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Orthonormal systems in Banach spaces and their applications

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1. *Introduction.*

By an orthonormal system in a general complex Banach space, we mean a collection $\{e_\alpha: \alpha \in \mathscr{A}\}$ of unit vectors such that, for each α, there is an hermitian (in the numerical range sense, see (4)) projection P_α whose range is lin (e_α) and such that $P_\alpha P_\beta = 0$, if $\alpha \neq \beta$. This paper is devoted to the study of orthonormal systems in general Banach spaces, and their applications to problems of characterizing isometries and hermitian operators.

We note first that our definition of an orthonormal system differs from that of Berkson (2), p. 116 (he requires the projections P_α to be perpendicular). However, we show in section 7 that the definitions are equivalent, although a good deal of work seems to be required to prove this. Orthonormal bases have been studied under the name normalized hyperorthogonal bases in (19), 355 (cf. (7)).

Sections 2 and 3 are devoted to elementary observations concerning hermitian projections. The most significant results of these sections are the Diagonalization theorem (2·4) and its consequence (2·6). These are obtained by generalizing techniques used previously in certain sequence space settings. In section 4, we obtain some results of Berkson and Tam by methods which seem more elementary than the original arguments; also we require some generalizations for future applications.

In section 5, we prove our fundamental results on orthonormal systems. It turns out that, in any Banach space X, there is a unique subspace $\hat{h}(X)$ which is the closed linear span of any maximal orthonormal system. Furthermore, $\hat{h}(X)$ may be decomposed into a direct sum of Hilbert spaces, which we call the Hilbert components of X.

These results are applied in section 6 to obtain theorems concerning the possible forms of hermitian operators and isometries on X. Some of these results have been previously obtained by Fleming and Jamison ((7), (8)); see also (18) and Tam (21). We feel the proofs here are rather simpler. Some applications of these results are also given. Thus Theorem 6·4 characterizes Hilbert spaces, while Theorem 6·5 characterizes the hermitian operators of rank one on a Banach function space. As already remarked, section 7 links our ideas with those of Berkson.

In sections 8 and 9, we consider a problem posed by Rolewicz (17). A norm on a Banach space X is maximal if there is no equivalent norm for which the group of isometries is strictly larger. Rolewicz shows that the spaces $L_p(0,1)$ $(1 \leqslant p < \infty)$, l_p $(1 \leqslant p < \infty)$ have maximal norms, and a finite-dimensional space has maximal norm if and only if it is a Hilbert space. He asks ((17), p. 259) whether $C[0,1]$ has maximal norm. In section 8 we show that $C[0,1]$, and indeed $C_0(S)$ for a large class of locally compact spaces S, have maximal norm. In section 9, we consider the same problem for spaces of real functions; here the techniques are necessarily quite different and the results rather weaker. We show that $C_{R,0}(S)$ has maximal norm when S is a manifold without boundary of dimension greater than or equal to two.

2. *Hermitian decomposition and splittings.*

DEFINITION 2·1. *An* (hermitian) splitting *of a Banach space is a family* $\{P_\alpha: \alpha \in \mathscr{A}\}$ *of hermitian projections such that* $P_\alpha P_\beta = 0$, *if* $\alpha \neq \beta$. *The* extent *of the splitting is the closed linear span of* $\{\cup P_\alpha(X): \alpha \in \mathscr{A}\}$. *If the extent of* $\{P_\alpha: \alpha \in \mathscr{A}\}$ *is* X, *then* $\{P_\alpha: \alpha \in \mathscr{A}\}$ *is called an* hermitian decomposition *of* X.

A closed linear subspace Y of X is called *split* if it is the range of an hermitian projection (necessarily unique, see (2) or (16)).

We start by listing some elementary results showing first that an hermitian decomposition is an unconditional Schauder decomposition.

PROPOSITION 2·2. *Let* $\{P_\alpha: \alpha \in \mathscr{A}\}$ *be a splitting of* X, *and let* $Y \subset X$ *be its extent. Then for* $x \in Y$

$$x = \sum_{\alpha \in \mathscr{A}} P_\alpha x.$$

Proof. For any finite subset $\mathscr{F}$ of $\mathscr{A}$, $P_{\mathscr{F}} = \sum_{\alpha \in \mathscr{F}} P_\alpha$ is an hermitian projection and so has norm one. Thus $\{P_{\mathscr{F}}: \mathscr{F} \subset \mathscr{A}\}$ is an equicontinuous family and so the set $\{x: P_{\mathscr{F}} x \to x\}$ is a closed linear subspace of X containing $P_\alpha(X)$ for $\alpha \in \mathscr{A}$. The result follows.

PROPOSITION 2·3. *Let* $\{P_\alpha: \alpha \in \mathscr{A}\}$ *be a splitting of* X *and let* $Y \subset X$ *be its extent. Then the following are equivalent.*

(i) *For* $x \in X$, $\sum_{\alpha \in \mathscr{A}} P_\alpha x$ *converges.*

(ii) Y *is split.*

(iii) *There is an hermitian decomposition of* X *containing* $\{P_\alpha: \alpha \in \mathscr{A}\}$.

These conditions are implied by:

(iv) Y *contains no subspace isomorphic to* c_0.

Proof. (i) ⇒ (ii). Let $Px = \sum_{\alpha \in \mathscr{A}} P_\alpha x$, $x \in X$. Then P is necessarily an hermitian projection whose range is Y.

(ii) ⇒ (iii). If P is an hermitian projection onto Y, adjoin $I - P$ to $\{P_\alpha: \alpha \in \mathscr{A}\}$.

(iii) ⇒ (i). Suppose $\{P_\alpha: \alpha \in \mathscr{A}\} \cup \{P_\beta: \beta \in \mathscr{B}\}$ is an hermitian decomposition of X. Then, for $x \in X$, $\sum_{\mathscr{A} \cup \mathscr{B}} P_\alpha x$ is unordered convergent and hence so is $\sum_{\alpha \in \mathscr{A}} P_\alpha x$.

(iv) ⇒ (i). If $\Sigma P_\alpha x$ does not converge, then there is a sub-series $\sum_{i=1}^{\infty} P_{\alpha_i} x$ which does not converge. This is, however, weakly unconditionally Cauchy, since

$$\Big\| \sum_{i \in \mathscr{F}} P_{\alpha_i} x \Big\| \leqslant \|x\|$$

for $\mathscr{F} \subset \mathscr{N}$ finite. Hence Y contains a copy of c_0 by a result of Bessaga and Pełczyński (3).

