \) such that

\[
\bigcup_{j=1}^{m(i)} C_{ij} = A_i
\]

\[
\mu(C_{ij}) = \mu(B_{ij}).
\]

Define \( Q_s : L_p(\Sigma_s) \to L_p(\Sigma_0) \) by \( Q_s(1_{B_{ij}}) = 1_{C_{ij}} \) for \( 1 \leq j \leq m(i) \) and \( 1 \leq i \leq k \). \( Q_s \) is an isometry, and \( Q_s f = f \) for \( f \in L_p(\Sigma_s \cap \Sigma_0) \). Now apply Lemma 3.1.
Example 7.3. If we take $\Omega$ a Polish space and $\Sigma$ the Borel sets in $\Omega$ and $\mu$ a nonatomic probability measure, then it is shown in [11] that $A(\Sigma_0) \cong L_p$ implies $L_p(\Sigma_0)$ is complemented in $L_p$.

As a special case consider $\Omega = (0, 1) \times (0, 1)$ with ordinary Lebesgue area measure and let $\Sigma_0$ be sets of the form $(0, 1) \times B$ where $B$ is a Borel subset of $(0, 1)$. Then $A(\Sigma_0) \cong L_p(L_p | C)$ whose $C$ is the space of constants in $L_p$. This is an $\ell_p$-space, but as seen in Example 6.7, $L_p | C$ is not an $\ell_p$-space.

Example 7.4. We now show how to construct an uncountable family of separable $p$-Banach spaces $(E_q : p < q \leq 1)$ so that

(7.4.1) $E_q$ is $p$-trivial [10] i.e. $\ell(L_p, E_q) = 0$

(7.4.2) There is a quotient map $Q : l_p \rightarrow E_q$ with $\ker Q \cong l_p$.

(7.4.3) The spaces $L_p(E_q)$ are mutually non-isomorphic $\ell_p$-spaces.

We start by letting $H$ be the subspace of $l_p$ spanned by the basic sequence $(e_{2m-1} + e_{2m} : m = 1, 2 \ldots)$ (where $(e_m)$ is the standard basis of $l_p$). Let $(A_m)$ be a Borel partitioning of $(0, 1)$ into sets of positive measure and suppose $A_m = B_{2m-1} \cup B_{2m}$ where $B_{2m-1} \cap B_{2m} = \emptyset$ and $\mu(B_{2m-1}) = \mu(B_{2m}) = 1/2\mu(A_m)$ where $\mu$ is Lebesgue measure on $(0, 1)$. Define an isometry $V : l_p \rightarrow L_p$ by

$$V(e_k) = \mu(B_k)^{-1/p} 1_{B_k} \quad k = 1, 2, \ldots$$

For $p < q \leq 1$, define $T_q : H \rightarrow l_p$ by

$$T_q (e_{2m-1} + e_{2m}) = 2^{1/q} e_m.$$ 

Then $\|T_q\| = 2^{1/q} - 1/p < 1$, and let $G_q = (I - T_q) H$. Then $G_q \cong l_p$. Define $E_q = l_p | G_q$. Then (7.4.2) is immediate, and (7.4.1) follows from the lifting theorems of [12].

Next we show $L_p(E_q)$ is an $\ell_p$-space. For each $m \in N$ we find $f_{2m-1}, f_{2m} \in L_p$ with

$$\|f_{2m-1}\|^p = \|f_{2m}\|^p = \frac{1}{2} \|T_q (e_{2m-1} + e_{2m})\|^p$$

and

$$f_{2m-1} + f_{2m} = VT_q (e_{2m-1} + e_{2m}).$$

Now there is an operator $U : L_p \rightarrow L_p$ with

$$U(1_{B_m}) = \mu(B_m)^{1/p} f_m \quad m \in N$$

and $\|U\|^p \leq \sup \|f_m\|^p < 1$. Clearly $VT_q x = UVx$ for $x \in H$: Thus $V(G_q) = (I - U) V(H)$ and $(I - U)$ is invertible.

Consider $L_p(V(G_q)) \subset L_p(L_p)$. By the above, there is an automorphism of $L_p(L_p)$ carrying $L_p(V(H_q))$ onto $L_p(V(H))$. However if we identify $L_p(L_p)$ as $L_p((0, 1) \times (0, 1))$ then $L_p(V(H))$ is identified with $L_p(\Sigma_0)$ where $\Sigma_0$ is the algebra generated by sets of the form $C \times B$ where $C$ is a Borel subset of $(0, 1)$ and $B$ is in the $\sigma$-algebra generated by $(B_k : k \in N)$. Thus $L_p(V(H))$ is locally comple-
meted. It is then also locally complemented in the smaller space $L_p(V(l_p))$ and thus $L_p(G_q)$ is locally complemented in $L_p(l_p)$ ($\cong L_p$). Hence $L_p$ of $G_q = L_p(E_q)$ is an $\ell_p$-space.

