EXTENDED ISAACS EQUATIONS FOR GAMES OF SURVIVAL

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§1. INTRODUCTION

In this note results due to the authors are described, full details of which appear in [3] and [4]. However, the results are also extended to payoffs of the form (2) below, which were first considered by Fleming in [5]. In fact, further extending of Fleming's idea, one is led to consider a value function $V^*(t,x,\xi)$ which is related to the value of the original game $V(t,x)$ by the identity

$$V^*(t,x,\xi) = e^{\xi}V(t,x).$$

Because V^* satisfies the Isaacs-Bellman equation (9) at points of differentiability the above identity implies that at the same points, V satisfies the 'extended Isaacs equation' (see (11) below). The results of Corollaries 3.2, 6.4 and Theorems 6.1, 6.2 and 7.1 are new.

SURVIVAL GAMES

As usual, we consider a dynamical system

$$\dot{x} = f(t,x,y,z), \quad x(t_0) = x_0. \tag{1}$$

Here $t \in [t_0, \infty)$, $x \in R^m$ and $y \in Y$, $z \in Z$ where Y and Z are compact metric spaces.

A terminal set $F \subset R \times R^m$ is supposed given. F is closed and $F \supset [T_0, \infty) \times R^m$ for some T_0. The game is a fixed time game if

$$F = [T_0, \infty) \times R^m.$$

Suppose t_F is the first time that the trajectory x(t) enters F; t_F is called the 'capture time.' The payoff is defined to be:

$$P = e^{\xi(t_F)} g(t_F, x(t_F)) + \int_{t_0}^{t_F} e^{\xi(t)} h(t,x,y,z)\,dt. \tag{2}$$

Here g and h are continuous real functions and ξ is a real variable such that:

$$\dot{\xi}(t) = k(t,x,y,z), \quad \xi(t_0) = 0. \tag{3}$$

f and k satisfy the usual Lipschitz and continuity conditions that ensure integrability. Throughout the game J_1 chooses his control variable $y \in Y$ in order to maximize the final payoff and J_2 chooses $z \in Z$ to minimize the final payoff.

If $g = 0$, $k = 0$ and $h = 1$, then $P = (t_F - t_0)$. That is, the payoff is the time to capture and the game is a 'pursuit-evasion game.' If $g = k = 0$ and $h \geq 0$, then the game is called a generalized pursuit-evasion game. In such games it is clear that J_2 is throughout trying to end the game and these games are discussed in [2]. However, we wish to discuss general games of the above form; such games are called 'games of survival.' We denote the game with dynamics (1) and payoff (2) by $G(t_0, x_0)$.

Note that the variable ξ can be considered as an additional trajectory variable and we could write $x^* = (x,\xi) \in R^{m+1}$.

If we introduce functions

$$g^*(t,x^*) = e^{\xi} g(t,x)$$

$$h^*(t,x^*,y,z) = e^{\xi} h(t,x,y,z)$$

and a terminal set $F^* = F \times R$, then the payoff is exactly of the form discussed in [3]. However, the initial condition $\xi(t_0) = 0$ for ξ is rather special; it will be convenient to consider the more general initial condition

$$\xi(t_0) = \xi_0. \tag{4}$$

Denoting the game with dynamics (1) and (3), with initial condition (4) for ξ, by $G^*(t_0,x_0,\xi_0)$, we notice that the payoff $P^*(y,z)$ in this game corresponding to a pair of control functions y,z is related to the payoff $P(y,z)$ in $G(t_0,x_0)$ by

$$P^*(y,z) = e^{\xi_0} P(y,z). \tag{5}$$

§2. NOTATION

Quantities associated with $G^*(t_0,x_0,\xi_0)$ will be indicated by an asterisk (*).

Write $\mathcal{M}_1(t_0)$ for the space of control functions for J_1 - that is, the set of measurable functions on $[t_0,\infty)$ with values in Y. $\mathcal{M}_2(t_0)$ denotes the control functions for J_2.

