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# Locally Complemented Subspaces and $\mathcal{L}_p$-Spaces for $0<p<1$

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**Abstract.** We develop a theory of $\mathcal{L}_p$-spaces for $0<p<1$, basing our definition on the concept of a locally complemented subspace of a quasi-BANACH space. Among the topics we consider are the existence of basis in $\mathcal{L}_p$-spaces, and lifting and extension properties for operators. We also give a simple construction of uncountably many separable $\mathcal{L}_p$-spaces of the form $\mathcal{L}_p(X)$ where $X$ is not a $\mathcal{L}_p$-space. We also give some applications of our theory to the spaces $H_p$, $0<p<1$.

## 1. Introduction

$\mathcal{L}_p$-spaces ($1\leqq p\leqq\infty$) were introduced by LINDENSTRAUSS and PELCZYŃSKI [15] as BANACH spaces whose local structure resembles that of the spaces $l_p$. Thus a BANACH space $X$ is an $\mathcal{L}_p$-space if there is a constant $\lambda$ such that for every finite dimensional subspace $F$ of $X$ there is a finite-dimensional subspace $G\supset F$ and a linear isomorphism $T: G\to l_p^{(n)}$ with $\|T\|\cdot\|T^{-1}\|\leqq\lambda$. The study of $\mathcal{L}_p$-spaces has proved to be rich and rewarding.

There has been little effort at a systematic treatment of $\mathcal{L}_p$-spaces for $0<p<1$. There is however, in the author's opinion, some interest in giving such a treatment. For example in [12], it is shown that the quotient $\mathcal{L}_p/1$ of $\mathcal{L}_p$ by a one-dimensional subspace is not an $\mathcal{L}_p$-space if $0<p<1$ and hence it cannot be isomorphic to $L_p$.

Suppose now $\Sigma_0$ is a sub-$\sigma$-algebra of the BOREL sets of (0, 1) and let $L_p(\Sigma_0)$ be the closed subspace of all $\Sigma_0$-measurable functions in $L_p$. We denote by $\Lambda(\Sigma_0)$ the quotient space $L_p/L_p(\Sigma_0)$. In [9] it is shown that, 'usually', $L_p(\Sigma_0)$ is uncomplemented in $L_p$ if $0<p<1$. Thus N. T. PECK raised the question whether $\Lambda(\Sigma_0)$ can be isomorphic to $L_p$ if $L_p(\Sigma_0)$ is uncomplemented, and equally whether $\Lambda(\Sigma_0)$ could be an $\mathcal{L}_p$-space.

The definition of an $\mathcal{L}_p$-space used in [12] is slightly different from the definition given above for $1\leqq p\leqq\infty$. It is merely required that $X$ contains an increasing net of finite-dimensional subspaces uniformly isomorphic to finite-dimensional $l_p$-spaces, whose union is dense. This distinction is unimportant for $p\geqq 1$, but for $0<p<1$ it is significant, for, as W. J. STILES pointed out to the author it is not

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clear that even $L_p(0<p<1)$ would satisfy the LINDENSTRAUSS-PEŁCZYŃSKI definition. This led us to consider whether there is an alternative indirect definition of $\mathfrak{L}_p$-spaces more suitable for $0<p<1$.

The crucial notion we introduce in this paper is that of a locally complemented subspace of a quasi-BANACH space. This idea is entirely natural we believe and leads to an attractive definition of $\mathfrak{L}_p$-spaces. Thus a quasi-BANACH $X$ is an $\mathfrak{L}_p$-space if and only if it is isomorphic to a locally complemented subspace of a space $L_p(\Omega, \Sigma, \mu)$. There is a local version of this definition (see Theorem 6.1 below); $X$ is an $\mathfrak{L}_p$-space if there is a uniform constant $\lambda$, such that whenever $F$ is a finite-dimensional subspaces of $X$ and $\varepsilon>0$ there are operators $S: F\to l_p$, $T: l_p\to X$ with $\|T\|\cdot\|S\|\leqq\lambda$ and $\|TSf-f\|\leqq\varepsilon\|f\|$ for $f\in F$. For $p=1$ or $\infty$, this simply reduces to the standard definition, but for $1<p<\infty$ $(p\neq 2)$ it gives a very slightly wider class (HILBERT spaces are $\mathfrak{L}_p$-spaces for $1<p<\infty$).

We now discuss the layout and main results of the paper. Section 2 is purely preparatory and in Section 3 we introduce the notion of a locally complemented subspace. In a BANACH space this has several equivalent attractive formulations; for example $N$ is a locally complemented subspace of $X$ if and only if $N^{**}$ is complemented in $X^{**}$. The Principle of Local Reflexivity plays an important role here, as it states that $X$ is locally complemented in $X^{**}$.

The absence of a bidual for non-locally convex quasi-BANACH spaces leads us to consider ultraproducts in Section 4, and we give a number of connections between these ideas. Section 5 contains our first main result that a locally complemented subspace of a quasi-BANACH space with a basis, under certain conditions, also has a basis; these conditions include the case of a weakly dense subspace. This result is similar in spirit to some results of JOHNSON, ROSENTHAL and ZIPPIN [7].

In Section 6, we introduce $\mathfrak{L}_p$-spaces and give some of their properties. We also show that if $0<p<1$, it is convenient to separate separable $L_p$-spaces into three categories – discrete, continuous and hybrid $\mathfrak{L}_p$-spaces. A separable $\mathfrak{L}_p$-space has a basis if and only if it is discrete, i.e., a locally complemented subspace of $l_p$. Separable $\mathfrak{L}_p$-spaces with trivial dual are called continuous and correspond to the locally complemented subspaces of $L_p$. We point out (Theorem 6.7) that the kernel of any operator from a $p$-BANACH space with a basis onto a continuous $\mathfrak{L}_p$-space (including $L_p$ itself) will again have a basis. We also produce a simple explicit example of a weakly dense subspace of $l_p$ $(0<p<1)$ failing to have a basis (or even the Bounded Approximation Property). In view of the results of DAVIE and ENFLO [3], [5] and recently SZANKOWSKI [22] the existence of such a subspace is hardly surprising; however the construction is very easy and the subspace has the additional property that every compact operator defined on it may be extended to $l_p$.

In Section 7 we show that the subspace $L_p(\Sigma_0)$ is locally complemented but not complemented (see [9]). A deduction is that in this case $\Lambda(\Sigma_0)$ is an $\mathfrak{L}_p$-space; however we have shown in [11] that, in the case where $(\Omega, \Sigma, \mu)$ is separable, that $\Lambda(\Sigma_0)\cong L_p$ implies that $L_p(\Sigma_0)$ is complemented. If we take the special example where $\Omega=(0,1)\times(0,1)$, $\Sigma$ is the BOREL sets of $(0,1)^2$ and $\Sigma_0$ is $\sigma$-algebra of sets of the form $B\times(0,1)$ for $B$ a BOREL subset of $(0,1)$, then $\Lambda(\Sigma_0)\cong L_p(L_p/1)$ where $L_p/1$ ([12]) is the quotient of $L_p$ by a single line. However $L_p/1$ is not an $\mathfrak{L}_p$-space. This shows also that for $0<p<1$ it is possible to have $L_p(X)$ an $L_p$-space without having $X$ as $\mathfrak{L}_p$-space, in contrast to the situation for $p=1$. We go on to construct an uncountable collection of separable $\mathfrak{L}_p$-spaces of this type.

In Section 8, we prove a number of lifting theorems (similar to those of [12]) and extension theorems for operators. For example if $X$ is a $p$-BANACH space and $N$ is a closed subspace such that $X/N$ is a continuous $\mathfrak{L}_p$-space then an operator $T: N\to Z$ can be extended to an operator $T_1: X\to Z$ under any one of three hypotheses: (1) $T$ is compact, (2) $Z$ is a $q$-BANACH space for some $q>p$ or (3) $Z$ is a pseudo-dual space. In each the extension is unique.

In Section 9, we give an application of these ideas to an example involving $H_p$ for $0<p<1$. Let $J_p$ for $0<p<1$ be the closed subspace of $H_p$ (regarded as a subspace of $L_p(\mathfrak{T})$, where $\mathfrak{T}$ is the unit circle with LEBESGUE measure) of all $f$ such that $\bar{f}\in H_p$. Exploiting a recent result of ALEKSANDROV [1] we show $J_p$ is isomorphic to a locally complemented subspace of $H_p\oplus\bar{H}_p$ (where $\bar{H}_p=\{f\in L_p(\mathfrak{T}): \bar{f}\in H_p\}$). We deduce that $J_p$ has $(BAP)$ and that as $H_p$ has a basis then so does $J_p$. We also quickly obtain the dual space of $J_p$; every continuous linear functional $\varphi\in J_p^*$ is of the form

$$\varphi(f)=\psi_1(f)+\psi_2(f)\quad f\in J_p$$

where $\psi_1\in H_p^*$ and $\psi_2\in\bar{H}_p^*$. We show that $J_p$ is non-isomorphic to $H_p$, but $L_p/H_p\cong L_p/J_p\cong H_p/J_p$. Finally we characterize translation-invariant operators $T: J_p\to J_p$ using the extension theorems of Section 8. We show that every translation-invariant operator $T: J_p\to J_p$ takes the form

$$Tf(z)=\sum_{n=1}^{\infty}c_nf(\omega_nz)+a_1\theta_0(f)+a_2\theta_\infty(f)$$

where $\omega_n\in\mathfrak{T}$, $\sum|c_n|^p<\infty$ and, $\theta_0(f)=f(0)$ regarding $f$ as a member of $H_p$ and $\theta_\infty(f)=\bar{f}(0)$ regarding $\bar{f}$ as a member of $H_p$.

## 2. Preliminaries

As usual a quasi-norm on real (or complex) vector space $X$ is a map $x\mapsto\|x\|$ $(x\in\mathbb{R})$ satisfying

(2.0.1) $\quad\|x\|>0\quad x\neq 0$

(2.0.2) $\quad\|\alpha x\|=|\alpha|\,\|x\|,\quad \alpha\in\mathbb{R}\quad$ (or $\mathbb{C}$), $\quad x\in X$

(2.03) $\quad\|x+y\|\leqq k\,(\|x\|+\|y\|)\quad x,y\in X$,

where $k$ is a constant independent of $x$ and $y$. A quasi-norm defines a locally bounded vector topology on $X$. A complete quasi-normed space is called a quasi-BANACH space. If, in addition the quasi-norm satisfies for some $p$, $0<p\leqq 1$,

(2.0.4) $\quad\|x+y\|^p\leqq\|x\|^p+\|y\|^p\quad x,y\in X$

then we say $X$ is a $p$-BANACH space. A basic theorem due to AOKI and ROLEWICZ asserts that every quasi-BANACH space may be equivalently renormed as a $p$-BANACH space for some $p$, $0<p\leqq 1$. We shall therefore assume without losing any generality that every quasi-BANACH space considered is a $p$-BANACH space for some suitable $p$ where $0<p\leqq 1$ (i.e. that (2.0.4) is satisfied).

If $(\Omega, \Sigma, \mu)$ is a measure space then by $L_p(\Omega, \Sigma, \mu)$ we denote the space of all real (or complex) $\Sigma$-measurable functions $f$ satisfying:

$$\|f\|_p = \{\int_\Omega |f|^p \, d\mu\}^{1/p} < \infty$$

$L_p(\Omega, \Sigma, \mu)$ is a $p$-BANACH space, after the standard identification of functions agreeing $\mu$-almost everywhere. If $\Sigma$ is the power set of $\Omega$ and $\mu$ is counting measure on $\Sigma$ (i.e. $\mu(A)$ is the cardinality of $A$ if $A$ is a finite subset of $\Omega$ and $\infty$ otherwise), then $L_p(\Omega, \Sigma, \mu)$ is written $l_p(\Omega)$. If $\Omega$ is countable this reduces to the standard sequence space $l_p$.