Finally we show these spaces are mutually non-isomorphic. If $p < r \leq 1$, there is no non-zero continuous linear operator from $E_q$ into $r$-Banach space if and only if $G_q$ is dense in $l_r$. If $r > q$, then $G_q$ is dense in $l_r$ since its closure contains the range of the invertible operator $A : l_r \rightarrow l_r$ given by

$$Ae_m = e_m - 2^{-\frac{1}{q}}(e_{2m-1} + e_{2m}).$$

(Here $\|A - I\|^r \equiv 2^{1-r/q}$ on $l_r$). On the other hand if $r < q$, then $\|Tx\| \equiv 2^{\frac{1}{q} - 1} \|x\|$ for $x \in H$ in $l_r$-norm, and so the closure of $G_q$ in $l_r$ has $e_{2m-1} + e_{2m} - 2^{1/q}e_m$ as a basis, equivalent to the usual $l_r$-basis. However $e_1 \notin G_q$ since if

$$e_1 = \sum_{m=1}^\infty c_m (e_{2m-1} + e_{2m} - 2^{1/q}e_m)$$

then solving coordinatewise $c_1 = -2^{-\frac{1}{q}}$, $c_2 = c_3 = 2^{-\frac{3}{q}}$, $c_4 = c_5 = c_6 = c_7 = -2^{-3/q}$ etc. and

$$\sum_{k=1}^\infty |c_k|^q = \sum_{n=1}^\infty 2^{n-1-n/r} = \infty.$$  

Thus the spaces $E_q$ are mutually non-isomorphic and even more, so are the spaces $L_p(E_q)$. Now by Theorem 8.4 of [11], the spaces $L_p(E_q)$ are mutually non-isomorphic.

Remarks. It can be shown that the containing $q$-Banach space of $E_q$ is isomorphic to $L_q$.

Also we note that if $G_q$ is the kernel of a quotient map of $l_p$ onto $L_p(E_q)$ then the spaces $G_q$ are mutually non-isomorphic discrete $\ell_p$-spaces. For suppose $S : G_q \rightarrow G_r$ is an isomorphism. Then since $G_q$ is locally complemented in $l_p$, and $l_p$ is an ultra-summand, Theorem 4.2 gives an extension $S_1 : l_p \rightarrow l_p$ of $S$. Similarly $S^{-1}$ has an extension $S_2 : l_p \rightarrow l_p$ and $S_2 S_1 : l_p \rightarrow l_p$ extends the identity from $G_q$ to itself. Since $G_q$ is weakly dense, $S_2 S_1 = I$, and similarly $S_1 S_2 = I$ so that $l_p/G_q \cong \cong l_p/G_r$, a contradiction.

8. Lifting theorems for continuous $\ell_p$-spaces

Lemma 8.1. Let $X$ be a continuous $\ell_p$-space and let $Y$ be an ultra-summand. Then $L(X, Y) = \{0\}$.

Proof. $X$ is isomorphic to a locally complemented subspace of $L_q(0, 1)$; $L_p(0, 1)$ is isomorphic to a locally complemented subspace of $L_p([0, 1])$ where $\Gamma$ is any set whose cardinality exceeds that of $Y$ by Theorem 7.2. Hence by Theorem 4.2 it suffices to consider maps $T : L_p([0, 1]) \rightarrow Y$. Suppose $f \notin L_p([0, 1])$ is simple. Then there is a set of functions $(r_\gamma : \gamma \in \Gamma)$ mutually independent and independent
of \( f \) so that \( \mu (r_{\gamma} = +1) = \mu (r_{\gamma} = -1) = \frac{1}{2} \) [We denote by \( \mu \) the product measure in \((0, 1)^{\Gamma}\)]. By Khintchine's inequality,

\[
(8.1.1) \quad \left\| \Sigma a_{\gamma} r_{\gamma} f \right\| \leq C \left( \Sigma |a_{\gamma}|^2 \right)^{\frac{1}{2}}
\]

for some constant \( C \) whenever \( a_{\gamma} \) is finitely non-zero. Since \( |\Gamma| > |Y| \), there are infinitely many \( \gamma \) with \( T(r_{\gamma} f) = g \) for some \( g \in Y \). By (8.1.1) we must have \( g = 0 \).