For $s \geq 0$ an s-delay strategy for J_1 is a map α: $\mathcal{M}_2(t_0) \rightarrow \mathcal{M}_1(t_0)$ such that if for any t_1:

$$z_1(t) = z_2(t) \quad \text{a.e. } t_0 \leq t \leq t_1$$

then

$$(\alpha z_1)(t) = (\alpha z_2)(t) \quad \text{a.e. } t_0 \le t \le t_1 + s.$$

Denote these strategies by $\Gamma_{t_0}(s)$. The s-delay strategies for J_2 are defined similarly and written $\Delta_{t_0}(s)$.

The value of a strategy α for J_1 in $G(t_0,x_0)$ is:

$$u(\alpha) = \inf_{z \in \mathcal{M}_2(t_0)} P(\alpha z, z)$$

and the value of the game to J_1 if he is able to use strategies with no delay is

$$U(t_0,x_0) = \sup_{\alpha \in \Gamma_{t_0}(0)} u(\alpha).$$

Similarly for $\beta \in \Delta_{t_0}(s)$

$$v(\beta) = \sup_{y \in \mathcal{M}_1(t_0)} P(y, \beta y)$$

and

$$V(t_0,x_0) = \inf_{\beta \in \Delta_{t_0}(0)} v(\beta).$$

The best values obtained using delay strategies are:

$$V^+(t_0,x_0) = \inf\Big[v(\beta); \quad \beta \in \bigcup_{s>0} \Delta_{t_0}(s)\Big]$$

and

$$V^-(t_0,x_0) = \sup\Big[u(\alpha); \quad \alpha \in \bigcup_{s>0} \Gamma_{t_0}(s)\Big].$$

Clearly, in general:

$$V^- \le V, \quad U \le V^+.$$

Later we wish to talk about the piecing together of delay rategies and it will be useful to have the following notion the 's-delay' value of a game:

Write $\mathcal{M}_1^s[t_0]$ for the measurable functions y: $[t_0, t_0 + s]$ Y. Then $\Gamma_{t_0}[s|y]$ is the set of strategies $\alpha \in \Gamma_{t_0}(s)$ such at $(\alpha z)(t) = y(t)$ a.e. for $t_0 \le t \le t_0 + s$ where $y \in \mathcal{M}_1^s[t_0]$ fixed. $\mathcal{M}_2^s[t_0]$ and $\Delta_{t_0}[s|z]$ are defined similarly.

The upper and lower s-delay values are then defined as:

$$Q_s^+(t_0, x_0) = \sup_{z \in \mathcal{M}_2^s[t_0]} \; \inf_{\beta \in \Delta_{t_0}[s|z]} v(\beta),$$

$$Q_s^-(t_0, x_0) = \inf_{y \in \mathcal{M}_1^s[t_0]} \; \sup_{\alpha \in \Gamma_{t_0}[s|y]} u(\alpha).$$

nally,

$$Q^+(t_0, x_0) = \inf_{s>0} Q_s^+(t_0, x_0)$$

$$Q^-(t_0, x_0) = \sup_{s>0} Q_s^-(t_0, x_0).$$

From the definitions note that

$$Q^- \le V^- \le V^+ \le Q^+. \tag{6}$$

Remark All the above values can be defined in $G^*(t_0, x_0, \xi_0)$ d in fact from (5) we have immediately that:

$$\begin{aligned} U^*(t_0, x_0, \xi_0) &= e^{\xi_0} U(t_0, x_0) \\ V^{+*}(t_0, x_0, \xi_0) &= e^{\xi_0} V^+(t_0, x_0) \\ Q^{+*}(t_0, x_0, \xi_0) &= e^{\xi_0} Q^+(t_0, x_0) \end{aligned} \tag{7}$$

and similarly for all other values defined above.

§3. DYNAMIC PROGRAMMING

Let $\tau: \mathcal{M}_1(t_0) \times \mathcal{M}_2(t_0) \to [t_0,\infty)$ be a map which prescribes a stopping time corresponding to any pair of control functions. Further, let θ be a real function (not necessarily continuous) on $R \times R^m$.