On the other hand if $(\Omega, \Sigma, \mu)$ is separable non-atomic probability space then $L_p(\Omega, \Sigma, \mu)$ can be identified isometrically with the function space $L_p(0, 1)$ and will be written $L_p$.

If $X$ is a quasi-BANACH space the $L_p(\Omega, \Sigma, \mu; X)$ will denote the space of $\Sigma$-measurable maps: $f : \Omega \to X$ with separable range satisfying:

$$\|f\|_p = \{\int_\Omega \|f(\omega)^p\| \, d\mu(\omega)\}^{1/p} < \infty .$$

Again $L_p(\Omega, \Sigma, \mu; X)$ is a quasi-BANACH space; if $X$ is a $p$-BANACH space, then it is also a $p$-BANACH space. If $\Omega = \boldsymbol{N}$ and $\mu$ is counting measure we write this space as $l_p(X)$, while if $(\Omega, \Sigma, \mu)$ is separable non-atomic probability space we write it as $L_p(X)$.

If $X$ is a $p$-BANACH space, then for any index set $I$, the space $l_\infty(I; X)$ is the space of "generalized sequences", $\{x_i\}_{i\in I}$ satisfying

$$\|\{x_i\}_{i\in I}\| = \sup_{i\in I} \|x_i\| < \infty .$$

$l_\infty(I; X)$ is also a $p$-BANACH space. If $\mathfrak{U}$ is a non-principal ultrafilter on $I$, then the ultraproduct $X_{\mathfrak{U}}$ of $X$ is the space $l_\infty(I; X)/C_{0,\mathfrak{U}}(I; X)$ where $C_{0,\mathfrak{U}}(I; X)$ is the closed subspace of $l_\infty(I; X)$ of all $\{x_i\}$ such that

$$\lim_{\mathfrak{U}} \|x_i\| = 0 .$$

It is often convenient to think of $X_{\mathfrak{U}}$ as the HAUSDORFF quotient of the space $l_\infty(I; X)$ with the "semi-quasi-norm"

$$\|\{x_i\}\|_{\mathfrak{U}} = \lim_{\mathfrak{U}} \|x_i\| .$$

We also shall identify $X$ as a subspace of $X_{\mathfrak{U}}$ by identifying each $x\in X$ with the constant sequence $x_i = x$ for $i\in I$.

The main theorem we shall require here is due to SCHREIBER [20] (the case $p\geqq 1$ is due to DACUNHA-CASTELLE and KRIVINE [2]).

**Theorem 2.1.** *Any ultraproduct of a space $L_p(\Omega, \Sigma, \mu)$ is isometrically isomorphic to $L_p(\Omega_1, \Sigma_1, \mu_1)$ for a suitably chosen measure space $(\Omega_1, \Sigma_1, \mu_1)$.*

Any separable $p$-BANACH space $X$ is a quotient of the space $l_p$. In the case $p=1$, LINDENSTRAUSS and ROSENTHAL [16] showed that there is a form of uniqueness of the quotient map of $l_1$ onto $X$. Precisely if $T_1 : l_1 \to X$ are any two quotient maps and $X$ is not isomorphic to $l_1$ then there is an automorphism $\tau : l_1 \to l_1$ such that $T_1 = T_2\tau$. STILES [21] asked whether this can be generalized to $l_p$ when $p<1$. In the stated form this is impossible, since as shown by STILES, $l_p$ contains a subspace $M$ which contains no copy of $l_p$ complemented in the whole space; then $l_p/M \cong l_p \oplus l_p/l_p \oplus M$ and there can be no isomorphism of $l_p$ onto $l_p \oplus l_p$ carrying $M$ to $l_p \oplus M$. However, excepting this case, the argument of LINDENSTRAUSS and ROSENTHAL can be extended. We therefore state for $0<p<1$:

**Theorem 2.2.** *Suppose $X$ is a separable $p$-BANACH space and suppose $T_1 : l_p \to X$ and $T_2 : l_p \to X$ are open mappings. Provided the kernels of $T_1$ and $T_2$ both contain copies of $l_p$ which are complemented in $l_p$, there is an automorphism $\tau : l_p \to l_p$ with $T_1 = T_2\tau$.*

The proof given in LINDENSTRAUSS-TZAFRIRI [18] p. 108 goes through undisturbed, once one observes that the operator $S$ defined therein is subjective for purely algebraic reasons (the proof in [18] appeals to duality) indeed given $x \in l_1$, $x - \hat{T}_1\hat{T}_2 x$ is clearly in $S(U)$ while $\hat{T}_1\hat{T}_2 x \in S(V)$.

A closed subspace $M$ of a quasi-BANACH space $X$ is said to have the HAHN-BANACH Extension Property (HBEP) if every continuous linear functional $\varphi \in M^*$ can be extended to a continuous linear functional $\tilde{\varphi} \in X^*$.

**Corollary 2.3.** *Suppose $X$ is a separable $p$-BANACH xspace non-isomorphic is $l_p$. Suppose $T_1 : l_p \to X$ and $T_2 : l_p \to X$ are two open mappings and suppose the kernel of $T_1$ has (HBEP). Then there is an automorphism $\tau$ of $l_p$ such that $T_1 = T_2\tau$.*

Proof. If ker $T_1$ has HBEP then $X$ is a $\mathcal{K}_p$-space as defined in [12] and so ker $T_2$ also has HBEP. But this means by results of STILES that both ker $T_1$ and ker $T_2$ contain copies of $l_p$, complemented in $l_p$.

We conclude by recalling some definitions. A quasi-BANACH space $X$ is a *pseudo-dual space* if there is HAUSDORFF vector topology on $X$ for which the unit ball is relatively compact. $X$ has the *Bounded Approximation Property* (BAP) if there is a sequence of finite-rank operation $T_n : X \to X$ such that $T_n x \to x$ for $x \in X$.

## 3. Locally complemented subspaces

We shall say that a closed subspace $E$ of a quasi-BANACH space $X$ is *locally complemented* in $X$ if there is a constant $\lambda$ such that whenever $F$ is a finite-dimensional subspace of $X$ and $\varepsilon > 0$ there is a linear operator $T = T_F : F \to E$ such that $\|T\| \leqq \lambda$ and $\|Tf - f\| \leqq \varepsilon \|f\|$ for $f \in E \cap F$.

By way of motivation let us observe that the Principle of Local Reflexivity for BANACH spaces (LINDENSTRAUSS and ROSENTHAL [17]) states that every BANACH space $X$ is locally complemented in its bidual $X^{**}$ (with $\lambda=1$).

We shall start with two rather technical lemmas which will be needed later to identify locally complemented subspaces.

**Lemma 3.1.** *Suppose $X$ is a quasi-BANACH space and that $E$ is closed subspace of $X$. Suppose there is an increasing net $X_\alpha$ of subspaces of $X$ so that $\cup(X_\alpha\cap E)$ is dense in $E$ and $\cup X_\alpha$ is dense in $X$. Suppose there are operators $Q_\alpha: X_\alpha\to E$ such that* $\sup\|Q_\alpha\|<\infty$ *and $Q_\alpha e\to e$ for $e\in\cup(X_\alpha\cap E)$. Then $E$ is locally complemented in $X$.*

Proof. Suppose $F\subset X$ is a finite-dimensional subspace and $\{f_1,\dots,f_n\}$ is a normalized basis of $F$ such that for some $m\leqq n$, $\{f_1,\dots,f_n\}$ is a basis of $E\cap F$. Then there is a constant $c>0$ such that for any $(a_1,\dots,a_n)$

$$\left\|\sum_{i=l}^{n}a_if_i\right\|\geqq c\left\{\sum_{i=l}^{n}|a_i|^p\right\}^{1/p}.$$

For fixed $0<\varepsilon<1$ select $\alpha$ and $g_1,\dots g_n\in X_\alpha$ so that $\|g_i-f_i\|^p\leqq\frac{1}{4}c^p\varepsilon^p$ for $1\leqq i\leqq n$ and $g_i\in E$ for $1\leqq i\leqq m$. Choose $\beta\geqq\alpha$ that $\|Q_\beta e-e\|^p\leqq\frac{1}{4}\varepsilon^p\|e\|^p$ for $e\in[g_1,\dots,g_m]$.

Then define $T: F\to E$ by

$$T\left(\sum_{i=l}^{n}a_if_i\right)=Q_\beta\left(\sum_{i=l}^{n}a_ig_i\right).$$

Then

$$\left\|\sum_{i=l}^{n}a_i(g_i-f_i)\right\|^p\leqq\frac{1}{4}c^p\varepsilon^p\sum|a_i|^p$$

$$\leqq\frac{1}{4}\varepsilon^p\left\|\sum a_if_i\right\|^p.$$

Thus $\|T\|^p\leqq2\|Q_\beta\|^p$ and if $e\in F\cap E$,

$$\|Te-e\|\leqq\varepsilon\|e\|.$$

**Lemma 3.2.** *Let $X$ be a quasi-BANACH space and suppose $E$ is a locally complemented subspace of $X$. Thus there is a constant $\lambda$ such that whenever $Y$ is a closed subspace of $X$ containing $E$ with* $\dim Y/E<\infty$ *there is a projection $P: Y\to E$ with $\|P\|\leqq\lambda$.*

**Remark.** Clearly the converse of Lemma 3.2 is immediate.

Proof. There is a constant $\lambda_0$ so that for every $\varepsilon>0$ and finite-dimensional subspace $F$ of $X$ there is a linear map $T: F\to E$ with $\|Tf-f\|\leqq\varepsilon\|f\|$ for $f\in E\cap F$ and $\|T\|\leqq\lambda_0$. We can suppose that $X$ is a $p$-BANACH space where $0<p\leqq1$.

Suppose $Q: Y \to E$ is any bounded projection (there is a bounded projection since dim $Y|E < \infty$). Suppose $\varrho = \|I - Q\|$ and choose $\varepsilon > 0$ so that

$$\varepsilon < (3\lambda_0^p + 4)^{-1} \varrho^{-1} .$$

Let $G = Q^{-1}(0)$ and let $\{g_1, \ldots, g_n\}$ is an $\varepsilon$-net for the unit ball of $G$. Let

$$\delta_i = d(g_i, E) = \inf_{e \in E} \|g_i - e\| \quad 1 \leqq i \leqq n$$

and choose $e_i \in E$ so that

$$\|g_i - e_i\|^p \leqq 2\delta_i^p \quad 1 \leqq i \leqq n .$$

Let $H$ be the linear span of $G$ and $\{e_1, \ldots, e_n\}$. Since $\|e_i\|^p \leqq 3$ there is a linear map $T: H \to E$ so that $\|T\| \leqq \lambda_0$ and

$$\|Te_i - e_i\| \leqq \varepsilon \quad 1 \leqq i \leqq n .$$

Suppose $g \in G$ and let $\|g\| = \theta$. For some $i$, $1 \leqq i \leqq n$

(3.2.1) $$\|g - \theta g_i\| \leqq \varepsilon \|g\|$$

and so

(3.2.2) $$\|g - \theta e_i\|^p \leqq (2\delta_i^p + \varepsilon^p) \|g\|^p$$

while

(3.2.3) $$\delta_i^p \|g\|^p \leqq d(g, E)^p + \varepsilon^p \|g\|^p .$$

Combining (3.2.2) and (3.2.3) we obtain:

(3.2.4) $$\|g - \theta e_i\|^p \leqq 2d(g, E)^p + 3\varepsilon^p \|g\|^p$$

Now define $P: Y \to E$ by $P = Q + T\ (I - Q)$. Then $P$ is a projection. Suppose $y \in Y$ and $\|y\| = 1$; let $g = y - Qy$ and $\theta = \|g\|$. Choose $i$ so that (3.2.1) holds. Then by (3.2.4)

$$\begin{aligned} \|Qy + \theta e_i\|^p &\leqq \|y\|^p + \|g - \theta e_i\|^p \\ &\leqq 1 + 2d(g, E)^p + 3\varepsilon^p \|g\|^p \\ &\leqq 3 + 3\varepsilon^p \varrho^p . \end{aligned}$$

On the other hand

$$\begin{aligned} \|Tg - \theta e_i\|^p &\leqq \lambda_0^p \|g - \theta e_i\|^p + \|g\|^p \|Te_i - e_i\|^p \\ &\leqq 2\lambda_0^p d(g, E)^p + (3\lambda_0^p + 1)\, \varepsilon^p \varrho^p . \end{aligned}$$

Thus we have

$$\|Py\|^p \leqq 2\lambda_0^p + 3 + \varepsilon^p \varrho^p\ (3\lambda_0^p + 4) \leqq 2\lambda_0^p + 4 .$$

Setting $\lambda^p = 2\lambda_0^p + 4$ we have the desired conclusion.