Remark. Theorem 2·19 of (2) may be improved by assuming only that X contains no copy of c_0; the proof is similar to the above Proposition.

We now come to our first main theorem which is an extension of a well-known result concerning diagonal maps on sequence spaces ((22)).

THEOREM 2·4. *Let $\{P_\alpha : \alpha \in \mathscr{A}\}$ be an hermitian decomposition of a Banach space X. Suppose $T \in B(X)$; then $\sum_{\alpha \in \mathscr{A}} P_\alpha T P_\alpha$ converges in the strong operator topology to an operator $D \in B(X)$ and $\|D\| \leqslant \|T\|$. If T is hermitian, then D is also hermitian.*

Proof. We shall first prove the result for a finite decomposition $\{P_1 \ldots P_n\}$; we show by induction that if

$$S_p = \sum_{j=1}^{n-p} \sum_{k=1}^{n-p} P_j T P_k + \sum_{j=n-p+1}^{n} P_j T P_j \quad (0 \leqslant p \leqslant n),$$

(summation over the empty set is taken to be zero when $p = 0$ or n), then $\|S_p\| \leqslant \|T\|$, and if $T \in \mathscr{H}(X)$ then also $S_p \in \mathscr{H}(X)$. Note that $S_0 = T$, and now suppose true for $p = q$, where $0 \leqslant q$. Then by Lemma 2·1 of (2), $I - 2P_{n-q}$ is an isometry of X into itself, with inverse again $I - 2P_{n-q}$. Hence

$$\|(I-2P_{n-q})\, S_q (I - 2P_{n-q})\| \leqslant \|T\|$$

and therefore

$$\|(I-2P_{n-q})\, S_q (I-2P_{n-q}) + S_q\| \leqslant 2\|T\|,$$

i.e.

$$\|S_q - P_{n-q} S_q - S_q P_{n-q} + 2P_{n-q} S_q P_{n-q}\| \leqslant \|T\|.$$

However,

$$S_q - P_{n-q} S_q - S_q P_{n-q} + 2P_{n-q} S_q P_{n-q} = S_{q+1}.$$

Also we note that if $S_q \in \mathscr{H}(X)$ then so does $(I-2P_{n-q})\, S_q (I-2P_{n-q})$ since

$$\exp\,(it(I-2P_{n-q})\, S_q(I-2P_{n-q})) = (I-2P_{n-q}) \exp\,(itS_q)\,(I-2P_{n-q}).$$

Hence, if $S_q \in \mathscr{H}(X)$ then so does S_{q+1}.

We conclude that $S_n = \sum_{j=1}^{n} P_j T P_j$ satisfies $\|S_n\| \leqslant \|T\|$, and if $T \in \mathscr{H}(X)$, $S_n \in \mathscr{H}(X)$.

Next we pass to the infinite case. Then, given $x \in X$ and $\epsilon > 0$, we can find a finite subset $\mathscr{F}_{\epsilon,x}$ of $\mathscr{A}$, such that if $\mathscr{B} \cap \mathscr{F}_{\epsilon,x} = \varnothing$ then $\|\sum_{\alpha \in \mathscr{B}} P_\alpha x\| \leqslant \epsilon$.

If $\mathscr{B}$ is finite and $\mathscr{B} \cap \mathscr{F}_{\epsilon,x} = \varnothing$ then

$$\|\sum_{\alpha \in \mathscr{B}} P_\alpha T P_\alpha + QTQ\| \leqslant \|T\|,$$

where $Q = I - \sum_{\alpha \in \mathscr{B}} P_\alpha$. Hence

$$\|(\sum_{\alpha \in \mathscr{B}} P_\alpha T P_\alpha + QTQ)\,(\sum_{\alpha \in \mathscr{B}} P_\alpha x)\| \leqslant \epsilon \|T\|,$$

i.e.

$$\|\sum_{\alpha \in \mathscr{B}} P_\alpha T P_\alpha x\| \leqslant \epsilon \|T\|$$

for $\mathscr{B} \cap \mathscr{F}_{\epsilon,x} = \varnothing$. Thus $\sum_{\alpha \in \mathscr{A}} P_\alpha T P_\alpha$ converges to an operator D in the strong operator topology.

Now for $\mathscr{F} \subset \mathscr{A}$ finite, we let $Q_{\mathscr{F}} = I - \sum_{\alpha \in \mathscr{F}} P_\alpha$. If for $x \in X$, $\mathscr{F} \supset \mathscr{F}_{\epsilon, x}$

$$\begin{aligned} \|Q_{\mathscr{F}} T Q_{\mathscr{F}} x\| &= \|(\sum_{\alpha \in \mathscr{F}} P_\alpha T P_\alpha + Q_{\mathscr{F}} T Q_{\mathscr{F}}) Q_{\mathscr{F}} x\| \\ &\leqslant \|T\| \, \|Q_{\mathscr{F}} x\| \\ &\leqslant \epsilon \|T\| \end{aligned}$$

and so the net $Q_{\mathscr{F}} T Q_{\mathscr{F}} \to 0$ in the strong operator topology. Hence

$$D = \lim_{\mathscr{F}} (\sum_{\alpha \in \mathscr{F}} P_\alpha T P_\alpha + Q_{\mathscr{F}} T Q_{\mathscr{F}})$$

in the strong operator topology. In particular $\|D\| \leqslant \|T\|$, and if $T \in \mathscr{H}(X)$, $D \in \mathscr{H}(X)$.

Remark. An examination of the proof shows that the only property of hermitian projections we use is that $\|I - 2P_\alpha\| = 1$. Therefore, we have

COROLLARY 2·5. *Theorem 2·4 holds if we only assume that $\{P_\alpha : \alpha \in \mathscr{A}\}$ is a collection of projections such that $P_\alpha P_\beta = 0$ for $\alpha \neq \beta$, $x = \sum_{\alpha \in \mathscr{A}} P_\alpha x$ for $x \in X$, and $\|I - 2P_\alpha\| = 1$ for $\alpha \in \mathscr{A}$.*

THEOREM 2·6. *Suppose P and Q are hermitian projections on X such that $PQ = 0$. Suppose $T \in \mathscr{H}(X)$. Then both $PTQ + QTP$ and $i(PTQ - QTP) \in \mathscr{H}(X)$.*

Proof. Let $R = I - (P + Q)$ and apply Theorem 2·4 to the decomposition $(P + Q, R)$ and (P, Q, R). We have that

$$PTP + QTQ + RTR \in \mathscr{H}(X) \quad \text{and} \quad PTP + QTQ + RTR + PTQ + QTP \in \mathscr{H}(X).$$

Hence $PTQ + QTP \in \mathscr{H}(X)$.