We can then define a game $G_\tau(t_0,x_0;\theta)$ with dynamics (1) and (3) but with a payoff

$$P_{\tau,\theta}(y,z) = \theta(\tau(y,z);x(\tau(y,z))) + \int_{t_0}^{\tau(y,z)} e^{\xi(t)} h(t,x(t),y(t),z(t))dt. \tag{8}$$

The various upper and lower values of this game can be defined as before and will be denoted

$U_\tau(t_0,x_0;\theta)$

$V_\tau^+(t_0,x_0;\theta)$

$Q_{s,\tau}^+(t_0,x_0;\theta)$, etc.

Instead of the initial condition $\xi(t_0) = 0$, we could consider a game $G_\tau^*(t_0,x_0,\xi_0;\theta)$ with initial condition $\xi(t_0) = \xi_0$ for ξ. The payoff $P_{\tau,\theta}^*$ is still given by (8) and the values

$U_\tau^*(t_0,x_0,\xi_0;\theta)$

$V_\tau^{+*}(t_0,x_0,\xi_0;\theta)$, etc.

can all be defined.

The stopping time τ is said to be non-anticipatory if given $y \in \mathcal{M}_1(t_0)$ and $z \in \mathcal{M}_2(t_0)$ and if:

$$\left.\begin{aligned} y'(t) &= y(t) \text{ a.e.} \\ z'(t) &= z(t) \text{ a.e.} \end{aligned}\right\} \text{ up to } \tau(y,z)$$

then $\tau(y',z') = \tau(y,z)$.

Note that the capture time t_F is non-anticipatory. We shall often write τ for $\tau(y,z)$.

It is interesting to consider games with stopping times in which the function θ is one of the values of the game played from the stopping position. The games $G_\tau^*(t_0,x_0,\xi_0;\theta)$ with

$$\theta = U^*,\ V^*,\ V^{+*},\ \text{etc.}$$

are of the form discussed in [3] and so we have the following dynamic programming results:

<u>Theorem 3.1</u> If τ is a non-anticipating stopping time, then:

$$U^*(t_0,x_0,\xi_0) = U_\tau^*(t_0,x_0,\xi_0;U^*)$$

$$V^*(t_0,x_0,\xi_0) = V_\tau^*(t_0,x_0,\xi_0;V^*).$$

However, for V^{+*} strategies do not fit so well together and

$$V^{+*}(t_0,x_0,\xi_0) \geq V_\tau^{+*}(t_0,x_0,\xi_0;V^{+*})$$

$$V^{-*}(t_0,x_0,\xi_0) \leq V^{-*}(t_0,x_0,\xi_0;V^{-*}).$$

If $Q_s^{\pm *}(t,x,\xi)$ converges to $Q^{\pm *}(t,x,\xi)$ uniformly on compacta, then

$$Q^{+*}(t_0,x_0,\xi_0) \leq Q_\tau^{+*}(t_0,x_0,\xi_0;Q^{+*})$$

$$Q^{-*}(t_0,x_0,\xi_0) \geq Q^{-*}(t_0,x_0,\xi_0;Q^{-*}).$$

However:

$$Q_s^{+*}(t_0,x_0,\xi_0) \geq Q_{s,\tau}^{+*}(t_0,x_0,\xi_0;U^*)$$

$$Q_s^{-*}(t_0,x_0,\xi_0) \leq Q_{s,\tau}^{-*}(t_0,x_0,\xi_0;V^*).$$

Because $U^*(t_0,x_0,\xi_0) = e^{\xi_0}U(t_0,x_0)$ these imply that for $G_\tau(t_0,x_0;\theta)$ with $\theta = U, V,$ etc.:

Corollary 3.2

$$U(t_0,x_0) = U_\tau(t_0,x_0;e^{\xi(\tau)}U)$$

$$V(t_0,x_0) = V_\tau(t_0,x_0;e^{\xi(\tau)}V)$$

$$V^+(t_0,x_0) \geq V_\tau^+(t_0,x_0;e^{\xi(\tau)}V^+), \text{ etc.}$$

§4. THE ISAACS-BELLMAN EQUATION

Using the dynamic programming relations it is proved as in [3] that for $G^*(t_0,x_0,\xi_0)$ the value functions at points of differentiability satisfy:

$$L^{+*}U^* = \frac{\partial U^*}{\partial t} + \min_z \max_y \left(\nabla U^* \cdot f + \frac{\partial U^*}{\partial \xi} k + h^*\right) = 0$$

$$L^{-*}V^* = \frac{\partial V^*}{\partial t} + \max_y \min_z \left(\nabla V^* \cdot f + \frac{\partial V^*}{\partial \xi} k + h^*\right) = 0 \qquad (9)$$

subject to the boundary conditions

$$U^*(t,x,\xi) = V^*(t,x,\xi) = g^*(t,x,\xi) = e^\xi g(t,x)$$

for $(t,x,\xi) \in \partial F^*$.

Also, for example,

$$L^{+*}V^{+*} \le 0$$
$$\text{and} \quad L^{-*}V^{-*} \ge 0. \tag{10}$$

However, the game we are really interested in is $G(t_0,x_0)$ where $\xi_0 = 0$ and we have observed that, for example,

$$U^*(t,x,\xi) = e^{\xi}U(t,x).$$

Consequently, $U^*(t,x,\xi)$ is differentiable if and only if $U(t,x)$ is differentiable and at points of differentiability $U(t,x)$ satisfies the 'extended Isaacs equation':

$$L^+U = \frac{\partial U}{\partial t} + \min_z \max_y \left(\nabla U\cdot f + kU + h\right) = 0. \tag{11}$$

Similarly, at points of differentiability

$$L^-V = \frac{\partial V}{\partial t} + \max_y \min_z \left(\nabla V\cdot f + kV + h\right) = 0.$$

U and V satisfy the boundary condition $U = V = g$ on ∂F. There are also results of the form:

$$L^+V^+ \le 0$$
$$L^-V^- \ge 0, \text{ etc.}$$

5. CONTINUITY OF VALUES

<u>Definition 5.1</u> Write

$$\bar{Q}_s^{+*}(t_0,x_0,\xi_0) = \limsup_{(t,x,\xi)\to(t_0,x_0,\xi_0)} Q_s^+(t,x,\xi)$$

$$= e^{\xi_0}\bar{Q}_s^+(t_0,x_0) = e^{\xi_0} \limsup_{(t,x)\to(t_0,x_0)} Q_s^+(t,x),$$

where the final equality defines $\bar{Q}_s^+$.

Definition 5.2 Suppose $(t,x,\xi) \in \partial F^*$ (i.e., $(t,x) \in \partial F$), U^* is continuous at (t,x,ξ) and

$$\lim_{s\to 0} \bar{Q}_s^{+*}(t,x,\xi) = g^*(t,x) = e^{\xi} g(t,x).$$

Then (t,x,ξ) is said to be Q^{+*} regular.

Note these conditions are equivalent to: U being continuous at $(t,x) \in \partial F$ and

$$\lim_{s\to 0} \bar{Q}_s^+(t,x) = g(t,x),$$

i.e., (cf. [3]) to (t,x) being a Q^+ regular point of ∂F.

∂F (resp. ∂F^*) is said to be Q^+ (resp. Q^{+*}) regular if each point is.

From [3] we quote:

Theorem 5.3 ∂F^* Q^{+*} regular implies Q^{+*} is continuous, and

$$\lim_{s\to 0} Q_s^{+*}(t,x) = Q^{+*}(t,x)$$

uniformly on compacta. Also

$$Q^{+*} = V^{+*}.$$

The same statements hold for Q^+.

§6. COMPARISON THEOREMS

Modifying Section 5 of [3] we can prove the following result.