**Lemma 3.3.** implies the following proposition whose proof we omit:

**Proposition 3.3.** *Suppose $X$ is a quasi-*Banach *space and $E \subset F$ are closed sub-spaces of $X$, If $F$ is locally complemented in $X$ and $E$ is locally complemented in $F$, then $E$ is locally complemented in $X$.*

We shall say a that closed subspace $E$ of a quasi-BANACH space $X$ has the *Compact Extension Property* (CEP) in $X$ if whenever $Z$ is a quasi-BANACH space and $K : E \approx Z$ is a compact operator then there is a compact operator $K_1 : X \to Z$ with $K_1 e = Ke$ for $e \in E$. An argument exactly as in Theorem 2.2 of [14] shows that if $E$ has (CEP) then for any fixed $r > 0$ there is a constant $\lambda$ so that whenever $Z$ is an $r$-BANACH space and $K : E \to Z$ is compact then we can determine $K_1$ so that $\|K_1\| \leqq \lambda \|K\|$.

**Theorem 3.4.** *If $E$ is a locally complemented subspace of $X$ then $E$ has (CEP).*

Proof. We shall not give full details here as this is a straightforward "LINDENSTRAUSS compactness argument". If $K : E \to Z$ is compact consider the net $\{KP_Y\}$ where $Y$ ranges over all subspaces of $X$ with $Y \supset E$ and $\dim Y|E < \infty$ and $P_Y : Y \to E$ is a uniformly bounded set of projections as in Lemma 3.2.

The next result is essentially known, but helps to clarify the situation for BANACH spaces.

**Theorem 3.5.** *Let $X$ be a* BANACH *space and let $E$ be a closed subspace of $X$. The following conditions on $E$ are equivalent:*

(1) *$E$ has (CEP) in $X$.*

(2) *$E$ is locally complemented in $X$.*

(3) *$E^{**}$ is complemented in $X^{**}$ under its natural embedding.*

(4) *There is a linear extension operator $L : E^* \to X^*$ such that $Le^*(e) = e^*(e)$ for $e \in E$ and $e^* \in E^*$.*

Proof. (2)$\Rightarrow$(1): Theorem 3.4.

(1)$\Rightarrow$(4): There is a constant $\lambda$ so that wherever $K : E \to Y$ is a compact operator into a BANACH space $Y$ then $K$ has extension $K_1 : X \to Y$ with $\|K_1\| \leqq \lambda \|K\|$.

Let $G$ be a finite-dimensional subspace of $E^*$ and let $G^\perp = \{e \in E : g(e) = 0$ for $g \in G\}$. Let $Y$ be the quotient space $E/G^\perp$ and $q : E \to Y$ be the quotient map. Then there exists a linear operator $K : X \to Y$ with $Ke = qe$ for $e \in E$ and $\|K\| \leqq \lambda$. Now $K^* : G \to X^*$, $\cdot$ $\|K^*\| \leqq \lambda$ and $K^* g(e) = g(e)$ for $g \in G$ and $e \in E$. The conclusion of (4) can then be obtained by a standard compactness argument.

(4)$\Rightarrow$(3) The adjoint $L^* : X^{**} \to E^{**}$ is a projection.

(3)$\Rightarrow$(2) This follows from Proposition 3.3 and the Principle of Local Reflexivity.

**Remark.** In general, so we shall see, the property (CEP) is strictly weaker than local complementation for a subspace.

## 4. Ultraproducts

The first part of the following theorem serves as a replacement in the non-locally convex setting for the Principle of Local Reflexivity.

**Theorem 4.1.** *Suppose $X$ is a quasi-*BANACH *space, $I$ is an index set and $\mathcal{U}$ is a non-principal ultrafilter on $I$.*

(1) *$X$ is locally complemented in $X_{\mathcal{U}}$.*

(2) *If $Y$ is a locally complemented subspace of $X$ then $Y_{\mathfrak{U}}$ is locally complemented in $X_{\mathfrak{U}}$.*

Proof. (1): Let $F$ be a finite-dimensional subspace of $X_{\mathfrak{U}}$ and let $\{f^{(1)}, \dots f^{(n)}\}$ be a basis of $F$. We shall regard $f^{(k)}$ as members of $l_\infty(I; X)$ by selecting representatives. For each $i \in I$, define $T_i : F \to X$ by

$$T_i\left\{\sum_{k=1}^{n} a_k f^{(k)}\right\} = \sum_{k=1}^{n} a_k f_i^{(k)} .$$

Clearly we have

$$\sup_{i \in I} \|T_i f\| < \infty$$

and

$$\lim_{\mathfrak{U}} \|T_i f\| = \|f\| \quad f \in F .$$

By an elementary compactness argument $\lim_{\mathfrak{U}} \|T_i\| = 1$. If $f \in F \cap X$ then

$$\lim_{\mathfrak{U}} Tf = f .$$

Again by a compactness argument we may select $i$ so that for any $\varepsilon > 0$, $\|T_i f - - f\|^p \leqq \varepsilon^p/2\|f\|^p$ $(f \in F \cap X)$ and $\|T_i\|^p \leqq 1 + \varepsilon^p/2$. Letting $S = (1+\varepsilon^p/2)^{-1/p}\, T_i$ we have $\|S\| \leqq 1$ and $\|Sf - f\| \leqq \varepsilon\|f\|$ for $f \in F \cap X$.

(2): Here we may suppose that for some $\lambda$, we have, for every subspace $W$ of $X$ containing $Y$ with $\dim W/Y < \infty$, a projection $P : W \to Y$ with $\|P\| \leqq \lambda$. Again let $F$ be a finite-dimensional subspace of $X_{\mathfrak{U}}$ and select a basis $\{f^{(1)}, \dots, f^{(n)}\}$ for $F$. For each $i \in I$ let $W_i = [Y, f_i^{(1)}, \dots, f_i^{(n)}]$ be the linear span of $Y$ and $f_i^{(1)}, \dots, f_i^{(n)}$. Let $P_i : W_i \to Y$ be a projection with $\|P_i\| \leqq \lambda$. Define $T : F \to Y_{\mathfrak{U}}$ by $Tf = \{P_i f_i\}_{i \in I}$ for $f \in F$. Then $\|T\| \leqq \lambda$ and if $f \in Y_{\mathfrak{U}}$ then $Tf = f$.

Let us define a quasi-Banach space $X$ to be an ultra-summand if $X$ is complemented in $X_{\mathfrak{U}}$ for every ultraproduct $X_{\mathfrak{U}}$ of $X$. Then we have:

**Theorem 4.2.** *Let $X$ be a quasi-*Banach *space and $E$ be a locally complemented subspace of $X$. Suppose $Y$ is an ultra-summand. Then any bounded linear operator $T_0 : E \to Y$ can be extended to a bounded linear operator $T : X \to Y$.*

Proof. For an index set $\mathfrak{I}$ we take the collection of subspaces $W$ of $X$ with $W \supset E$ and $\dim W/E < \infty$. We let $\mathfrak{U}$ be any ultrafilter on $\mathfrak{I}$ containing all subsets of $\mathfrak{I}$ of the form $\{W \in \mathfrak{I} : W \supset W_0\}$ for $W_0 \in \mathfrak{I}$. For each $W \in \mathfrak{I}$ there is a projection $P_W : W \to E$ so that $\sup \|P_W\| = \lambda < \infty$.

Define $\hat{T} : X \to Y_{\mathfrak{U}}$ by

$$(\hat{T}x)_W = 0 \qquad x \notin W$$
$$\qquad = T_0 P_W x \quad x \in W .$$

Then $\hat{T}$ factors to a linear map into $Y_{\mathfrak{U}}$ and $\|\hat{T}\| \leqq \lambda\|T_0\|$. If $Q : Y_{\mathfrak{U}} \to Y$ is any projection then $T = Q\hat{T}$ provides the desired extension.

**Proposition 4.3.** *A complemented subspace of a pseudo-dual space is an ultra-summand.*

Proof. Suppose $Y$ is a pseudo-dual space and $P: Y \to X$ is a projection onto a closed subspace $X$ of $Y$. We may assume the unit ball of $Y$ is compact in a HAUSDORFF vector topology $\gamma$. If $X_{\mathfrak{U}}$ is any ultraproduct of $X$ then we can define $Q: X_{\mathfrak{U}} \to X$ by

$$Q(\{x_i\}) = P\,(\gamma - \lim_{\mathfrak{U}} x_i)\,.$$

Then $Q$ is a projection of $X_{\mathfrak{U}}$ onto $X$.

**Theorem 4.4.** *Consider the following properties of a quasi-*BANACH *space $X$:*

(1) *$X$ is an ultra-summand.*

(2) *Whenever $X$ is a locally complemented subspace of a quasi-*BANACH *space $Z$ then $X$ is complemented in $Z$.*

(3) *$X$ is isomorphic to a complemented subspace of a pseudo-dual space .*

*Then* (1) *and* (2) *are equivalent in general. If $X$ has $(BAP)$ then* (1), (2) *and* (3) *are equivalent.*

Proof. (1)$\Leftrightarrow$(2): This follows directly from Theorems 4.1 and 4.2.

(2)$\Rightarrow$(3) when $X$ has (BAP): Suppose $T_n: X \to X$ is a sequence of finite-rank operators with $T_n x \to x$ for $x \in X$. Then $\sup \|T_n\| = \lambda < \infty$. Form the space $Z$ of all sequence $\xi = (\xi_n)_{n=1}^\infty$ where $\xi_n \in T_n(X)$ such that $\|\xi\| = \sup \|\xi_n\| < \infty$. Then $Z$ is a pseudo-dual space since its unit ball is compact for co-ordinatewise convergence. Define $J: X \to Z$ by $Jx = (T_n x)_{n=1}^\infty$. Then $J$ is an isomorphic embedding of $X$ into $Z$. Define $Q_k: Z \to J(X)$ by $Q_k(\xi) = J\xi_k$; then $\|Q_k\| \leqq \|J\|$ and $Q_k u \to u$ for $u \in J(X)$. By Lemma 3.1, $J(X)$ is locally complemented in $Z$ and hence is complemented in $Z$.