Now as $P \in \mathscr{H}(X)$, by Lemma 4, p. 57 of (4)

$$i[P(PTQ + QTP) - (PTQ + QTP)P] \in \mathscr{H}(X)$$

where $QP = 0$ by (2) Theorem 2·13, i.e.

$$i(PTQ - QTP) \in \mathscr{H}(X).$$

3. *The hermitian elements.*

DEFINITION 3·1. *An element x of a complex Banach space X is said to be* hermitian *if there is an hermitian projection P_x whose range is the linear span of x. The set of hermitian elements is denoted by $h(X)$ and its closed linear span by $\hat{h}(X)$.*

On X we can induce a duality map $X \to X^*$ $(x \to x^*)$ with properties

(i) $\|x^*\| = \|x\|$,

(ii) $x^*(x) = \|x\|^2$,

(iii) if $x \in h(X)$, then $\|x\|^2 P_x = x^* \otimes x$.

Note here that condition (iii) is not usually imposed but can easily be satisfied. Then the duality map induces a semi-inner product $[\ ,\ ]$ on X defined by $[x, y] = y^*(x)$ (see (12)). This definition is unique for $y \in h(X)$.

PROPOSITION 3·2. *$h(X)$ is a closed subset of X.*

Proof. Suppose $x_n \in h(X)$ and $x_n \to x$. We may suppose $x \neq 0$ since $0 \in h(X)$, and we may therefore also suppose $\inf\|x_n\| = \theta > 0$. Let $g_n = \|x_n\|^{-2} x_n^*$, so that $g_n \otimes x_n$ is an hermitian projection. We have

$$\|g_n\| = \|x_n\|^{-1} \leqslant \theta^{-1}$$

so that there is a weak*-limit point g of $\{g_n\}$ in X^*. Now for $n \in \mathscr{N}$

$$\begin{aligned} |1 - g_n(x)| &= |g_n(x - x_n)| \\ &\leqslant \theta^{-1}\|x - x_n\| \to 0. \end{aligned}$$

Hence $g(x) = 1$, and $g \otimes x$ is a non-zero projection.

For any $z \in X$ and $t \in \mathrm{R}$

$$\|z + (e^{it} - 1)\, g_n(z)\, x_n\| = \|z\|,$$

since $g_n \otimes x_n$ is an hermitian projection. Letting $n \to \infty$, since $g(z)$ is a limit point of $g_n(z)$,

$$\|z + (e^{it} - 1)\, g(z)\, x\| = \|z\|,$$

i.e. $g \otimes x$ is hermitian.

PROPOSITION 3·3. *Suppose $x, y \in h(X)$ and $[x, y] = 0$. Then $[y, x] = 0$.*

Proof. Since $[x, y] = 0$ we have $P_y x = 0$ and hence $P_y P_x = 0$. By Theorem 2·13 of (2), $P_x P_y = 0$, i.e. $[y, x] = 0$.

The following definition of an orthonormal system is related to a definition of Berkson (2), p. 116. It is, however, important to realize that Berkson requires the projections P_x to be 'perpendicular', a formally stronger condition than being hermitian. We shall show later that if x is hermitian then P_x is perpendicular, so that the definition given below of a complete orthonormal system is equivalent to Berkson's Definition 4·1.

DEFINITION 3·4. *A collection $\{e_\alpha : \alpha \in \mathscr{A}\}$ of elements of a Banach space X is an* orthonormal system *if $[e_\alpha, e_\beta] = \delta_{\alpha\beta}$ for $\alpha, \beta \in \mathscr{A}$ and $\{e_\alpha : \alpha \in \mathscr{A}\} \subset h(X)$. The* extent *of an orthonormal system is the closed linear span of the $\{e_\alpha : \alpha \in \mathscr{A}\}$. An orthonormal system is* complete *(or an* orthonormal basis*) if its extent is equal to X.*

DEFINITION 3·5. *A closed subspace $Y \subset X$ is* orthonormal *if it is the extent of an orthonormal system. A split orthonormal subspace is* orthogonal. *A collection $(Y_\alpha : \alpha \in \mathscr{A})$ of orthogonal subspaces of X is* mutually orthogonal *if the associated hermitian projections $P_\alpha : X \to Y_\alpha$ form a splitting.*

An orthonormal system $(e_\alpha : \alpha \in \mathscr{A})$ induces naturally an hermitian splitting defined by $P_\alpha x = [x, e_\alpha]\, e_\alpha$ for $\alpha \in \mathscr{A}$. The following is simply a restatement of Propositions 2·2 and 2·3.

PROPOSITION 3·6. *Let $\{e_\alpha : \alpha \in \mathscr{A}\}$ be an orthonormal system in X which has extent Y. Then*

(i) $\sum_{\alpha \in \mathscr{A}} [x, e_\alpha]\, e_\alpha$ *converges for $x \in Y$,*

(ii) $\sum_{\alpha \in \mathscr{A}} [x, e_\alpha]\, e_\alpha$ *converges for all $x \in X$ if and only if Y is orthogonal,*

(iii) *If Y contains no subspace isomorphic to c_0, then Y is orthogonal.*

Example 3·7. Let $X = l_\infty$ the space of bounded sequences. Then c_0 is an orthonormal subspace but is not orthogonal, as there is no bounded projection of l_∞ onto c_0 (see (20)).

We note also that a complete orthonormal system is simply an unconditional Schauder basis for which the unconditional basis constant is one.

4. *Characterizations of Hilbert subspaces.*

The results of this section are very slight improvements of results due to Berkson ((1), (2)) and Tam (21). The proofs are in some cases rather more elementary and for this reason we give them in detail.