Theorem 6.1 Suppose $\theta_1(t,x)$ is C^1 on R^{m+1} - int F,

$$\theta_1(t,x) \geq g(t,x) \text{ on } \partial F \tag{12}$$

and

$$L^+\theta_1 = \frac{\partial\theta_1}{\partial t} + \min_z \max_y \left(\nabla\theta_1 \cdot f + k\theta_1 + h\right) \leq 0 \tag{13}$$

for $(t,x) \in R^{m+1}$ - int F. Then

$$Q^+(t,x) \leq \theta_1(t,x). \tag{14}$$

Proof Writing

$$\theta_1^*(t,x,\xi) = e^\xi \theta_1(t,x)$$

and replacing g by g*, f by F*, conditions (12) and (13) are satisfied by θ_1^*. By Theorem 5.3 of [3]:

$$Q^{+*}(t,x,\xi) \leq \theta_1^*(t,x,\xi).$$

Dividing by e^ξ (14) follows.

Similarly we prove:

Theorem 6.2 Suppose

$$\theta_2(t,x) \leq g(t,x) \text{ on } \partial F$$

and

$$L^+\theta_2 = \frac{\partial\theta_2}{\partial t} + \min_z \max_y \left(\nabla\theta_2 \cdot f + k\theta_2 + h\right) \geq 0$$

on R^{m+1} - int F. Then

$$\theta_2(t,x) \leq U(t,x) \quad \text{on } R^{m+1} - \text{int } F.$$

<u>Proof</u> This result is obtained by writing $\theta_2(t,x,\xi) = e^{\xi}\theta_2(t,x)$ and quoting the result for $U^*(t,x,\xi)$ given by Theorem 5.4 of [3].

A consequence of the above results is:

<u>Theorem 6.3</u> Suppose there are C^1 functions $\theta_1(t,x)$ and $\theta_2(t,x)$ such that $\theta_1 = \theta_2 = g$ on ∂F and $L^+\theta_1 \le 0 \le L^+\theta_2$ on R^{m+1} - int F. Then ∂F is Q^+ regular (or, equivalently, ∂F^* is Q^{+*} regular).

Further (see section 4 of [4]) $U(t,x)$ is Lipschitz continuous on ∂F and so uniformly Lipschitz continuous in any bounded region of R^{m+1}.

A well known theorem of Rademacher says that a uniformly Lipschitz function is differentiable almost everywhere; consequently from equations (11) we have:

<u>Corollary 6.4</u> Under the hypotheses of Theorem 6.3 the value $U(t,x)$ of the game $G(t,x)$ is almost everywhere a solution of

$$\frac{\partial U}{\partial t} + \min_z \max_y \left(\nabla U \cdot f + kU + h\right) = 0$$

with the boundary condition $U = g$ on ∂F. Similarly, V almost everywhere satisfies

$$\frac{\partial V}{\partial t} + \max_y \min_z \left(\nabla V \cdot f + kV + h\right) = 0.$$

For the general payoff (2) the following result follows as in [3].

<u>Theorem 6.5</u> If the following Isaacs condition holds for all $(p,q) \in R^{m+1}$, $z \in R^m$, $t \in [0,1]$

$$\min_{z} \max_{y} (p \cdot f + qk + h) = \max_{y} \min_{z} (p \cdot f + ak + h)$$

and if ∂F is Q^+ and Q^- regular, then

$$Q^+ = V^+ = V^- = Q^-.$$

§7. NONLINEAR EQUATIONS

Consider the Cauchy problem

$$Lu = u_t + G(t,x,u,\nabla u) = 0, \quad u = g \text{ on } \partial F \tag{15}$$

where $F \subset R^{m+1}$ is as described in Section 1. We now indicate how the results of [4] can be extended to the case where G is also a function of u. We suppose G is uniformly Lipschitz in all variables and denote Lipschitz constants by K.