**Theorem 4.5.** *Suppose $E$ is a locally complemented subspace of $X$. Then $X/E$ is isomorphic to a locally complemented subspace of an ultraproduct $X_{\mathfrak{U}}$ of $X$.*

Proof. Again let $\mathfrak{J}$ be the collection of all subspaces $W$ of $X$ with $W \supset E$ and $\dim W/E < \infty$. Let $\mathfrak{U}$ be an ultrafilter on $\mathfrak{J}$ containing all subsets of the form $\{W : W \subset W_0\}$ for $W_0 \in \mathfrak{J}$. There exist projections $P_W: W \to E$ so that $\sup \|P_W\| = \lambda < \infty$. Define $Q: X \to X_{\mathfrak{U}}$ by

$$\begin{aligned}(Qx)_W &= 0 & x \notin W\\ &= x - P_W x & x \in W\,.\end{aligned}$$

Again $Q$ is linear into $X_{\mathfrak{U}}$ (after factoring out sequences tending to zero through $\mathfrak{U}$) and $\|Q\| \leqq (1+\lambda^p)^{1/p}$ (where we assume $X$ to be a $p$-BANACH space). If $x \in E$ then $Qx = 0$ and clearly in general,

$$\|Qx\| \geqq d(x, E)\,.$$

Thus $Q$ factors to an embedding of $X/E$ into $X_{\mathfrak{U}}$. It remains to show that $Q(X)$ is locally complemented in $X_{\mathfrak{U}}$.

Let $F$ be a finite-dimensional subspace of $X_{\mathfrak{U}}$ with a basis $\{f^{(1)}, \ldots, f^{(n)}\}$. For each $W \in \mathfrak{J}$ define $T_W: F \to Q(X)$ by

$$T_W\left(\sum_{j=1}^n a_{(j)} f^{(j)}\right) = \sum_{j=1}^n a_j Q f_W^{(j)}\,.$$

Now $\sup \|T_W\| < \infty$ and $\lim_{\mathfrak{U}} \|T_W\| \leqq \|Q\|$ as in the proof of Theorem 4.1. If $f \in Q(X) \cap F$ then $f = Qx$ for some $x \in X$. Hence

$$T_W f = Q\,(x - P_W x)$$

eventually (as $W \to \infty$ through $\mathfrak{U}$). Thus $T_W f = Qx = f$ eventually.

Now we can clearly choose $W \in \mathfrak{J}$ so that $T_W f = f$ for $f \in Q(X) \cap F$ and $\|T_W\| \leqq 2\|Q\|$, thus showing $Q(X)$ is locally complemented in $X_{\mathfrak{U}}$.

## 5. Bases

If a quasi-Banach space $X$ has (BAP) then it is possible to give a generalizition of Theorem 3.5.

**Theorem 5.1.** *Suppose $X$ is a quasi-*Banach *space with (BAP): Then a closed subspace $E$ of $X$ is locally complemented if and only if $E$ has both (BAP) and (CEP).*

Proof. Suppose first that $E$ has (BAP) and (CEP). Then where is a sequence $T_n : E \to E$ of finite-rank operators with $T_n e \to e$ for $e \in E$ and $\sup_n \|T_n\| < \infty$. Now by (CEP) (and remarks following the definition) there is a uniformly bounded sequence of operators $Q_n : X \to T_n(E)$ such that $Q_n e = T_n e$ for $e \in E$. Now by Lemma 3.1, $E$ is locally complemented.

Conversely supposed $T_n : X \to X$ are finite-rank operators satisfying $T_n x \to x$ for $x \in X$ and $\sup \|T_n\| < \infty$. If $E$ is locally complemented there are uniformly bounded projections $P_n : E + T_n(X) \to E$. Define $Q_n = P_n T_n$; then $\sup \|Q_n\| < \infty$, $Q_n(X) \subset E$ and $Q_n e \to e$ for $e \in E$. Thus $E$ has (BAP); it has (CEP) by Theorem 3.4.

**Remark.** See below Example 6.7.

**Corollary 5.2.** *If $X$ has (BAP) and $E$ is locally complemented in $X$ there is a sequence of operators $S_n : X \to E$ such that* $\sup \|S_n\| < \infty$ *and $S_n e \to e$ for $e \in E$.*

Now suppose $X$ has a basis. It is unlikely that in general every complemented subspace of $X$ has a basis. This would require for Banach spaces the equivalence of (BAP) and the existence of a basis; see Lindenstrauss and Tzafriri [18] p. 38 and p. 92. However under certain circumstances we shall show that a locally complemented subspace does have a basis.

Suppose $X$ has a basis $(b_n)$ and $E$ is a closed subspace of $X$. Let $\Gamma$ be the linear span in $X^*$ of the biorthogonal functionals $(b_n^*)$. We shall say that $E$ is *residual* in $X$ if there is a uniformly bounded sequence of operators $T_n : X \to E$ such that $T_n^* \gamma \to \gamma$ for $\gamma \in \Gamma$ in the weak*-topology (i.e. $\gamma(T_n x) \to \gamma(x)$ for $x \in X$).

We shall denote by $P_m$ the partial summation operators with respect to the basis i.e.

$$P_m x = \sum_{k=1}^{m} b_k^*(x)\, b_k\,.$$

Let $X_0$ be the algebraic linear span of $(b_k)_{k=1}^\infty$.

Our main theorem will be that every residual locally complemented subspace of $X$ has a basis. This theorem is similar in spirit to result of JOHNSON, ROSENTHAL and ZIPPIN [7] on the existence of bases in BANACH spaces. Our proof will be achieved in several steps; the first is:

**Lemma 5.3.** *Suppose $E_0$ is a residual locally complemented subspace of $X$. Then there is a residual locally complemented subspace $E$ of $X$ isomorphic to $E_0$ and uniformly bounded sequences of finite-rank operators $S_n : X \to E \cap X_0 T_n : X \to E \cap X_0$ such that*

(5.3.1) $S_n e \to e \quad e \in E$

(5.3.2) $T_n^* \gamma \to \gamma \quad \text{weak*}, \quad \gamma \in \Gamma$.

Proof. Since $E_0$ is residual and locally complemented there are uniformly bounded operators $\hat{S}_n : X \to E_0$, $\hat{T}_n : X \to E_0$ so that $\hat{S}_n e_0 \to e_0$ for $e_0 \in E_0$ and $\hat{T}_n^* \gamma \to \gamma$ weak* for $\gamma \in \Gamma$.

Choose a countable dimensional dense subspace of $E_0$, $E_{00}$ say, such that $\hat{S}_n(X_0) \subset E_{00}$ for $n \in \boldsymbol{N}$ and $\hat{T}_n(X_0) \subset E_{00}$ for $n \in \boldsymbol{N}$. Since $\Gamma$ separates the points of $E_{00}$ it is possible to chose a Hamel basis $(w_n : n \in \boldsymbol{N})$ of $E_{00}$ such that the biothogonal functionals $\varphi_n \in \Gamma$. Now for each $n \in \boldsymbol{N}$ choose $m(n) \in \boldsymbol{N}$ so that

$$\|w_n - P_{m(n)} w_n\|^p \leqq 2^{-(n+1)} \|\varphi_n\|^p .$$

Let $v_n = w_n - P_{m(n)} w_n$ and define $K : X \to X$ by

$$Kx = \sum_{n=1}^{\infty} \varphi_n(x) \, v_n .$$

Then $\|K\| < 1$ and so $A = I - K$ is invertible. Now let $E = A(E_0)$.

Clearly $\{A\hat{S}_n A^{-1} : n \in \boldsymbol{N}\}$ is uniformly bounded and $A\hat{S}_n A^{-1} e \to e$ for $e \in E$. Let $S_n = A\hat{S}_n A^{-1} P_n$; then $\{S_n : n \in \boldsymbol{N}\}$ is a uniformly bounded sequence of finite-rank operators and $S_n e \to e$ for $e \in E$.

If $\gamma \in \Gamma$ and $x \in X$

$$\gamma(A\hat{T}_n A^{-1} x) = \gamma(\hat{T}_n A^{-1} x) - \gamma(K\hat{T}_n A^{-1} x)$$

and

$$\gamma(\hat{T}_n A^{-1} x) \to \gamma(A^{-1} x) \quad \text{as} \quad n \to \infty .$$

On the other hand

$$\gamma(K\hat{T}_n A^{-1} x) = \sum_{j=1}^{\infty} \varphi_j(T_n A^{-1} x) \, \gamma(v_j) .$$

Now

$$|\varphi_j(\hat{T}_n A^{-1} x)| \; |\gamma(v_j)| \leqq C \, \|\varphi_j\| \, \|v_j\|$$

where

$$C = (\sup \|\hat{T}_n\|) \, \|A^{-1}\| \, \|\gamma\| \, \|x\| .$$

Hence, by a form of the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \gamma(K\hat{T}_n A^{-1}x) = \sum_{j=1}^{\infty} \varphi_j(A^{-1}x)\,\gamma(v_j)$$

noting that

$$\lim_{n\to\infty} \varphi_j(\hat{T}_n A^{-1}x) = \varphi_j(A^{-1}x) \quad \text{since} \quad \varphi_j \in \Gamma .$$

Thus

$$\lim_{n\to\infty} \gamma(A\hat{T}_n A^{-1}x) = \gamma(A^{-1}x) - \gamma(KA^{-1}x) = \gamma(x) .$$

Now let $T_n = \hat{T}_n P_n$; then (5.3.2) follows immediately.

**Lemma 5.4.** *If $E$ satisfies the conclusions of Lemma 5.3, then there is a uniformly bounded sequence of finite-rank operators $V_n : X \to E \cap X_0$ such that*

(5.4.1) $\quad V_n e \to e \qquad e \in E$

(5.4.2) $\quad P_n V_n = P_n \quad n \in \mathbf{N}$.

Proof. Let $W_n = S_n + T_n - T_n S_n$. Then for fixed $k$,

$$P_k T_n x = \sum_{i=1}^{k} T_n^* b_i^*(x)\, b_i$$

and $\|P_k T_n - P_k\| \to 0$, as $n \to \infty$. Hence $P_k W_n \to P_k$ as $n \to \infty$. Choose $m(k)$ an increasing sequence so that

$$\|P_k W_{m(k)} - P_k\|^p < \frac{1}{2} k^{-p} \quad k = 1, 2, \ldots$$

Then on $[b_1, \ldots, b_k]$, $P_k W_{m(k)}$ is invertible with inverse $A_k$ with

$$\|A_k - I\|^p \leqq \frac{1}{2} k^{-p} \left(1 - \frac{1}{2} k^{-p}\right)^{-1} \leqq k^{-p} .$$

Let $V_k = W_{m(k)} A_k P_k$. Then $P_k V_k = P_k$ and $\{V_k\}$ is uniformly bounded. If $e \in E$

$$V_k e - P_k e = (W_{m(k)} A_k - I)\, P_k e$$

so that

$$\|V_k e - P_k e\|^p \leqq \|W_{m(k)}\|^p\, k^{-p} \|e\| + \|(W_{m(k)} - I)\, P_k e\|^p$$

and

$$W_{m(k)} - I = (T_{m(k)} - I)\,(I - S_{m(k)}) .$$

Hence

$$\|(W_{m(k)} - I)\, e\| \to 0$$

and

$$\|(W_{m(k)} - I)\,(e - P_k e)\| \to 0 .$$

Thus $V_k e \to e$ for $e \in E$.

**Lemma 5.5.** *If $E$ satisfies the conclusions of Lemma 5.4., then we can find a constant $\lambda$, an increasing sequence of positive integers $(h_n : n=0, 1, 2, \ldots)$ with $h_0=0$ and $h_1=1$, and a (not necessarily continuous) linear operator $T : X_0 \to X_0$ such that*

(5.5.1) *If* $G_n=[b_i : h_{n-1}<i\leqq h_n]$ *for* $n\geqq 1$, *then* $T(G_n)\subset G_{n+1}$ .

(5.5.2) *If* $g\in G_n$ *then* $\|Tg\|\leqq\lambda\|g\|$ *and* $\|Tg\|^p\leqq\lambda^p\,(d(g, E)^p+\lambda^{-2np}\|g\|^p)$

(5.5.3) *If* $x\in X_0$, *then* $x-Tx\in E$ .