LEMMA 4·1. *Let $\|\,.\,\|$ be a norm on $\mathbb{C}^2$ such that $\|(1,0)\| = \|(0,1)\| = 1$. If $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ is the matrix of an hermitian operator on $\mathbb{C}^2$, then $b = \bar{a}$.*

Proof. Let $S = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$; we suppose $S \neq 0$. Then $\exp(itS)$ is an isometry for $t \in \mathbb{R}$. In particular $S^2 \neq 0$, and so $ab \neq 0$. As S has only real eigenvalues, $ab = c^2$, where $c > 0$. Then

$$\exp(itS) = \begin{pmatrix} \cos ct & iac^{-1}\sin ct \\ ibc^{-1}\sin ct & \cos ct \end{pmatrix}.$$

Let $t = \dfrac{\pi}{2c}$; then

$$\exp\left(i\frac{\pi}{2c}S\right) = \begin{pmatrix} 0 & iac^{-1} \\ ibc^{-1} & 0 \end{pmatrix}$$

is an isometry. As $\|(1,0)\| = \|(0,1)\| = 1$, $c = |a| = |b|$ and $b = \bar{a}$.

PROPOSITION 4·2. *Let X be a Banach space and $\{e_1, e_2\}$ an orthonormal system in X. Suppose there exists $T \in \mathscr{H}(X)$ such that $[Te_1, e_2] \neq 0$. Then whenever $A = (a_{jk})_{j=1,2;\,k=1,2}$ is an hermitian matrix, the operator $\sum_{j,k} a_{jk} e_j^* \otimes e_k$ is hermitian, and for any $\xi_1, \xi_2 \in \mathbb{C}$,*

$$\|\xi_1 e_1 + \xi_2 e_2\|^2 = |\xi_1|^2 + |\xi_2|^2.$$

Proof. Let $\alpha = [Te_1, e_2]$, and $\beta = [Te_2, e_1]$. By Theorem 2·6, the operator $\alpha e_1^* \otimes e_2 + \beta e_2^* \otimes e_1$ is hermitian. Restricting to lin (e_1, e_2), we obtain by Lemma 4·1 that $\beta = \bar{\alpha}$. Again by 2·6, $i(\alpha e_1^* \otimes e_2 - \bar{\alpha} e_2^* \otimes e_1) \in \mathscr{H}(X)$ and so by the real-linearity of $\mathscr{H}(X)$ both $e_1^* \otimes e_2 + e_2^* \otimes e_1$ and $i(e_1^* \otimes e_2 - e_2^* \otimes e_1)$ belong to $\mathscr{H}(X)$. So also do $e_1^* \otimes e_1$ and $e_2^* \otimes e_2$, and hence the first part of the assertion is proved.

Suppose $\xi_1, \xi_2 \in \mathbb{C}$ and $|\xi_1|^2 + |\xi_2|^2 = 1$. Then the matrix

$$U = \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix}$$

is unitary. Hence $U = \exp(iA)$, where $A = (a_{jk})$ is hermitian. Then $\Sigma a_{jk} e_j^* \otimes e_k \in \mathscr{H}(X)$ and so $\xi_1 e_1^* \otimes e_1 + \bar{\xi}_1 e_2^* \otimes e_2 + \xi_2 e_1^* \otimes e_2 - \bar{\xi}_2 e_2^* \otimes e_1$ is an isometry. In particular applying this operator to e_1, $\|\xi_1 e_1 + \xi_2 e_2\| = 1$, and we quickly obtain the result.

Remark. This is a slight generalization of Lemma 6 of (21).

PROPOSITION 4·3. *Suppose* $\{e_1, e_2\}$ *is an orthonormal system, and there is a linear combination* $\alpha e_1 + \beta e_2 \in h(X)$ *with* $\alpha\beta \neq 0$. *Then there exists* $T \in \mathscr{H}(X)$ *with* $[Te_1, e_2] \neq 0$ *and the conclusions of* 4·2 *are valid. In particular,* $\text{lin}(e_1, e_2) \subset h(X)$.

Proof. Let P be the hermitian projection onto $\text{lin}(\alpha e_1 + \beta e_2)$. Let P_0 be its restriction to $\text{lin}(e_1, e_2)$. Then P_0 must have a non-diagonal matrix and so either $[Pe_1, e_2] \neq 0$ or $[Pe_2, e_1] \neq 0$ (and hence both are non-zero). Note that $\text{lin}(e_1, e_2) \subset h(X)$ follows from Proposition 4·2.

COROLLARY 4·4. (Berkson (1).) *Suppose* $h(X) = X$; *then* X *is isometric to a Hilbert space.*

Proof. By a result of Jordan and von Neumann ((11)) it is sufficient to show the result for every two-dimensional subspace of X. If X_0 is two-dimensional we may select $e_1 \in X_0$ with $\|e_1\| = 1$ and e_2 such that $[e_2, e_1] = 0$, with $\|e_2\| = 1$. The result follows from 4·3.

5. *The Hilbert components of* X.

DEFINITION 5·1. *Let* $\{H_\lambda : \lambda \in \Lambda\}$ *be the collection of maximal linear subspaces of* $h(X)$. *Then* $\{H_\lambda : \lambda \in \Lambda\}$ *are the* Hilbert components *of* X.

We remark that since $h(X)$ is closed, each H_λ is also closed and as $h(H_\lambda) = H_\lambda$, each H_λ is (isometrically) a Hilbert space, justifying our terminology.

LEMMA 5·2. *Suppose* $x, y \in h(X)$ *and* $[x, y] \neq 0$. *Then* $\text{lin}(x, y) \subset h(X)$.

Proof. We may assume x and y linearly independent. Let $e_1 = \|x\|^{-1}x$ and $f = y - [y, e_1]e_1$. We note that $f \neq 0$. Since $y^* \otimes y \in \mathscr{H}(X)$ we may apply Theorem 2·4 to deduce that $S = P(y^* \otimes y)P + Q(y^* \otimes y)Q \in \mathscr{H}(X)$, where $P = e_1^* \otimes e_1$ and $Q = I - P$. Now $P(y^* \otimes y)P = \lambda P$, where $\lambda \in \mathsf{R}$ since it is an eigenvalue of S. Thus $Q(y^* \otimes y)Q \in \mathscr{H}(X)$, i.e. $Q^*y^* \otimes Qy \in \mathscr{H}(X)$. Now $Qy = f \neq 0$; suppose $Q^*y^* = 0$. Then by Theorem 2·6, both $P^*y^* \otimes Qy$ and $iP^*y^* \otimes Qy \in \mathscr{H}(X)$ so that $P^*y^* \otimes Qy = 0$. Thus $P^*y^* = 0$ and so $y^* = 0$, which is a contradiction. Thus $Q^*y^* \otimes Qy \neq 0$ and $f \in h(X)$. Letting $e_2 = \|f\|^{-1}f$ we have an orthonormal system $\{e_1, e_2\}$. Now the lemma follows from Proposition 4·3.

