$$\sum a^2 = 2, \quad \sum b^2 = 5, \quad \sum ab = 3, \quad D = 1,$$

and
$$7^2 = 3^2 + 6^2 + 2^2.$$

The diagonal square matrix

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

yields the well known formula

$$(a^2 + b^2)^2 = (a^2 - b^2)^2 + (2ab)^2$$

for Pythagorean numbers.

Formula (2) shows that the square of the sum of the squares of $2n$ numbers a_i, b_i, some of which might be zeros, is a sum of

$$2 + \frac{n(n-1)}{2}$$

squares generally.

In the case $n = 2$ and $\sum ab \neq 0$, the same formula can be very useful for authors and teachers when preparing some numerical problems of three-dimensional coordinate geometry.

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3146. Quadratic forms that are perfect squares

If a and b are given integers, for how many integral values of x is $x^2 + ax + b$ a perfect square?

If $x = X$ is a solution,

$$X^2 + aX + b = Y^2$$

$$\therefore \quad (X + \tfrac{1}{2}a)^2 - Y^2 = \tfrac{1}{4}a^2 - b$$

$$\therefore \quad (2X - 2Y + a)(2X + 2Y + a) = a^2 - 4b = D \text{ say.}$$

(i) If $D = 0$, then $a = 2c$, $b = c^2$

and
$$(x^2 + ax + b) \equiv (x + c)^2.$$

$\therefore$ The number of solutions is infinite.

(ii) If a is odd, then $\quad a^2 \equiv 1 \pmod 4$

$$\therefore \quad D \equiv 1 \pmod 4$$

If
$$2X - 2Y + a = f_1$$
$$2X + 2Y + a = f_2$$

then $f_1 f_2 = D$ and $4X = f_1 + f_2 - 2a$.

But every factor pair (f_1, f_2) of D is such that

$$f_1 \equiv f_2 \equiv \pm 1 \pmod 4$$

since $D \equiv 1 \pmod 4$

$$\therefore \qquad f_1 + f_2 \equiv 2 \pmod 4$$

$\therefore$ since a is odd

$$f_1 + f_2 - 2a \equiv 0 \pmod 4.$$

$\therefore$ To every factor pair (f_1, f_2) of D there corresponds a solution $\frac{1}{4}(f_1 + f_2 - 2a)$. Since no two factor pairs have the same sum, these solutions are distinct; also, to every solution $x = X$, there corresponds a factor pair $(2X - 2Y + a,\ 2X + 2Y + a)$. Hence the number of solutions equals the number of factor pairs $= N$, say.

If $D = 1$ the factor pairs are $(+1, +1)$ and $(-1, -1)$ giving $N = 2$.

If $|D| = 3^{\alpha_2} 5^{\alpha_3} \dots p_n^{\alpha_n}$ where the largest prime factor of $|D|$ is p_n, the nth positive prime, then the number of factors (positive or negative) of D is $2 \prod_2^n (1 + \alpha_r)$, since all numbers of the form

$$\pm 3^{\beta_2} 5^{\beta_3} \dots p_n^{\beta_n} \quad \text{with} \quad 0 \leqslant \beta_r \leqslant \alpha_r$$

are factors.

If D is not a perfect square, there are, therefore, $\prod_2^n (1 + \alpha_r)$ factor pairs and $N = \prod_2^n (1 + \alpha_r)$.

If $D = z^2$, there are (excluding $\pm z$) $2 \prod_2^n (1 + \alpha_r) - 2$ factors giving $\prod_2^n (1 + \alpha_r) - 1$ factor pairs. In addition there are the factor pairs $(+z, +z)$, $(-z, -z)$.

$$\therefore \quad N = \prod_2^n (1 + \alpha_r) + 1.$$

(iii) If a is even, then $a = 2c$

$$(X - Y + c)(X + Y + c) = c^2 - b = \delta \text{ say.}$$

Then if

$$X - Y + c = g_1,$$
$$X + Y + c = g_2,$$
$$g_1 g_2 = \delta \quad \text{and} \quad 2X = g_1 + g_2 - 2c.$$

$\therefore$ There is a solution for every factor pair (g_1, g_2) of δ if $g_1 + g_2$ is even.

Let $$\delta = 2^m 3^{\alpha_2} 5^{\alpha_3} \dots p_n^{\alpha_n}.$$

If $m = 0$, every factor is odd and hence $g_1 + g_2$ is even for all pairs. Hence $N = \prod_2^n (1 + \alpha_r)$ unless δ is a perfect square, when

$$N = \prod_2^n (1 + \alpha_r) + 1.$$

If $m = 1$, then one factor is odd and one even.

Hence their sum is odd and $N = 0$.

If $m \geqslant 2$, for pairs giving solutions, both factors must be even. Hence the number of solutions is the number of factor pairs of

$$\tfrac{1}{4}\delta = 2^{m-2} 3^{\alpha_2} 5^{\alpha_3} \dots p_n^{\alpha_n}$$

$$\therefore\ N = (m-1) \prod_2^n (1 + \alpha_r)$$

unless δ is a perfect square when

$$N = (m-1) \prod_2^n (1 + \alpha_r) + 1$$

But $$4\delta = D = 2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} \dots p_n^{\alpha_n}$$

where $\alpha_1 = m + 2$.

$\therefore$ Conclusion

(1) If $D = 0$ then N is infinite.

(2) If $D = 1$ then $N = 2$.

If $D = \pm 2^{\alpha_1} 3^{\alpha_2} \dots p_n^{\alpha_n}$ and t be defined such that $t = 0$ unless D is a perfect square, when $t = 1$, [thus $t = \tfrac{1}{2}\{1 - (-1)^{\prod_1^n (1+\alpha_r)}\}$] then:

(3) If $\alpha_1 = 0$, $N = \prod_2^n (1 + \alpha_r) + t$.

(4) $\alpha_1 = 1$ is impossible.

(5) If $\alpha_1 = 2$, $N = \prod_2^n (1 + \alpha_r) + t$.

(6) If $\alpha_1 \geqslant 3$, $N = (\alpha_1 - 3) \prod_2^n (1 + \alpha_r) + t$.

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