Proof. Choose $\lambda$ sufficiently large so that $\lambda^p>2$,

$$\|P_m-P_n\|\leqq\lambda \quad m, n\geqq 0$$

(where $P_0=0$) and

$$\|V_n\|\leqq\lambda \qquad n\in \boldsymbol{N} .$$

$$\|I-V_n\|\leqq\lambda \quad n\in \boldsymbol{N} .$$

Next observe that if $x\in X$ and $\varepsilon>0$ then we can find $e\in E$ so that

$$\|x-e\|^p<d(x, E)^p+\varepsilon\|x\|^p$$

and as $(I-V_n)\,e\to 0$, for large enough $n$ we have

(5.5.4) $\|(I-V_n)\,x\|^p<\lambda^p(d(x, E)^p+\varepsilon\|x\|^p)$ .

By obvious compactness argument if $F$ is a finite-dimensional subspace of $X$ we can choose $n\in\boldsymbol{N}$ so that (5.5.4) holds for any $x\in F$.

Using this remark it is possible to construct two increasing sequences of positive integers $\{h_n : n=0, 1, 2, \ldots\}$ and $\{m_n : n=1, 2, 3, \ldots\}$ so that $h_0=0$, $h_1=1$, $m_n\geqq h_n$ and.

(5.5.5) $(I-V_{m(n)})\,(G_n)\subset G_{n+1}$

(5.5.6) $\|(I-V_{m(n)})\,g\|^p\leqq\lambda^p\,(d(g, E)^p+\lambda^{-2np}\|g\|^p)\quad g\in G_n$ .

Here we have used the fact that $P_{m(n)}\,(I-V_{m(n)})=0$.

Let $T : X_0\to X_0$ be the linear map defined by $Tg=(I-V_{m(n)})\,g$ for $g\in G_n$. Then the lemma follows.

**Theorem 5.6.** *If $E$ is a residual locally complemented subspace of a space $X$ with a basis; then $E$ also has a basis.*

Proof. We may assume that $E$ satisfies the conclusions of Lemma 5.5. We start with some observations where we let $T^0=I$.

(5.6.1) $P_nT^j=0 \qquad h_j\geqq n$

(5.6.2) $P_nT^j=P_nT^jP_n \qquad h_j<n, \quad j\geqq 0$

(5.6.3) $TP_{h_j}-P_{h_j}T=TQ_j \quad j=1, 2, 3, \ldots$

where

$$Q_j=P_{h_j}-P_{h_{j-1}} \qquad (j=1, 2, 3, \ldots) .$$

Note that $(I-T)\,(X_0)\subset E$ and define $w_n=(I-T)\,b_n$. Clearly $b_n^*(w_n)=1$ so that

$w_n \neq 0$. We shall show that $(w_n)$ is a basis for $E$. To this end we define a sequence of operators $U_n : X \to E$ by

$$U_n = (I-T)\, P_n \left( \sum_{j=0}^{k-1} T^j \right) P_n$$

where $h_{k-1} < n \leqq h_k$. If $x \in X_0$,

$$U_n x = (I-T)\, P_n \sum_{j=0}^{k-1} T^j x \,.$$

by (5.5.1). If $1 \leqq l \leqq n$

$$U_n w_l = (I-T)\, P_n \left( \sum_{j=0}^{k-1} T^j (I-T)\, b_l \right)$$
$$= (I-T)\, P_n\, (I-T^k)\, b_l = (I-T)\, b_l = w_l \,.$$

Similarly if $l > n$, $U_n w_l = 0$. Similar calculations show that $U_m U_n = U_n U_m = U_n$ whenever $m \geqq n$. Thus to show $(w_n)$ is a basis it will suffice to show $U_n e \to e$ for $e \in E$.

First we make a preliminary calculation; suppose $x \in X_0$ and $k \geqq 1$. Let

$$y = \sum_{j=0}^{k-1} T^j Q_{k-j} x \,.$$

Then $y \in G_k$ and $\sum_{j=0}^{k-1} (T^j - I)\, Q_{k-j}\, x \in E$. Hence

$$d(y, E) = d(P_{h_k} x, E) \,.$$

Also

$$\|T^j Q_{k-j} x\| \leqq \lambda^j \|Q_{k-j} x\|$$

by (5.5.2). Thus

$$\|y\|^p \leqq \left( \sum_{j=0}^{k-1} \lambda^{(j+1)p} \right) \|x\|^p \leqq \lambda^{(k+1)p} \|x\|^p$$

since $\lambda^p > 2$. Returning to 5.5.2, we have

$$\left\| T \left( \sum_{j=0}^{k-1} T^j Q_{k-j} x \right) \right\|^p \leqq \lambda^p \left( d(P_{h_k} x, E)^p + \lambda^{(1-k)p} \|x\|^p \right) \tag{5.6.4}$$

and in particular.

$$\left\| T \left( \sum_{j=0}^{k-1} T^j Q_{k-j} \right) \right\|^p \leqq \lambda^{2p} + \lambda^{(2-k)p} \leqq \lambda^{3p} \tag{5.6.5}$$

Now for any $k$

$$P_{h_k} - U_{h_k} = T Q_k \sum_{j=0}^{k-1} T^j P_{h_k}$$

by (5.6.3). Thus

$$P_{h_k} - U_{h_k} = T\left(\sum_{j=0}^{k-1} T^j Q_{k-j}\right) P_{h_k}$$

so that

$$\|P_{h_k} - U_{h_k}\| \leqq \lambda^4 .$$

Finally $\|U_{h_k}\| \leqq \lambda^5$ for all $k$. If $e \in E$

$$\|P_{h_k}e - U_{h_k}e\| \leqq \lambda^p \left[(d(P_{h_k}e, E))^p + \lambda^{(1-k)p}\|P_{h_k}e\|^p\right]$$

by (5.6.4) and so $P_{h_k}e - U_{h_k}e \to 0$ i.e. $U_{h_k}e \to e$.

If $h_{k-1} < n \leqq h_k$, then

$$U_{h_k} - U_n = (I-T)\,(P_{h_k} - P_n)\, Q_k\left(\sum_{j=0}^{k-1} T^j\right) P_{h_k}$$

$$= (I-T)\,(P_{h_k} - P_n) + (I-T)\,(P_{h_k} - P_n)\, Q_k\left(\sum_{j=1}^{k-1} T^j\right) P_{h_k} .$$

Now

$$Q_k\left(\sum_{j=1}^{k-1} T^j\right) P_{h_k} = \sum_{j=1}^{k-1} T^j Q_{k-j} P_{h_k} = T\left(\sum_{j=0}^{k-2} T^j Q_{k-1-j}\right) P_{h_k} .$$

Hence by (5.6.5)

$$\|U_{h_k} - U_n\|^p \leqq (\lambda^p + 1)\, \lambda^p\, [1 + \lambda^{3p}] \leqq \lambda^{7p} .$$

We conclude $\|U_n\| \leqq \lambda^8$ for all $n \in N$. If $h_{k-1} < n \leqq h_k$ then for $e \in E$

$$e - U_n e = (I - U_n)\,(e - U_{h_{k-1}} e)$$

$$\to 0 \quad \text{and} \quad n \to \infty .$$

Thus $(w_n)$ is a basis for $E$.

**Theorem 5.7.** *If $X$ is a quasi-*Banach *space with a basis and $E$ is a weakly dense locally complemented subspace of $X$ then $E$ also has a basis.*

Proof. There is a uniformly bounded sequence of operators $S_n : X \to E$ with $S_n e \to e$ for $e \in E$. Then if $\gamma \in \Gamma$, consider the map $A : X \to l_\infty$ defined by $Ax = (\gamma\,(x - S_n x))_{n=1}^\infty$. Since $l_\infty$ is locally convex then $A^{-1}(c_0)$ is weakly closed. However $A^{-1}(c_0) \supset E$ is weakly dense so that $A(X) \subset c_0$ i.e. $S_n^* \gamma \to \gamma$. Thus $E$ is also residual with $T_n = S_n$.

## 6. $\mathfrak{L}_p$-spaces when $0 < p < 1$

We shall say that a quasi-Banach space $X$ is an *$\mathfrak{L}_p$-space* for $0 < p < 1$ if it is isomorphic to a locally complemented subspace of a space $L_p(\Omega, \Sigma, \mu)$ where $(\Omega, \Sigma, \mu)$ is measure space.

Let us note that the standard definition of an $\mathfrak{L}_p$-space for $1 \leqq p \leqq \infty$ due to LINDENSTRAUSS and PEŁCZYŃSKI [15] is local in character. $X$ is an $\mathfrak{L}_p$-space ($1 \leqq p \leqq \infty$) if for some constant $\lambda$ and for every finite-dimensional subspace $F$ of $X$ there is a finite-dimensional subspace $G$ containing $F$ an isomorphism $S: G \to l_p^{(n)}$ (where $n = \dim G$) with $\|S\| \cdot \|S^{-1}\| \leqq \lambda$. The problem with this definition for $0 < p < 1$ (pointed out to us by W. J. STILES) is that it is by no means clear that even $L_p(0, 1)$ satisfies this condition. A possible alternative would be to define $X$ to be an $\mathfrak{L}_p$-space if there is a constant $\lambda$ and an increasing net of finite-dimensional subspace $(E_\alpha : \alpha \in A)$ with $UE_\alpha$ dense in $X$ and isomorphisms $S_\alpha : E_\alpha \to l_p^{(n_\alpha)}$ with $\|S_\alpha\| \cdot \|S_\alpha^{-1}\| \leqq \lambda$. This definition was adopted in [12]. It is a consequence of Theorem 6.1 below that every such space is an $\mathfrak{L}_p$-space in our sense here, but we do not know whether the converse holds.

In our opinion, the definition given above would serve as a natural definition for all $p$, $0 < p \leqq \infty$. However for $1 < p < \infty$, it would make a BANACH space $X$ an $\mathfrak{L}_p$-space if and only if it is a complemented subspace of a space $L_p(\Omega, \Sigma, \mu)$. The standard definition makes $X$ an $L_p$-space if it is a complemented non-HILBERTian subspace of a space $L_p(\Omega, \Sigma, \mu)$ [17]. For $p=1$ or $p=\infty$ our definition is the same as the standard one. The equivalence follows easily from Theorem 3.5 and results in [17] (Corollary to Theorem 3.2, and Theorem III (a)).

Note that every $\mathfrak{L}_p$-space is (isomrophic to) a $p$-BANACH space when $0 < p \leqq 1$. The following theorem lists several equivalent formulations of the statement that $X$ is an $\mathfrak{L}_p$-space.

**Theorem 6.1.** *Let $X$ be a $p$-BANACH space where $0 < p \leqq 1$. The following conditions on $X$ are equivalent:*

(1) *$X$ is an $\mathfrak{L}_p$-space*

(2) *$X$ is isomorphic to a locally complemented subspace of some $\mathfrak{L}_p$-space*

(3) *$X$ is isomorphic to the quotient of a $\mathfrak{L}_p$-space by a locally complemented subspace*

(4) *$X$ is isomorphic to the quotient of a space $l_p(I)$ by a locally complemented subspace.*

(5) *Whenever $Z$ is a $p$-BANACH space and $Q: Z \to X$ is an open map then $\ker Q$ is locally complemented in $Z$.*

(6) *There is a constant $\lambda$ such that whenever $F$ is a finite-dimensional subspace of $X$ and $\varepsilon > 0$ there are linear operators $S: F \to l_p$, $T: l_p \to X$ with $\|S\| \cdot \|T\| \leqq \lambda$ and $\|TSf - f\| \leqq \varepsilon \|f\|$ for $f \in F$.*

Proof. (1)$\Leftrightarrow$(2) follows from Proposition 3.3. Since every $p$-BANACH space is a quotient of $l_p(I)$ for some index set $I$, we have (5)$\Rightarrow$(4)$\Rightarrow$(3). To conclude the proof we shall show (1)$\Rightarrow$(6), (6)$\Rightarrow$(5) and (3)$\Rightarrow$(1).