THEOREM 5·3. *The spaces* $\{H_\lambda : \lambda \in \Lambda\}$ *form a mutually orthogonal collection.*

Proof. By Proposition 3·6, each H_λ is an orthogonal subspace (select an orthonormal basis). Next suppose $x \in H_\lambda$ and $y \in H_\mu$, where $\lambda \neq \mu$ and $[y, x] \neq 0$ (and $[x, y] \neq 0$ by 3·3). Suppose also $z \in H_\mu$; then for some small enough $\alpha \in \mathsf{R}$, $[y + \alpha z, x] \neq 0$. Hence, $\text{lin}(x, y + \alpha z) \subset h(X)$ and $\text{lin}(x, y) \subset h(X)$; thus $\text{lin}(x, z) \subset h(X)$. As this is valid for any $z \in H_\mu$, $x \in H_\mu$.

By the same reasoning, for any $z \in H_\mu$, $y + \alpha z \in H_\lambda$ for some α, and also $y \in H_\lambda$. Hence $H_\mu \subset H_\lambda$, which is a contradiction. We conclude that $[x, y] = 0$; it follows quickly that the spaces $\{H_\lambda\}$ are mutually orthogonal.

It is now easy to see that any orthonormal system decomposes into an orthonormal system in each H_λ; conversely orthonormal systems may be constructed as the unions of such systems in each H_λ. We conclude:

COROLLARY 5·4. *The extent of any maximal orthonormal system is* $\hat{h}(X)$.

COROLLARY 5·5. *X possesses a complete orthonormal system if and only if* $\hat{h}(X) = X$.

If X has an orthonormal basis, then it is possible to show (by considering finite-dimensional hermitian operators and their associated groups of isometries) that the norm of $\sum_{\lambda\in\Lambda} x_\lambda$, where $x_\lambda \in H_\lambda$, depends only on $\{\|x_\lambda\| : \lambda\in\Lambda\}$. It follows easily that X has an H-decomposition in the sense of (7); conversely such a decomposition implies the existence of an orthonormal basis. It follows from Theorem 4·7 of (7) that X has an orthonormal basis precisely if $X \in \mathscr{S}$ (see (7) and (8) for definitions).

6. *Hermitian operators and isometries.*

THEOREM 6·1. *Suppose X is a complex Banach space, with $h(X) \neq \{0\}$. Let $(H_\lambda : \lambda\in\Lambda)$ be the Hilbert components of X. Let $(\Pi_\lambda : \lambda\in\Lambda)$ be the associated hermitian projection.*

(i) *If $U: X\to X$ is an isometry then $U(h(X)) = h(X)$, and there is a bijection $\gamma: \Lambda\to\Lambda$ such that $U(H_\lambda) = H_{\gamma(\lambda)}$.*

(ii) *If $T: X\to X$ is hermitian then $T\Pi_\lambda = \Pi_\lambda T$ $(\lambda\in\Lambda)$.*

Proof. (i) If $x\in h(X)$ with $\|x\| = 1$, then $U^{-1}(x^*\otimes x)U$ is hermitian and so $Ux\in h(X)$. Hence for given λ, $U(H_\lambda)\subset h(X)$ and so $U(H_\lambda)\subset H_\mu$, some μ. However $U^{-1}(H_\mu)\subset H_{\lambda'}$ by a similar argument and hence $\lambda = \lambda'$ and $U(H_\lambda) = H_\mu$. Letting $\mu = \gamma(\lambda)$, we obtain the desired mapping.

(ii) For $t\in \mathsf{R}$, $\exp(itT)(H_\lambda) = H_{\mu(t)}$.
For $x\in H_\lambda$, and t such that $\mu(t)\neq\lambda$,

$$\|(\exp(itT)-I)x\| \geqslant \|\Pi_\lambda(\exp(itT)-I)x\| = \|x\|.$$

As $\lim_{t\to 0}\|\exp(itT)-I\| = 0$, $\mu(t) = \lambda$ for small enough t, and hence, as Π_λ is unique,

$$\exp(-itT)\Pi_\lambda\exp(itT) = \Pi_\lambda.$$

Expanding, we obtain (ii).

Theorem 6·1 obviously facilitates the identification of hermitian operators when $h(X)\neq\{0\}$. In particular, we can completely determine $\mathscr{H}(X)$ when X has an orthonormal basis. The following two results are known but we believe the proofs are rather simpler.

THEOREM 6·2 (Fleming–Jamison (7)). *Suppose X is a Banach space with an orthonormal basis. Let $(H_\lambda : \lambda\in\Lambda)$ be the Hilbert components of X. Then for $T\in B(X)$, $T\in\mathscr{H}(X)$ if and only if $T(H_\lambda)\subset H_\lambda$ and T is hermitian as an operator on the Hilbert space H_λ, for each $\lambda\in\Lambda$.*

Proof. It is a trivial consequence of 6·1 that if $T\in\mathscr{H}(X)$ then $T(H_\lambda)\subset H_\lambda$ and T is hermitian on H_λ. Now suppose conversely that $T(H_\lambda)\subset H_\lambda$ and T is hermitian on each H_λ. For each λ, there is a net $S_{n,\lambda}$ of finite-dimensional hermitian operators on H_λ such that $S_{n,\lambda}\to T$ is the weak-operator topology of H_λ. Then

$$S_{n,\lambda} = \sum_{j=1}^{k(n)} a_j e_j^*\otimes e_j,$$

where $(e_j\colon 1 \leqslant j \leqslant k(n))$ is an orthonormal system in H_λ and $a_j \in \mathbb{R}$. Hence $S_{n,\lambda}\Pi_\lambda\colon X \to X$ is hermitian and therefore, taking weak-operator limits, $T\Pi_\lambda$ is hermitian on X. Again taking weak-operator limits, since $T = \Sigma T\Pi_\lambda$, T is hermitian.