Theorem 8.2. Let \( X \) be a \( p \)-Banach space, and let \( N \) be a closed subspace of \( X \) such that \( X/N \) is a continuous \( \mathbb{L}_p \)-space. Let \( Z \) be any quasi-Banach space and let \( T : N \to Z \) be a bounded linear operator. Each of the following conditions implies \( T \) has a unique extension \( T_1 : X \to Z \):

1. \( T \) is compact (and then \( T_1 \) is compact)
2. \( Z \) is \( q \)-convex for some \( q > p \)
3. \( Z \) is an ultra-summand.

Proof. Let \( Q : l_p(I) \to X \) be a quotient map, and consider \( S : Q^{-1}(N) \to Z \).

Then \( Q^{-1}(N) \) is locally complemented and so in cases (1) and (3) \( S \) has an extension \( S_1 : l_p(I) \to Z \) which is compact in case (1). In case (2) we appeal to the non-separable version of Theorem 5.1. There is a uniformly bounded set of finite-rank operators \( V_{x} : l_p(I) \to Q^{-1}(N) \) so that \( V_{x} x \to x \) for \( x \in Q^{-1}(N) \). Since \( Q^{-1}(N) \) is weakly dense, we have \( \| V_{x} x - x \|_1 \to 0 \) for \( x \in l_p(I) \) where \( \| \cdot \|_1 \) is the \( l_1 \)-norm on \( l_p(I) \).

If \( u \in Q^{-1}(N) \) and \( \| u \|_1 < \varepsilon \) then we can write \( u = v_1 + \ldots + v_n \) where the \( v_i \)'s have disjoint support and \( \varepsilon^p \leq \| v_i \|_p^p \leq 2^p \varepsilon^p \) for \( i \leq n - 1 \), with \( \| v_n \|_p \leq 2 \varepsilon^p \). Thus

\[
\| SV_{x} u \| \leq \left( \sum_{i=1}^{n} \| V_{x} v_i \|_q^q \right)^{1/q} \leq 2^{1/p} n^{1/q} \| V_{x} \| \varepsilon^q .
\]

Hence

\[
\| S u \| \leq 2^{1/p} C n^{1/q} \varepsilon
\]

where \( C = \sup \| V_{x} \| \). Now

\[
(n - 1) \equiv \| u \|_p^p \varepsilon^{-p}
\]

so that

\[
\| S u \| \leq 2^{1/p} C (1 + \| u \|_p^p \varepsilon^{-p})^{1/q} \varepsilon .
\]

We conclude that if \( x \in l_p(I) \), since \( \{ V_{x} x \} \) is bounded and \( l_1 \)-Cauchy, \( SV_{x} x \) converges in \( Z \). Defining \( S_1 x = \lim SV_{x} x \) for \( x \in l_p(I) \) we obtain our extension.

The extension \( S_1 \) factors to \( T_1 : X \to Z \). In each case the extension is unique. In case (1) there are no compact operators on \( X/N \) since (using Theorem 3.4) there are no compact operators on \( L_p \). In case (2) use Lemma 8.1. In case (3) uniqueness follows from the construction.
Theorem 8.3. Let $X$ be a continuous $\ell_p$-space and let $Z$ be a $p$-Banach space. Let $N$ be a closed subspace of $Z$ which is either $q$-convex for $q>p$ or an ultra-summand. Then any bounded linear operator $T: X \rightarrow Z/N$ has a unique lift $T_1: X \rightarrow Z$ (so that $qT_1 = T$ where $q: Z \rightarrow Z/N$ is the quotient map).

Proof. Let $V: l_p \rightarrow X$ be a quotient map. Then there is a lifting $S: l_p \rightarrow Z$ of $TV: X \rightarrow Z/N$. Consider $S: \ker V \rightarrow N$. Since $\ker V$ is weakly dense and locally complemented in $l_p$, then in either case there is an extension $S_1: l_p \rightarrow N$. Consider $(S - S_1): l_p \rightarrow Z$; $S - S_1$ factors to the desired lift $T_1$. Again uniqueness follows from $\mathfrak{z}(X, N) = \{0\}$. In the case when $N$ is $q$-convex this follows from using 8.2 to extend to any operator from $X$ into $N$ to an operator from $L_\ell$ into $N$.

Remarks. Compare Theorem 4.2 of [17] with Theorems 8.2 and 8.3. It is possible to derive a statement similar to that of Theorem 4.2 in [17] from Theorem 8.2 for $\ell_p$-spaces when $p<1$, but it no longer characterizes $\ell_p$-spaces. This is because (see Example 6.7) the (CEP) does not imply local complementation for subspaces of $l_p$ when $p<1$.