(1)$\Rightarrow$(6): We suppose $X$ is a locally complemented subspace of $L_p(\Omega, \Sigma, \mu)$. Let $\lambda$ be a constant so that whenever $Y \supset X$ there is a projection $P_Y$: $Y \to X$ with $\|P_Y\| \leqq \lambda$. Suppose $F \subset X$ is a finite-dimensional subspace and $\varepsilon > 0$. By a routine approximation argument there is a finite subalgebra $\Sigma_0$ of $\Sigma$ and a linear map $S : F \to L_p(\Omega, \Sigma_0, \mu)$ with $\|S\| \leqq 1$ and $\|Sf - f\| \leqq \lambda^{-1}\varepsilon \, \|f\|$. Let $Y = X + L_p(\Omega, \Sigma_0, \mu)$ and let $T = P_Y | L_p(\Omega, \Sigma_0, \mu)$. Then $\|T\| \leqq \lambda$ and $\|TSf - f\| \leqq \varepsilon \, \|f\|$ for $f \in F$. Since $L_p(\Omega, \Sigma_0, \mu)$ is isometric to a subspace of $l_p$ which is the range of a norm-one projection, (1) follows.

(6) $\Rightarrow$ (5). For convenience we may suppose $Q$ is a quotient map. Let $F$ be a subspace of $Z$ of dimension $n$ with a basis $f_1, \ldots, f_n$ where $\|f_i\| = 1$ for $1 \leqq i \leqq n$. Suppose $0 < \varepsilon < 1$ and let $\alpha > 0$ be a constant so that

$$\left\| \sum_{i=1}^{n} a_i f_i \right\| \geqq \alpha (\Sigma |a_i|^p)^{1/p}$$

for all $a_1, \ldots, a_n$. Choose operators $S : Q(F) \to l_p$ and $T : l_p \to X$ with $\|T\| \cdot \|S\| \leqq \lambda$ and

$$\|TSQf - Qf\| \leqq \frac{1}{2} \alpha\varepsilon \|Qf\| \quad f \in F .$$

Since $l_p$ is projective for $p$-Banach spaces there is an operator $T_1 : l_p \to Z$ with $\|T_1\| = \|T\|$ and $QT_1 = T$. Define $R : F \to Z$ by $R = I - T_1 SQ$. Then $QR = (I - TS)\, Q$ and $\|QR\| \leqq \frac{1}{2} \alpha\varepsilon$. Thus we can find $g_1, \ldots, g_n \in Z$ with $Qg_i = QRf_i$ and $\|g_i\| < \alpha\varepsilon$. Define $L : F \to Z$ by $Lf_i = g_i$. Then

$$\left\| L\left( \sum_{i=1}^{n} a_i f_i \right) \right\| \leqq \alpha\varepsilon \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} \leqq \varepsilon \left\| \sum_{i=1}^{n} a_i f_i \right\| .$$

Hence $\|L\| \leqq \varepsilon$. Let $V = R - L$; then $V(F) \subset \ker Q$ and $\|V\| \leqq (\lambda^p + 2)^{1/p}$. If $f \in F \cap \cap \ker Q$, we have $Rf = f$ and $\|f - Vf\| \leqq \varepsilon \|f\|$. Hence $\ker Q$ is locally complemented in $Z$.

(3) $\Rightarrow$ (1). We may suppose $X$ is the quotient of a space $Y$ by a locally complemented subspace, where $Y$ is itself a locally complemented subspace of $L_p(\Omega, \Sigma, \mu)$.

By Theorem 4.5, $X$ is isomorphic to a locally complemented subspace of an ultraproduct $Y_{\mathfrak{U}}$ of $Y$. By Theorem 4.1, $Y_{\mathfrak{U}}$ is isomorphic to a locally complemented subspace of $(L_p(\Omega, \Sigma, \mu))_{\mathfrak{U}}$, which by Schreiber's Theorem 2.1 is a space $L_p(\Omega_1, \Sigma_1, \mu_1)$. Hence by Proposition 3.3, $X$ is a $\mathfrak{L}_p$-space.

Separable infinite-dimensional spaces $L_p(\Omega, \Sigma, \mu)$ $0 < p < 1$ are isomorphic to one of the spaces $l_p$, $L_p$ or $l_p \oplus L_p$. Based on this, we define, for $0 < p < 1$, a *discrete* $\mathfrak{L}_p$*-space* to be a separable $\mathfrak{L}_p$-space isomorphic to a locally complemented subspace of $l_p$. We also define $X$ to be a *continuous* $\mathfrak{L}_p$-space if it is isomorphic to a locally complemented subpace of $L_p$.

We shall say that a separable $\mathfrak{L}_p$-space is a *hybrid* $\mathfrak{L}_p$*-space* if it is neither discrete nor continuous,

**Theorem 6.4.** *Let $X$ be a separable $\mathfrak{L}_p$-space where $0 < p < 1$. The following conditions on $X$ are equivalent:*

(1) *$X$ is a discrete $\mathfrak{L}_p$-space*

(2) *$X$ has (BAP)*

(3) *$X$ has a basis.*

Proof. (1)$\Rightarrow$(3) Suppose $X$ is a locally complemented subspace of $l_p$, which is non-isomorphic to $l_p$. Let $l_p/X=Y$, so that $Y$ is also an $\mathcal{L}_p$-space. Then there is a quotient map $U: l_p \to Y$ which takes the unit vector basis $(e_n)$ of $l_p$ to a sequence $Ue_n$ dense in the unit ball of $Y$. By Corollary 2.3, ker $U \cong X$, and is locally complemented in $l_p$.

For each $n \in N$ select for $1 \leqq k \leqq n$, $u_{n,k} \in l_p$ with $u_{n,k} \in [e_{n+1}, e_{n+2,\ldots}]$, $Un_{n,k}=Ue_k$ and $\|u_{n,k}\| \leqq 2\|Ue_k\|$. Define $T_n: l_p \to \ker U$ by

$$T_n\left(\sum_{i=1}^{\infty} a_i e_i\right)=\sum_{i=1}^{n} a_i (e_i-u_{n,i}) .$$

Then $\|T_n\| \leqq (1+2^p)^{1/p}$ and $T_n^*\gamma \to \gamma$ weak* for $\gamma$ in the linear span of the biorthogonal functionals $(e_k^*)$. Hence ker $U$ is residual, and we can apply Theorem 5.6 to deduce that ker $U$ (and hence $X$) has a basis.

(3)$\Rightarrow$(2): Immediate

(2)$\Rightarrow$(1): We may suppose $X$ is a locally complemented subspace of $l_p \oplus L_p$. From the proof of Theorem 5.1 it is easy to see there is a uniformly bounded sequence of finite-rank operators $S_n: l_p \oplus L_p \to X$ with $S_n x \to x$ for $x \in X$. Let $P: l_p \oplus L_p \to l_p \oplus L_p$ by defined by $P(u, v)=(u, 0)$. Clearly $S_n=S_nP$ and so $S_nP_x \to \to x$ for $x \in X$. Thus $P$ maps $X$ isomorphically onto a space $P(X)$ of $l_p$ $(=l_p \oplus \{0\})$. Now $PS_nPx \to x$ for $x \in P(X)$ and so by Lemma 3.1, $P(X)$ is also locally complemented in $l_p$. Thus $X$ is a discrete $\mathcal{L}_p$-space.

**Remark.** Every separable $\mathcal{L}_p$-space $(1 \leqq p \leqq \infty)$ has a basis [7].

**Theorem 6.5.** *Let $X$ be a separable $\mathcal{L}_p$-space where $0<p<1$. Then $X$ is continuous if and only if $X^*=\{0\}$.*

Proof. If $X$ is locally complemented in $L_p$, then $X$ has HBEP i.e. $X^*=\{0\}$. Conversely if $X \subset l_p \oplus L_p$ and $X^*=\{0\}$ then $X \subset \{0\} \oplus L_p$.

A nice property of continuous $\mathcal{L}_p$-spaces is given by:

**Theorem 6.6.** *Let $X$ be a $p$-Banach space and let $Y$ be a continuous $\mathcal{L}_p$-space. Suppose $Q: X \to Y$ is an open mapping. Then (a) if $X$ has (BAP),* ker $Q$ *has (BAP) and (b) if $X$ has a basis,* ker $Q$ *has a basis.*

Proof. Since $Y^*=\{0\}$, ker $Q$ is weakly dense in $X$. Simply apply Theorems 5.1, 5.7 and 6.1 (5).

**Example 6.7** Let $C$ denote the subspace of $L_p$ of constant functions. Since $C$ fails to have HBEP, $C$ is not locally complemented. Thus $L_p/C$ is not a $\mathcal{L}_p$-space (see [12], where essentially this argument is invoked to show $L_p/C \not\cong L_p$). However $L_p/C$ is isomorphic to a subspace of $L_p$ by the embedding $T: L_p/C \to L_p$ $[(0,1)\times \times(0,1)]$ given by ([13])

$$Tqf(s, t)=f(s)-f(t) \quad s, t \in (0, 1) .$$

where $q: L_p \to L_p/C$ is the quotient map. Let $Y$ be this subspace of $L_p$. Now let $Q: l_p \to L_p$ be any quotient map and let $Z \subset l_p$ be defined by $Z=Q^{-1}(Y)$. We claim that $Z$ has (CEP). Indeed if $T: Z \to W$ is a compact operator then $T|\ker Q$

is compact and as $\ker Q$ is locally complemented it has an extension $T_1 : l_p \to W$ which is also compact. Now $T_1 - T$ factors to a compact operator on $Y \cong L_p|C$. Since there are no non-zero compact operators on $L_p$, $T_1 = T$ on $Z$.

However $Z$ is not locally complemented, since if it were $l_p|Z \cong L_p|C$ would be an $\mathfrak{L}_p$-space. We conclude that $Z$ also fails (BAP) by Theorem 5.1.

## 7. Example of $\mathfrak{L}_p$-spaces

If $0 < p < 1$, it is rather easy to construct numerous mutually non-isomorphic examples of separable $\mathfrak{L}_p$-spaces. This contrasts with the case $p = 1$ (see [6]). The construction used by Johnson and Lindenstrauss in [6] can be adapted to the case $p < 1$ to construct examples which are in general hybrids. We shall however take another route to construct examples. The following observation is routine:

**Theorem 7.1.** *Let $X$ be a separable $\mathfrak{L}_p$-space. Then $L_p(X)$ is a continuous $\mathfrak{L}_p$-space.*

As we shall see, the converse of Theorem 7.1 is false, for $0 < p < 1$. We can construct examples where $X$ is not an $\mathfrak{L}_p$-space but $L_p(X)$ is. For $p = 1$ this is impossible since $L_1(X)$ contains a complemented copy of $X$, when $X$ is locally convex.

**Theorem 7.2.** *Let $(\Omega, \Sigma, \mu)$ be a non-atomic measure space and let $\Sigma_0$ be a sub-$\sigma$-algebra of $\Sigma$. For $0 < p < 1$, let $L_p(\Sigma_0)$ denote the closed subspace of $L_p(\Omega, \Sigma, \mu)$ of all $\Sigma_0$-measurable functions, and let $\Lambda(\Sigma_0)$ denote the quotient $L_p(\Omega, \Sigma, \mu)/L_p(\Sigma_0)$. Then the following statements are equivalent:*

(1) *$L_p(\Sigma_0)$ is locally complemented in $L_p(\Omega, \Sigma, \mu)$*
(2) *$\Lambda(\Sigma_0)$ is an $\mathfrak{L}_p$-space*
(3) *$L_p(\Sigma_0)^* = \{0\}$*
(4) *$\mu \mid \Sigma_0$ is non-atomic.*

Proof. It follows from Theorem 6.1 that (1) and (2) are equivalent, and the equivalence of (3) and (4) is classical (cf. [4]). Since (1) implies that $L_p(\Sigma_0)$ has (HBEP) we have (1) ⇒ (3). We complete the proof by showing (4) ⇒ (1).