COROLLARY 6·3 (Tam (21)). *Suppose X is not isometric to a Hilbert space and $(e_\alpha\colon \alpha \in \mathscr{A})$ is a symmetric orthonormal basis of X (i.e. for every bijection $\pi\colon \mathscr{A} \to \mathscr{A}$, there is an isometry $U_\pi\colon X \to X$ such that $U_\pi e_\alpha = e_{\pi(\alpha)}$). Then T is hermitian on X if and only if T has a representation*

$$Tx = \sum_{\alpha \in \mathscr{A}} a_\alpha [x, e_\alpha]\, e_\alpha,$$

where $a_\alpha \in \mathbb{R}$ and $\sup|a_\alpha| < \infty$.

Proof. Let $(H_\lambda\colon \lambda \in \Lambda)$ be the Hilbert components of X, and let $\mathscr{A}_\lambda = \{\alpha\colon e_\alpha \in H_\lambda\}$. Clearly if $\alpha, \beta \in \mathscr{A}_\lambda$ then $\pi(\alpha)$ and $\pi(\beta)$ belong to the same $\mathscr{A}_\mu$ for every bijection $\pi\colon \mathscr{A} \to \mathscr{A}$. Hence, either each $\mathscr{A}_\lambda$ is a singleton or there is only one Hilbert component; the latter is impossible, since it would imply that X is a Hilbert space. Hence each $\mathscr{A}_\lambda$ is a singleton and each H_λ is one-dimensional; the result then follows by 6·2.

Rolewicz (17) defines a norm on a Banach space X to be *convex-transitive* if, whenever $\|x_0\| = 1$, the unit ball of X is the closed convex cover of $\{Ux_0\colon U \in \mathscr{G}\}$ where $\mathscr{G}$ is the group of isometries of X. He shows that the Banach spaces $L_p(0, 1)$ for $1 \leqslant p \leqslant \infty$ are convex-transitive but $C[0, 1]$ is not.

THEOREM 6·4. *Suppose X has a convex-transitive norm and $h(X) \neq \{0\}$. Then X is a Hilbert space.*

Proof. Suppose $x_0 \in h(X)$ with $\|x_0\| = 1$. Then $\{Ux_0\colon U \in \mathscr{G}\}$ is contained in $h(X)$ and hence by convex-transitivity $\hat{h}(X) = X$. Thus X has an orthonormal basis. Let $(H_\lambda\colon \lambda \in \Lambda)$ be the Hilbert components of X, with the associated hermitian projections Π_λ. If $x \in H_{\lambda_0}$, then for any $U \in \mathscr{G}$, $\sum_\lambda \|\Pi_\lambda Ux\| = \|x\|$ (since $Ux \in H_\mu$ some μ). Hence for any $y \in \overline{co}(Ux\colon U \in \mathscr{G})$,

$$\sum_\lambda \|\Pi_\lambda y\| \leqslant \|x\|.$$

In particular taking $\|x\| = 1$, we see that if $\|y\| = 1$ then

$$\sum_\lambda \|\Pi_\lambda y\| \leqslant 1.$$

Hence for $y \in X$

$$\|y\| = \sum_\lambda \|\Pi_\lambda y\|.$$

However, if Λ is not a singleton, we may take $x_1 \in H_\lambda$ and $x_2 \in H_\mu$ with $\lambda \neq \mu$ and $\|x_1\| = \|x_2\| = 1$. Then for $U \in G$

$$\max_\lambda \|\Pi_\lambda U(\tfrac{1}{2}(x_1 + x_2))\| = \tfrac{1}{2}$$

and hence for all $y \in \overline{co}(U(\tfrac{1}{2}(x_1 + x_2))\colon U \in G)$

$$\max_\lambda \|\Pi_\lambda y\| \leqslant \tfrac{1}{2}.$$

This is a contradiction, and so Λ is a singleton, i.e. X is a Hilbert space.

To conclude the section, we classify $h(X)$ for a Banach function space. Let (Ω, Σ, μ)

be a σ-finite positive measure space, and let X be a Banach space of μ-measurable functions on Ω (where we identify functions differing only on a set of μ-measure zero) such that

(1) If $f \in X$, g is μ-measurable and $|g| \leqslant |f|$ a.e., then $g \in X$ and $\|g\| \leqslant \|f\|$.

(2) If $f_n \geqslant 0$ and $f_n \in X$ are such that $f_n \uparrow f$ a.e. and $\sup_n \|f_n\| < \infty$, then $f \in X$ and $\|f\| = \lim_{n\to\infty} \|f_n\|$. (See (14) or (15) for a discussion of these conditions.)

THEOREM 6·5. *Let X be a Banach function space satisfying* (1) *and* (2) *above. Let* $f \in X$ *and* $\Omega_0 = \{\omega : f(\omega) \neq 0\}$. *Then* $f \in h(X)$ *if and only if*

(i) *there is a μ-measurable function* $k\colon \Omega_0 \to \mathsf{R}^+$ *such that if* $g = 0$ *on* $\Omega - \Omega_0$ *and* $g \in X$,

$$\|g\|^2 = \int_{\Omega_0} |g|^2 \,.\, k\, d\mu;$$

(ii) *if* $g_1 = 0$ *on* Ω_0, $g_2 = g_3 = 0$ *on* $\Omega - \Omega_0$, *and* $\|g_2\| = \|g_3\|$ *then* $\|g_1 + g_2\| = \|g_1 + g_3\|$.

Proof. If (i) and (ii) are satisfied then the projection

$$Pg = \left\{\int_{\Omega_0} g\bar{f}k\, d\mu\right\} f$$

is hermitian. This may be verified by showing that $\|e^{itP}\| \equiv 1$, $t \in \mathsf{R}$.

Conversely suppose $f \in h(X)$. By assumption (1) on X, if ϕ is μ-measurable on Ω, $|\phi| = 1$ a.e., then the multiplication operator $M_\phi g = \phi \,.\, g$ is an isometry. Hence for $A \in \Sigma$, the projection $P_A g = \chi_A \cdot g$ (where χ_A is the characteristic function of A) is hermitian.