For suitable bounds (α,α_0) and (β,β_0) we introduce the control sets

$$Y = \{y = (y',y_0);\ |y'| \leq \alpha,\ |y_0| \leq \alpha_0\}$$

$$Z = \{z = (z',z_0);\ |z'| \leq \beta,\ |z_0| \leq \beta_0\}$$

and the functions

$$f(t,x,y,z) = \frac{G(t,x,y',y_0)y'}{1 + |y'|^2} + z'$$

$$h(t,x,y,z) = \frac{G(t,x,y',y_0)}{1 + |y'|^2} - y' \cdot z' - y_0 z_0 \tag{16}$$

$$k(t,x,y,z) = z_0.$$

As in Fleming [5], Lemma 3, it can be shown that given (α,α_0) there are bounds (β,β_0) such that $\max_{Y} \min_{Z} (p \cdot f + uk + h)$

$= G(t,x,u,p)$. In fact, $\beta_0 = K$ and β depends on K and T_0. The bounds α and α_0 are a priori bounds for $|\nabla u|$ and u, respectively.

Theorem 7.1 Consider the nonlinear boundary value problem (15). Suppose there are C^1 functions θ_1, θ_2 satisfying $\theta_1 = \theta_2 = g$ on ∂F and

$$L\theta_1 \geq 0 \geq L\theta_2 \quad \text{on } R^{m+1} - \text{int } F.$$

Suppose further that for any time t_0, θ_i and $|\nabla\theta_i|$, i = 1, 2, are bounded for $t \geq t_0$. Then there is a generalized solution of the boundary value problem (15).

Proof The bounds for θ_i and $|\nabla\theta_i|$ give a priori bounds α and α_0 and so determine a control set Y. The bounds β and β_0 for Z are determined as described above.

For any position (t_0,x_0) consider the differential game $G(t_0,x_0)$ with dynamics

$$\dot{x}(t) = f(t,x,y,z);\ x(t_0) = x_0$$

$$\dot{\xi}(t) = k(t,x,y,z);\ \xi(t_0) = 0$$

and payoff

$$P(y,z) = e^{\xi(t_F)} g(t_F, x(t_F)) + \int_{t_0}^{t_F} e^{\xi(t)} h(t,x,y,z)\,dt$$

where f, h and g are the functions defined in (16).

It is also convenient to introduce the related differential game $G^*(t_0,x_0,\xi_0)$ in which $\xi(t_0) = \xi_0$. As above, write $h^* = e^{\xi}h$, $g^* = e^{\xi}g$ and $F^* = F \times R$. Then, for example,

$$V^*(t,x,\xi) = e^{\xi}V(t,x)$$

and the functions $\theta_1^*(t,x,\xi) = e^{\xi}\theta_1(t,x)$ and $\theta_2^*(t,x,\xi) = e^{\xi}\theta_2(t,x)$ will be upper and lower solutions of the lower Isaacs-Bellman equation of $G^*(t,x,\xi)$.

Now as $\dot{\xi}(t) = k = z_0$ and $|z_0| \leq \beta_0 = K$, we have $|\xi(t)| \leq T_0 - t_0$. Also $|\theta_i(t,x)| \leq \alpha$ for $t \geq t_0$, so we see that θ_1^* and θ_2^* have bounded derivatives on R^{m+2} - int F^* for $t \geq t_0$. Consequently, by Theorem 5.1 of [4] $V^*(t,x,\xi)$ is almost everywhere a solution of

$$\frac{\partial V^*}{\partial t} + \max_y \min_z \left(\nabla V^* \cdot f + \frac{\partial V^*}{\partial \xi} k + h^*\right) = 0$$

and

$$V^* = g^* \text{ on } \partial F^*.$$

But $V^*(t,x,\xi) = e^{\xi}V(t,x)$ so

$$\frac{\partial V}{\partial t} + \max_y \min_z (\nabla V \cdot f + kV + h) = 0 \quad \text{a.e.}$$

i.e.,

$$\frac{\partial V}{\partial t} + G(t,x,V,\nabla V) = 0 \quad \text{a.e.}$$

and

$$V = g \text{ on } \partial F.$$

That is, V is a generalized solution of (15). Further, as in [4] it can be shown the differential game solution is unique.

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