Consider the net $\Sigma_\alpha$ (under containment) of finite subalgebras of $\Sigma$. For each $\Sigma_\alpha$, let $A_1, \ldots, A_k$ be the atoms of $\Sigma_\alpha \cap \Sigma_0$ and let $(B_{ij} : 1 \leqq j \leqq m(i))$ be the atoms of $\Sigma_\alpha$ contained in $A_i$. Then there are disjoint sets $(C_{ij} : 1 \leqq j \leqq m(i))$ in $\Sigma_0$ such that

$$\bigcup_{j=1}^{m(i)} C_{ij} = A_i$$

$$\mu(C_{ij}) = \mu(B_{ij}) \, .$$

Define $Q_\alpha : L_p(\Sigma_\alpha) \to L_p(\Sigma_0)$ by $Q_\alpha(1_{B_{ij}}) = 1_{C_{ij}}$ for $1 \leqq j \leqq m(i)$ and $1 \leqq i \leqq k$. $Q_\alpha$ is an isometry, and $Q_\alpha f = f$ for $f \in L_p(\Sigma_\alpha \cap \Sigma_0)$. Now apply Lemma 3.1.

**Example 7.3.** If we take $\Omega$ a Polish space and $\Sigma$ the Borel sets in $\Omega$ and $\mu$ a nonatomic probability measure, then it is shown in [11] that $\Lambda(\Sigma_0) \cong L_p$ implies $L_p(\Sigma_0)$ is complemented in $L_p$.

As a special case consider $\Omega = (0, 1) \times (0, 1)$ with ordinary Lebesgue area measure and let $\Sigma_0$ be sets of the form $(0, 1) \times B$ where $B$ is a Borel subset of $(0, 1)$. Then $\Lambda(\Sigma_0) \cong L_p(L_p \mid C)$ whose $C$ is the space of constants in $L_p$. This is an $\mathfrak{L}_p$-space, but as seen in Example 6.7, $L_p \mid C$ is not an $\mathfrak{L}_p$-space.

**Example 7.4.** We now show how to construct an uncountable family of separable $p$-Banach spaces $(E_q : p < q \leq 1)$ so that

(7.4.1) $E_q$ is $p$-trivial [10] i.e. $\mathfrak{L}(L_p, E_q) = 0$

(7.4.2) There is a quotient map $Q : l_p \to E_q$ with $\ker Q \cong l_p$.

(7.4.3) The spaces $L_p(E_q)$ are mutually non-isomorphic $\mathfrak{L}_p$-spaces.

We start by letting $H$ be the subspace of $l_p$ spanned by the basic sequence $(e_{2m-1} + e_{2m} : m = 1, 2 \ldots)$ (where $(e_m)$ is the standard basis of $l_p$). Let $(A_m)$ be a Borel partitioning of $(0, 1)$ into sets of positive measure and suppose $A_m = B_{2m-1} \cup B_{2m}$ where $B_{2m-1} \cap B_{2m} = \Phi$ and $\mu(B_{2m-1}) = \mu(B_{2m}) = 1/2\mu(A_m)$ where $\mu$ is Lebesgue measure on $(0, 1)$. Define an isometry $V : l_p \to L_p$ by

$$V(e_k) = \mu(B_k)^{-1/p} 1_{B_k} \quad k = 1, 2, \ldots$$

For $p < q \leq 1$, define $T_q : H \to l_p$ by

$$T_q\,(e_{2m-1} + e_{2m}) = 2^{1/q} e_m\ .$$

Then $\|T_q\| = 2^{1/q - 1/p} < 1$, and let $G_q = (I - T_q)\,H$. Then $G_q \cong l_p$. Define $E_q = l_p \mid G_q$. Then (7.4.2) is immediate, and (7.4.1) follows from the lifting theorems of [12].

Next we show $L_p(E_q)$ is an $\mathfrak{L}_p$-space. For each $m \in \mathbf{N}$ we find $f_{2m-1}, f_{2m} \in L_p$ with

$$\|f_{2m-1}\|^p = \|f_{2m}\|^p = \frac{1}{2}\, \|T_q\,(e_{2m-1} + e_{2m})\|^p$$

and

$$f_{2m-1} + f_{2m} = VT_q\,(e_{2m-1} + e_{2m})\ .$$

Now there is an operator $U : L_p \to L_p$ with

$$U(1_{B_m}) = \mu(B_m)^{1/p} f_m \qquad m \in \mathbf{N}$$

and $\|U\|^p \leq \sup \|f_m\|^p < 1$. Clearly $VT_q x = UVx$ for $x \in H$: Thus $V(G_q) = (I - U)\ V(H)$ and $(I - U)$ is invertible.

Consider $L_p(V(G_q)) \subset L_p(L_p)$. By the above, there is an automorphism of $L_p(L_p)$ carrying $L_p(V(H_q))$ onto $L_p(V(H))$. However if we identify $L_p(L_p)$ as $L_p((0, 1) \times (0, 1))$ then $L_p(V(H))$ is identified with $L_p(\Sigma_0)$ where $\Sigma_0$ is the algebra generated by sets of the form $C \times B$ where $C$ is a Borel subset of $(0, 1)$ and $B$ is in the $\sigma$-algebra generated by $(B_k : k \in \mathbf{N})$. Thus $L_p(V(H))$ is locally complemeted. It is then also locally complemented in the smaller space $L_p(V(l_p))$ and thus $L_p(G_q)$ is locally complemented in $L_p(l_p)$ $(\cong L_p)$. Hence $L_p(l_p \mid G_q) = L_p(E_q)$ is an $\mathfrak{L}_p$-space.

Finally we show these spaces are mutually non-isomorphic. If $p < r \leqq 1$, there is no non-zero continuous linear operator from $E_q$ into $r$-Banach space if and only if $G_q$ is dense in $l_r$. If $r > q$, then $G_q$ is dense in $l_r$ since its closure contains the range of the invertible operator $A : l_r \to l_r$ given by

$$Ae_m = e_m - 2^{-1/q}(e_{2m-1} + e_{2m}) .$$

(Here $\|A - I\|^r \leqq 2^{1-r/q}$ on $l_r$). On the other hand if $r < q$, then $\|Tx\| \leqq 2^{r/q-1}\|x\|$ for $x \in H$ in $l_r$-norm, and so the closure of $G_q$ in $l_r$ has $e_{2m-1} + e_{2m} - 2^{1/q}e_m$ as a basis, equivalent to the usual $l_r$-basis. However $e_1 \notin G_q$ since if

$$e_1 = \sum_{m=1}^{\infty} c_m (e_{2m-1} + e_{2m} - 2^{1/q}e_m)$$

then solving co-ordinatewise $c_1 = -2^{-1/q}$, $c_2 = c_3 = 2^{-2/q}$, $c_4 = c_5 = c_6 = c_7 = -2^{-3/q}$ etc. and

$$\sum_{k=1}^{\infty} |c_k|^r = \sum_{n=1}^{\infty} 2^{n-1-nr/q} = \infty .$$

Thus the spaces $E_q$ are mutually non-isomorphic and even more, so are the spaces $l_p(E_q)$. Now by Theorem 8.4 of [11], the spaces $L_p(E_q)$ are mutually non-isomorphic.

**Remarks.** It can be shown that the containing $q$-Banach space of $E_q$ is isomorphic to $L_q$.

Also we note that if $G_q$ is the kernel of a quotient map of $l_p$ onto $L_p(E_q)$ then the spaces $G_q$ are mutually non-isomorphic discrete $\mathfrak{L}_p$-spaces. For suppose $S : G_q \to G_r$ is an isomorphism. Then since $G_q$ is locally complemented in $l_p$, and $l_p$ is an ultra-summand, Theorem 4.2 gives an extension $S_1 : l_p \to l_p$ of $S$. Similarly $S^{-1}$ has an extension $S_2 : l_p \to l_p$ and $S_2S_1 : l_p \to l_p$ extends the identity from $G_q$ to itself. Since $G_q$ is weakly dense, $S_2S_1 = I$, and similarly $S_1S_2 = I$ so that $l_p/G_q \cong$ $\cong l_p/G_r$, a contradiction.

## 8. Lifting theorems for continuous $\mathfrak{L}_p$-spaces

**Lemma 8.1.** *Let $X$ be a continuous $\mathfrak{L}_p$-space and let $Y$ be an ultra-summand. Then $L(X, Y) = \{0\}$.*

Proof. $X$ is isomorphic to a locally complemented subspace of $L_p(0, 1)$; $L_p(0, 1)$ is isomorphic to a locally complemented subspace of $L_p[(0, 1)^\Gamma]$ (where $\Gamma$ is any set whose cardinality exceeds that of $Y$) by Theorem 7.2. Hence by Theorem 4.2 it suffices to consider maps $T : L_p[(0, 1)^\Gamma] \to Y$. Suppose $f \in L_p[(0, 1)^\Gamma]$ is simple. Then there is a set of functions $(r_\gamma : \gamma \in \Gamma)$ mutually independent and independent of $f$ so that $\hat{\mu}\,(r_\gamma=+1)=\hat{\mu}\,(r_\gamma=-1)=\frac{1}{2}$ [We denote by $\hat{\mu}$ the product measure in $(0,1)^\Gamma$]. By KHINTCHINE's inequality,

$$(8.1.1) \qquad \|\Sigma a_\gamma(r_\gamma f)\| \leqq C(\Sigma|a_\gamma|^2)^{\frac{1}{2}}$$

for some constant $C$ whenever $a_\gamma$ is finitely non-zero. Since $|\Gamma|>|Y|$, there are infinitely many $\gamma$ with $T(r_\gamma f)=g$ for some $g\in Y$. By (8.1.1) we must have $g=0$. Thus for some $\gamma\in\Gamma$, $T(r_\gamma f)=0$, and $\|(1+r_\gamma)\,f\|_p=2^{1-1/p}\|f\|$. Thus $\|Tf\|\leqq 2^{1-1/p}\|T\|\cdot\|f\|$. Hence $\|T\|\leqq 2^{1-1/p}\|T\|$, i.e. $T=0$.

**Theorem 8.2.** *Let $X$ be a $p$-BANACH space, and let $N$ be a closed subspace of $X$ such that $X/N$ is a continuous $\mathcal{L}_p$-space. Let $Z$ be any quasi-BANACH space and let $T:N\to Z$ be a bounded linear operator. Each of the following conditions implies $T$ has a unique extension $T_1:X\to Z$*

(1) *$T$ is compact (and then $T_1$ is compact)*
(2) *$Z$ is $q$-convex for some $q>p$*
(3) *$Z$ is an ultra-summand.*

Proof. Let $Q:l_p(I)\to X$ be a quotient map, and consider $S:Q^{-1}(N)\to Z$. Then $Q^{-1}(N)$ is locally complemented and so in cases (1) and (3) $S$ has an extension $S_1:l_p(I)\to Z$ which is compact in case (1). In case (2) we appeal to the non-separable version of Theorem 5.1. There is a uniformly bounded set of finite-rank operators $V_\alpha:l_p(I)\to Q^{-1}(N)$ so that $V_\alpha x\to x$ for $x\in Q^{-1}(N)$. Since $Q^{-1}(N)$ is weakly dense, we have $\|V_\alpha x-x\|_1\to 0$ for $x\in l_p(I)$ where $\|\cdot\|_1$ is the $l_1$-norm on $l_p(I)$.