Let H_f be the Hilbert component of f. By 6·1 $P_A f \in H_f$ for $A \in \Sigma$. If $A \cap B = \varnothing$ then $P_A P_B = 0$ and hence the projections P_A and P_B are orthogonal when restricted to H_f. In particular

$$\|P_A f\|^2 + \|P_B f\|^2 = \|P_{A\cup B} f\|^2.$$

Let $\nu(A) = \|P_A f\|^2$ for $A \in \Sigma$. Then by using condition (2), ν is a positive measure. Clearly $\nu(\Omega - \Omega_0) = 0$, and ν is absolutely continuous with respect to μ. Hence by Radon–Nikodym theorem $\nu = k \,.\, |f|^2 \,.\, \mu$, where $k\colon \Omega \to \mathsf{R}^+$ is zero outside Ω_0.

If g is a simple function

$$\|g \,.\, f\|^2 = \int |g|^2 d\nu$$

$$= \int |g \,.\, f|^2 k\, d\mu.$$

Taking pointwise limits we obtain that if $g = 0$ outside Ω_0

$$\|g\|^2 = \int |g|^2 k\, d\mu.$$

To prove condition (ii) it is again only necessary in view of (2) to prove it when $\sup |g_2 f^{-1}| < \infty$ and $\sup |g_3 f^{-1}| < \infty$ (when $0/0 = 0$). Let $L_\infty(f)$ be the closure of the space $\{f.g: g \in L_\infty(\mu)\}$ in X. We note that if ϕ is real and $\phi \in L_\infty(\mu)$, then M_ϕ is an hermitian operator on X. Then $M_\phi f \in H_f$ and so $L_\infty(f) \subset H_f$. Suppose $g_2, g_3 \in L_\infty(f)$; then there is a finite-rank hermitian operator S on $L_\infty(f)$ (which is a Hilbert space) such that

$$e^{iS}g_2 = g_3.$$

Then
$$Sx = \sum_{j=1}^{n} a_j[x,f_j]f_j \quad x \in L_\infty(f),$$

where $f_j \in H_f$, $\|f_j\| = 1$ and $a_j \in \mathbb{R}$. Extend S to $\bar{S}$ on X by

$$\bar{S}x = \sum_{j=1}^{n} a_j[x,f_j]f_j \quad x \in X.$$

Then $\bar{S}$ is hermitian, and $\|e^{i\bar{S}}\| = 1$.

Let $\phi(\omega) = 1$ for $\omega \in \Omega_0$, $\phi(\omega) = -1$ for $\omega \in \Omega - \Omega_0$. Then M_ϕ is an isometry and $M_\phi^{-1}(f_j^* \otimes f_j)M_\phi$ is an hermitian projection for each j, $1 \leqslant j \leqslant n$. However,

$$M_\phi^{-1}(f_j^* \otimes f_j)M_\phi = M_\phi^* f_j^* \otimes M_\phi f_j$$

(note $M_\phi^2 = I$). As $M_\phi f_j = f_j$, $M_\phi^* f_j^* = f_j^*$. Thus

$$\begin{aligned}[g_1,f_j] &= [M_\phi g_1, f_j] \\ &= -[g_1,f_j] \quad 1 \leqslant j \leqslant n.\end{aligned}$$

Hence $\bar{S}g_1 = 0$ and $e^{iS}(g_1+g_2) = g_1+g_3$. Hence $\|g_1+g_2\| = \|g_1+g_3\|$.

Remark. For many examples these conditions hold if and only if Ω_0 is a single point (an atom).

7. *Perpendicular projections* (*after Berkson*).

In this short section we relate our notion of an orthonormal system to that of Berkson (2), p. 116. Following Berkson ((2), p. 112) we say a collection $\mathscr{F}$ of hermitian projections is an (*i.c*)-*family* if given $P_1, P_2 \in \mathscr{F}$ there exists $P_0 \in \mathscr{F}$ with

$$\mathscr{R}(P_0) = \mathscr{R}(P_1) \cap \mathscr{R}(P_2),$$

and if $I - P \in \mathscr{F}$ whenever $P \in \mathscr{F}$. A projection is *perpendicular* if it belongs to every maximal (i.c)-family.

THEOREM 7·1. *Every hermitian projection of rank one is perpendicular.*

Proof. We suppose $\mathscr{F}$ is a maximal (i.c)-family. Let $(H_\lambda: \lambda \in \Lambda)$ be the Hilbert components of X and $\Pi_\lambda: X \to X$ be associated hermitian projections. Then Π_λ commutes with every hermitian projection, and if $R_\lambda: H_\lambda \to H_\lambda$ is an hermitian projection then $R_\lambda \circ \Pi_\lambda$ is hermitian on X (6·1 (ii) and the proof of 6·2).

If $P \in \mathscr{H}(X)$ is a projection then so is

$$P(I - \sum_{\lambda \in \Lambda_0} \Pi_\lambda) + \sum_{\lambda \in \Lambda_0} R_\lambda \circ \Pi_\lambda,$$

where $\Lambda_0 \subset \Lambda$ is finite and each $R_\lambda: H_\lambda \to H_\lambda$ is an hermitian projection. Let $\mathscr{F}^*$ be the collection of hermitian projections of this form for $P \in \mathscr{F}$. We show $\mathscr{F}^*$ is an (i.c)-family.

First observe

$$I-P(I-\sum_{\lambda\in\Lambda_0}\Pi_\lambda)-\sum_{\lambda\in\Lambda_0}R_\lambda\circ\Pi_\lambda=(I-P)(I-\sum_{\lambda\in\Lambda_0}\Pi_\lambda)+\sum_{\lambda\in\Lambda_0}(I-R_\lambda)\circ\Pi_\lambda.$$

Next suppose $P_1, P_2\in\mathscr{F}$ and

$$Q_1=P_1(I-\sum_{\lambda\in\Lambda_0}\Pi_\lambda)+\sum_{\lambda\in\Lambda_0}R_\lambda^{(1)}\circ\Pi_\lambda$$

$$Q_2=P_2(I-\sum_{\lambda\in\Lambda_0}\Pi_\lambda)+\sum_{\lambda\in\Lambda_0}R_\lambda^{(2)}\circ\Pi_\lambda$$

(we suppose Λ_0 is the same for both Q_1 and Q_2 without loss of generality). Suppose $P_0\in\mathscr{F}$ and $\mathscr{R}(P_0)=\mathscr{R}(P_1)\cap\mathscr{R}(P_2)$, and suppose for $\lambda\in\Lambda_0$, $R_\lambda^{(0)}$ is an hermitian projection on H_λ such that $\mathscr{R}(R_\lambda^{(0)})=\mathscr{R}(R_\lambda^{(1)})\cap\mathscr{R}(R_\lambda^{(2)})$. Then let

$$Q_0=P_0(I-\sum_{\lambda\in\Lambda_0}\Pi_\lambda)+\sum_{\lambda\in\Lambda_0}R_\lambda^{(0)}\circ\Pi_\lambda.$$

Then $Q_0\in\mathscr{F}^*$ and $\mathscr{R}(Q_0)=\mathscr{R}(Q_1)\cap\mathscr{R}(Q_2)$.