9. Some applications to $H_p$

Now we consider the space $L_\ell(\mathfrak{z}, m)$ where $\mathfrak{z}$ is the unit circle in the complex plane and $dm = d\theta/2\pi$ is normalized Lebesgue measure on the circle. The closure of the polynomials in $L_\ell(\mathfrak{z})$ is denoted by $H_p$. It is easy to show for $0<p<\infty$ that $H_p$ has (BAP), and it has recently been shown that it has a basis [23]. Also $H_p$ is a pseudo-dual space.

In [15] it is shown that $H_1$ is not an $\ell_1$-space.

**Proposition 9.1.** $H_p$ is not a $\ell_p$-space for $0<p<1$.

Proof. $H_p$ has (BAP) but does not embed into $l_p$ since it contains copies of $l_2$ (see Theorem 6.4).

Let us denote by $H_p$ the space of polynomials in $\mathfrak{z}$, i.e. the space of complex conjugates of $H_p$-functions. Let $J_p = H_p \cap H_p$ the linear span of the real $H_p$-functions. Recently Aleksandrov [1] showed that $H_p + H_p = L_\ell(\mathfrak{z})$ if $0<p<1$ (this is clearly false when $p=1$ but true trivially for $1<p<\infty$).

This means we can set up a map $U: H_p \oplus H_p \rightarrow L_\ell(\mathfrak{z})$ defined by $U(f, g) = f + g$. Then $\ker U = \{(f, g) : f = -g\}$ is isomorphic to $J_p$.

**Proposition 9.2.** (1) $J_p$ is not an ultra-summand and is therefore non-isomorphic to $H_p$.

(2) $J_p$ has a basis.

Proof. These remarks follow from the fact that the $\ker U$ must be locally complemented in $H_p \oplus H_p$, but is clearly weakly dense. We use of course Theorems 6.1, 4.4, 5.1 and 5.7. For (2) we use the fact that $H_p$ has a basis [23].
Theorem 9.3 (see [24]). The spaces $L_p(\mathbb{S})/J_p$, $L_p(\mathbb{S})/H_p$ and $H_p/J_p$ are isomorphic.

Proof. By considering the automorphism $z \rightarrow \bar{z}$ of the circle, we clearly obtain $L_p/H_p \cong L_p/\bar{H}_p$. If we define projections in $L_p$ by

$$ Pf(z) = \frac{1}{2} (f(z) + f(-z)) $$
$$ Qf(z) = \frac{1}{2} (f(z) - f(-z)) $$

then $P$ and $Q$ each leave $H_p$ invariant. Thus $L_p/H_p \cong P(L_p)/P(H_p) \oplus Q(L_p)/Q(H_p)$. Now if $Tf(z) = f(z^2)$, $T$ maps $L_p$ onto $P(L_p)$ and $H_p$ onto $P(H_p)$, isometrically. Similarly $Tf(z) = zf(z^2)$ maps $L_p$ onto $Q(L_p)$ and $H_p$ onto $Q(H_p)$. Thus $L_p/H_p \cong L_p/H_p \oplus L_p/H_p$.

Now use the Aleksandrov map $U : H_p \oplus \bar{H}_p \rightarrow L_p$. Since $U^{-1}(H_p) = J_p \oplus H_p$, we have $L_p/H_p \cong H_p/J_p$. Since $U^{-1}(J_p) = J_p \oplus J_p$, $L_p/J_p \cong H_p/J_p = \bar{H}_p/J_p$. However $\bar{H}_p/J_p \cong L_p/H_p$ by the above reasoning and so $L_p/J_p \cong L_p/H_p \cong H_p/J_p$.

Theorem 9.4. Suppose $X$ is an ultral-summand or is $q$-convex for some $q \geq p$ and $T : J_p \rightarrow X$ is a bounded linear operator. Then $T$ can be expressed in the form

$$ Tf = S_1 f + S_2 f \quad f \in J $$

where $S_1 : H_p \rightarrow X$ and $S_2 : \bar{H}_p \rightarrow X$ are bounded linear operators.

Proof. Define $T_1 : \ker U \rightarrow X$ by

$$ T_1(f, -f) = Tf \quad f \in J_p. $$

Extend $T_1$ by Theorem 8.2 to give an operator $V : H_p \oplus \bar{H}_p \rightarrow X$. Write

$$ S_1 f = V(f, 0) \quad f \in H_p $$
$$ S_2 f = -V(0, f) \quad f \in \bar{H}_p. $$

Corollary 9.5. (Aleksandrov). Every continuous linear functional $\varphi$ on $J_p$ is the form

$$ \varphi(f) = \psi_1(f) + \psi_2(f) $$

where $\psi_1 \in H_p^*$ and $\psi_2 \in \bar{H}_p^*$.