If $u\in Q^{-1}(N)$ and $\|u\|_1<\varepsilon$ then we can write $u=v_1+\ldots+v_n$ where the $v_i$'s have disjoint support and $\varepsilon^p\leqq\|v_i\|^p\leqq 2\varepsilon^p$ for $i\leqq n-1$, with $\|v_n\|^p\leqq 2\varepsilon^p$. Thus

$$\|SV_\alpha u\|\leqq\left(\sum_{i=1}^{n}\|V_\alpha v_i\|^q\right)^{1/q}\leqq 2^{1/p}n^{1/q}\,\|V_\alpha\|\,\varepsilon\,.$$

Hence

$$\|Su\|\leqq 2^{1/p}Cn^{1/q}\varepsilon$$

where $C=\sup\|V_\alpha\|$. Now

$$(n-1)\leqq\|u\|^p\varepsilon^{-p}$$

so that

$$\|Su\|\leqq 2^{1/p}C\,(1+\|u\|^p\varepsilon^{-p})^{1/q}\varepsilon\,.$$

We conclude that if $x\in l_p(I)$, since $\{V_\alpha x\}$ is bounded and $l_1$-CAUCHY, $SV_\alpha x$ converges in $Z$. Defining $S_1x=\lim\limits_\alpha SV_\alpha x$ for $x\in l_p(I)$ we obtain our extension.

The extension $S_1$ factors to $T_1:X\to Z$. In each case the extension is unique. In case (1) there are no compact operators on $X/N$ since (using Theorem 3.4) there are no compact operators on $L_p$ [8]. In case (2) use Lemma 8.1. In case (3) uniqueness follows from the construction.

**Theorem 8.3.** *Let $X$ be a continuous $\mathfrak{L}_p$-space and let $Z$ be a $p$-BANACH space. Let $N$ be a closed subspace of $Z$ which is either $q$-convex for $q > p$ or an ultra-summand. Then any bounded linear operator $T : X \to Z/N$ has a unique lift $T_1 : X \to Z$ (so that $qT_1 = T$ where $q : Z \to Z/N$ is the quotient map).*

Proof. Let $V : l_p \to X$ be a quotient map. Then there is a lifting $S : l_p \to Z$ of $TV : X \to Z/N$. Consider $S : \ker V \to N$. Since $\ker V$ is weakly dense and locally complemented in $l_p$, then in either case there is an extension $S_1 : l_p \to N$. Consider $(S - S_1) : l_p \to Z$; $S - S_1$ factors to the desired lift $T_1$. Again uniqueness follows from $\mathfrak{L}(X, N) = \{0\}$. In the case when $N$ is $q$-convex this follows from using 8.2 to extend to any operator from $X$ into $N$ to an operator from $L_p$ into $N$.

**Remarks.** Compare Theorem 4.2 of [17] with Theorems 8.2 and 8.3. It is possible to derive a statement similar to that of Theorem 4.2 in [17] from Theorem 8.2 for $\mathfrak{L}_p$-spaces when $p < 1$, but it no longer characterizes $\mathfrak{L}_p$-spaces. This is because (see Example 6.7) the (CEP) does not imply local complementation for subspaces of $l_p$ when $p < 1$.

## 9. Some applications to $H_p$

Now we consider the space $L_p(\mathfrak{T}, m)$ where $\mathfrak{T}$ is the unit circle in the complex plane and $dm = d\theta/2\pi$ is normalized Lebesgue measure on the circle. The closure of the polynomials in $L_p(\mathfrak{T})$ is denoted by $H_p$. It is easy to show for $0 < p < \infty$ that $H_p$ has (BAP), and it has recently been shown that it has a basis [23]. Also $H_p$ is a pseudo-dual space.

In [15] it is shown that $H_1$ is not an $\mathfrak{L}_1$-space.

**Proposition 9.1.** *$H_p$ is not a $\mathfrak{L}_p$-space for $0 < p < 1$.*

Proof. $H_p$ has (BAP) but does not embed into $l_p$ since it contains copies of $l_2$ (see Theorem 6.4).

Let us denote by $\bar{H}_p$ the space of polynomials in $\bar{z}$, i.e. the space of complex conjugates of $H_p$-functions. Let $J_p = H_p \cap \bar{H}_p$ the linear span of the real $H_p$-functions. Recently Aleksandrov [1] showed that $H_p + \bar{H}_p = L_p(\mathfrak{T})$ if $0 < p < 1$ (this is clearly false when $p = 1$ but true trivially for $1 < p < \infty$).

This means we can set up a map $U : H_p \oplus \bar{H}_p \to L_p(\mathfrak{T})$ defined by $U(f, g) = f + g$. Then $\ker U = \{(f, g) : f = -g\}$ is isomorphic to $J_p$.

**Proposition 9.2.** (1) *$J_p$ is not an ultra-summand and is therefore non-isomorphic to $H_p$*

(2) *$J_p$ has a basis.*

Proof. These remarks follow from the fact that the $\ker U$ must be locally complemented in $H_p \oplus \bar{H}_p$, but is clearly weakly dense. We use of course Theorems 6.1, 4.4, 5.1 and 5.7. For (2) we use the fact that $H_p$ has a basis [23].

**Theorem 9.3** (see [24]). *The spaces $L_p(\mathfrak{T})/J_p$, $L_p(\mathfrak{T})/H_p$ and $H_p/J_p$ are isomorphic.*

Proof. By considering the automorphism $z \to \bar{z}$ of the circle, we clearly obtain $L_p/H_p \cong L_p/\bar{H}_p$. If we define projections in $L_p$ by

$$Pf(z) = \frac{1}{2}(f(z) + f(-z))$$

$$Qf(z) = \frac{1}{2}(f(z) - f(-z))$$

then $P$ and $Q$ each leave $H_p$ invariant. Thus $L_p/H_p \cong P(L_p)/P(H_p) \oplus Q(L_p)/Q(H_p)$. Now if $Tf(z) = f(z^2)$, $T$ maps $L_p$ onto $P(L_p)$ and $H_p$ onto $P(H_p)$, isometrically. Similarly $T_1 f(z) = zf(z^2)$ maps $L_p$ onto $Q(L_p)$ and $H_p$ onto $Q(H_p)$. Thus $L_p/H_p \cong$ $\cong L_p/H_p \oplus L_p/H_p$.

Now use the Aleksandrov map $U: H_p \oplus \bar{H}_p \to L_p$. Since $U^{-1}(\bar{H}_p) = J_p \oplus \bar{H}_p$, we have $L_p/\bar{H}_p \cong H_p/J_p$. Since $U^{-1}(J_p) = J_p \oplus J_p$, $L_p/J_p \cong H_p/J_p \oplus \bar{H}_p/J_p$. However $\bar{H}_p/J_p \cong L_p/H_p$, by the above reasoning and so $L_p/J_p \cong L_p/H_p \cong H_p/J_p$.

**Theorem 9.4.** *Suppose $X$ is an ultra-summand or is $q$-convex for some $q > p$ and $T: J_p \to X$ is a bounded linear operator. Then $T$ can be expressed in the form*

$$Tf = S_1 f + S_2 f \quad f \in J$$

*where $S_1: H_p \to X$ and $S_2: \bar{H}_p \to X$ are bounded linear operators.*

Proof. Define $T_1: \ker U \to X$ by

$$T_1(f, -f) = Tf \quad f \in J_p .$$

Extend $T_1$ by Theorem 8.2 to give an operator $V: H_p \oplus \bar{H}_p \to X$. Write

$$S_1 f = V(f, 0) \qquad f \in H_p$$

$$S_2 f = -V(0, f) \quad f \in \bar{H}_p .$$

**Corollary 9.5.** (Aleksandrov). *Every continuous linear functional $\varphi$ on $J_p$ is the form*

$$\varphi(f) = \psi_1(f) + \psi_2(f)$$

*where $\psi_1 \in H_p^*$ and $\psi_2 \in \bar{H}_p^*$.*

**Remark.** Aleksandrov proves this directly [1].

Finally we apply our methods to characterize translation-invariant operators on $J_p$. An operator $T: X \to L_p$, where $X$ is a translation-invariant subspace of $L_p$, is translation-invariant if

$$T(f_\omega) = (Tf)_\omega \quad \omega \in \mathfrak{T}$$

where $f_\omega(z) = f(\omega z)$ .

Oberlin [19] has shown that every translation-invariant operator $T: L_p \to L_p$ is of the form

$$Tf = \sum_{n=1}^{\infty} c_n f_{\omega_n}$$

where $\sum_{n=1}^{\infty} |c_n|^p < \infty$, and $\omega_n \in \mathfrak{T}$. Clearly any such operator restricts to a translation-invariant endomorphism of $J_p$.

For $f \in H_p$, $f$ can be realized as the boundary values of a function $\hat{f}$ analytic in the open unit disc. Denote by $\theta_0(f)$ the value of $\hat{f}$ at 0. Then the map $f \mapsto \theta_0(f) \cdot 1$ is a translation-invariant linear operator on $J_p$, as is $f \mapsto \overline{\theta_0(\bar{f})} \cdot 1$ (where we exploit the fact the $\bar{f} \in H_p$ for $f \in J_p$) Let $\theta_\infty(f) = \overline{\theta_0(\bar{f})}$.

**Theorem 9.6.** *Let $T: J_p \to J_p$ be a translation-invariant linear operator. Then takes the form:*

$$(9.6.1) \qquad Tf(z) = \sum_{n=1}^{\infty} c_n f(\omega_n z) + a_1 \theta_0(f) + a_2 \theta_\infty(f)$$

*where $\omega_n \in \mathfrak{T}$, $\Sigma |c_n|^p < \infty$ and $a_1, a_2 \in C$.*

Proof. First define $T_1 : \ker U \to J_p$ by

$$T_1(f, -f) = Tf .$$

Then since $H_p$ is an ultra-summand, we can find a unique extension $S_1 : H_p \oplus \bar{H}_p \approx H_p$.

If $\omega \in \mathfrak{T}$, then $(f, g) \mapsto (S_1(f, g))_\omega$ extends $(f, -f) \mapsto (Tf)_\omega = T_1(f_\omega, -f_\omega)$, as does $(f, g) \mapsto S_1(f_\omega, g_\omega)$. Hence by uniqueness $(S_1(f, g))_\omega = S_1(f_\omega, g_\omega)$.

For $n \geqq 0$ choose $g_n(z) = z^{-n} \in \bar{H}_p$. Then $S_1(0, g_n)_\omega = \omega^{-n} S_1(0, g)$ and $S_1(0, g) \in H_p$. Hence $S_1(0, g_n) = 0$ if $n > 1$, and $S_1(0, g_0)$ is constant. We conclude that

$$S_1(0, f) = \alpha \theta_\infty(f) .$$

for some $\alpha \in C$. Thus we have

$$S_1(f, g) = V_1 f + \alpha \theta_\infty(g)$$

where $V_1 : H_p \to H_p$ is translation-invariant. Similarly $T_1$ extends to a translation invariant operator $S_2 : H_p \oplus \bar{H}_p \to \bar{H}_p$ of the form

$$S_2(f, g) = \beta \theta_0(f) + V_2 f .$$

On $\ker U$ $S_1 = S_2$. Thus there is an operator $R : L_p(\mathfrak{T}) \to L_p(\mathfrak{T})$ such that

$$RU = S_1 - S_2$$

and $R$ is clearly translation-invariant. Hence $R$ is of the form

$$Rf(z) = \sum_{n=1}^{\infty} c_n f(\omega_n z)$$

where $\Sigma |c_n|^p < \infty$ and $\omega_n \in \mathfrak{T}$.

If $f \in H_p$,

$$Rf = V_1 f - \beta \theta_0(f) .$$

Hence

$$V_1 f = Rf + \beta \theta_0(f)$$

and if $f \in J_p$

$$Tf = S_1(f, -f) = Rf + \beta \theta_0(f) - \alpha \theta_\infty(f)$$

which is of the form (9.6.1).

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