Remark. Aleksandrov proves this directly [1].

Finally we apply our methods to characterize translation-invariant operators on $J_p$. An operator $T : X \rightarrow L_p$, where $X$ is a translation-invariant subspace of $L_p$, is translation-invariant if

$$ T(f_{\omega}) = (Tf)_{\omega} \quad \omega \in \mathbb{S} $$

where $f_{\omega}(z) = f(\omega z)$.

Oberlin [19] has shown that every translation-invariant operator $T : L_p \rightarrow L_p$ is of the form

$$ Tf = \sum_{n=1}^{\infty} c_n f_{\omega_n} $$
where \( \sum_{n=1}^{\infty} |c_n|^p < \infty \), and \( \omega_n \in \mathcal{S} \). Clearly any such operator restricts to a translation-invariant endomorphism of \( J_p \).

For \( f \in H_p \), \( f \) can be realized as the boundary values of a function \( \tilde{f} \) analytic in the open unit disc. Denote by \( \theta_0(f) \) the value of \( f \) at 0. Then the map \( f \mid \rightarrow \theta_0(f) \cdot 1 \) is a translation-invariant linear operator on \( J_p \) as is \( f \mid \rightarrow \theta_0(f) \cdot 1 \) (where we exploit the fact the \( \tilde{f} \in H_p \) for \( f \in J_p \)). Let \( \theta_q(f) = \theta_0(\tilde{f}) \).

**Theorem 9.6.** Let \( T : J_p \rightarrow J_p \) be a translation-invariant linear operator. Then takes the form:

\[
Tf(z) = \sum_{n=1}^{\infty} c_n f(\omega_n z) + a_1 \theta_0(f) + a_2 \theta_q(f)
\]

where \( \omega_n \in \mathcal{S} \), \( \sum |c_n|^p < \infty \) and \( a_1, a_2 \in \mathcal{C} \).

**Proof.** First define \( T_1 : \ker U \rightarrow J_p \) by

\[
T_1(f, -f) = Tf.
\]

Then since \( H_p \) is an ultra-summand, we can find a unique extension \( S_1 : H_p \oplus \overline{H}_p \rightarrow H_p \).

If \( \omega \in \mathcal{S} \), then \( (f, g) \mid \rightarrow (S_1(f, g))_\omega \) extends \( (f, -f) \mid \rightarrow (Tf)_\omega = T_1(f_\omega, -f_\omega) \), as does \( (f, g) \mid \rightarrow S_1(f_\omega, g_\omega) \). Hence by uniqueness \( (S_1(f, g))_\omega = S_1(f_\omega, g_\omega) \).

For \( n \geq 0 \) choose \( g_n(z) = z^{-n} \in H_p \). Then \( S_1(0, g_n)_\omega = \omega^{-n} S_1(0, g) \) and \( S_1(0, g) \in \mathcal{E}_H \). Hence \( S_1(0, g_n) = 0 \) if \( n > 1 \), and \( S_1(0, g_0) \) is constant. We conclude that

\[
S_1(0, f) = a \theta_q(f)
\]

for some \( a \in \mathcal{C} \). Thus we have

\[
S_1(f, g) = V_1 f + a \theta_q(g)
\]

where \( V_1 : H_p \rightarrow H_p \) is translation-invariant. Similarly \( T_1 \) extends to a translation invariant operator \( S_2 : H_p \oplus \overline{H}_p \rightarrow \overline{H}_p \) of the form

\[
S_2(f, g) = \beta \theta_0(f) + V_2 f.
\]

On \( \ker U \) \( S_1 = S_2 \). Thus there is an operator \( R : L_p(\mathcal{S}) \rightarrow L_p(\mathcal{S}) \) such that

\[
RU = S_1 - S_2
\]

and \( R \) is clearly translation-invariant. Hence \( R \) is of the form

\[
Rf(z) = \sum_{n=1}^{\infty} c_n f(\omega_n z)
\]

where \( \sum |c_n|^p < \infty \) and \( \omega_n \in \mathcal{S} \).

If \( f \in H_p \),

\[
Rf = V_1 f - \beta \theta_0(f)
\]

Hence

\[
V_1 f = Rf + \beta \theta_0(f)
\]

and if \( f \in J_p \)

\[
Tf = S_1(f, -f) = Rf + \beta \theta_0(f) - a \theta_q(f)
\]

which is of the form (9.6.1).
Kalton, Locally Complemented Subspaces and $L_p$-Spaces

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