Thus $\mathscr{F}^*$ is an (i.c.)-family and hence $\mathscr{F}^*=\mathscr{F}$. As $0\in\mathscr{F}$, every hermitian projection of rank one is in $\mathscr{F}$.

8. *Bounded groups of operators.*

Let X be a real or complex Banach space, and let $\mathscr{G}$ be any bounded subgroup of the general linear group $GL(X)$ of all bounded invertible operators on X. Then X can be renormed equivalently so that each $T\in\mathscr{G}$ is an isometry, by

$$\|x\|^*=\sup_{T\in\mathscr{G}}\|Tx\|.$$

Following a definition of Rolewicz (17), p. 251, we say that a bounded subgroup of $GL(X)$ is *maximal* if it is not contained in any larger bounded subgroup, and a norm $\|.\|$ on X is *maximal* if its group of isometries is maximal. By the above remark, corresponding to every maximal subgroup $\mathscr{G}$ there is at least one maximal norm for which $\mathscr{G}$ is the group of isometries.

Rolewicz shows ((17), pp. 251–252) that on a finite-dimensional space, a norm is maximal if and only if it is a Euclidean norm (he demonstrates this only for the real case, but the complex case is proved similarly). Thus the maximal bounded subgroups of $GL(X)$ are similar to orthogonal or unitary groups. For the (real or complex) spaces c_0, l_p $(1\leqslant p<\infty)$, $L_p(0,1)$ $(1\leqslant p<\infty)$, Pełczyński and Rolewicz ((17), pp. 252–265) have shown that the standard norms are maximal. However, for spaces of continuous functions less is known; in (17) p. 260 it is shown that if K is the Cantor set, then both $C(K)$ and $C_{\mathrm{R}}(K)$ have maximal norms (by $C(K)$ we denote the continuous complex-valued functions on K). Rolewicz asks ((17), p. 259) if the norm on $C[0,1]$ is maximal.

In this section, we shall settle this question by using the results of the previous sections, and indeed we establish much more general results on spaces $C(S)$ (for real-valued functions, see the following section).

Let S be a locally compact Hausdorff space and let $C_0(S)$ be the space of continuous complex functions on S vanishing at infinity. The standard norm on $C_0(S)$ we denote by $\| \|_\infty$,

$$\|f\|_\infty = \sup_{s\in S} |f(s)|.$$

The dual of $C_0(S)$, we denote by $C_0^*(S)$; this can be identified with the space of finite regular Borel measures on S. (A complex measure μ defined on the Borel sets of S is regular if for a Borel set B and $\epsilon > 0$ there exists an open set $G \supset B$ and a compact set $C \subset B$ such that $|\mu|(G-C) \leqslant \epsilon$.)

On $C_0(S)$, every isometry takes the form

$$Uf(s) = \theta(s)f(\gamma(s)),$$

where $\theta: S \to \mathbb{C}$ is continuous and satisfies $|\theta(s)| \equiv 1$, while $\gamma: S\to S$ is a homeomorphism (see Torrance (23) or (5) p. 93). Thus the group $\mathscr{G}$ of isometries on $C_0(S)$ may be considered as the semi-direct product of the group $\mathscr{M}$ of multipliers

$$M_\theta f(s) = \theta(s)f(s),$$

where $|\theta(s)| \equiv 1$ and the group $\hat{\Gamma}$ induced by the group Γ of homeomorphisms of S,

$$T_\gamma f(s) = f(\gamma(s)).$$

THEOREM 8·1. *Let $\|.\|$ be a norm on $C_0(S)$ (equivalent to $\|.\|_\infty$) such that every $M_\theta \in \mathscr{M}$ is an isometry. Then there is an equivalence relation $\sim$ on S such that for some $n\in N$, we have card $\{s': s' \sim s\} \leqslant n$ for all $s\in S$, and $\nu \in h(C_0^*(S))$ if and only if*

$$\nu = \sum_{s'\sim s} \xi_{s'} \delta_{s'}$$

for some $s\in S$, where $\xi_{s'}\in \mathbb{C}$ and $\delta_{s'}$ denotes the unit mass at s'.

Proof. For any bounded continuous complex function ϕ on S, we define

$$\|\phi\|_{op} = \sup(\|\phi.f\| : \|f\| \leqslant 1).$$

Then $C_b(S)$ (bounded continuous functions) becomes a Banach algebra, and every real function in $C_b(S)$ is hermitian since, if ϕ is real, $\|e^{it\phi}\|_{op} \equiv 1$ by assumption. Hence by the Vidav–Palmer theorem ((4)), $C_b(S)$ is a B^*-algebra, and it follows that

$$\|\phi\|_{op} = \sup_{s\in S} |\phi(s)|.$$

In particular, we have $\|g\| \leqslant \|h\|$ for $g, h\in C_0(S)$ whenever $|g| \leqslant |h|$ everywhere.

For $\mu, \nu \in C_0(S)^*$ with $|\mu| \leqslant |\nu|$ we have

$$\begin{aligned}\|\mu\| &= \sup\left\{\int g\,d\mu: \|g\| \leqslant 1\right\}\\ &= \sup\left\{\int \phi g\,d\mu: \|g\| \leqslant 1\,\|\phi\|_{op} \leqslant 1\right\}\\ &= \sup\left\{\int |g|\,d|\mu| : \|g\| \leqslant 1\right\}\\ &\leqslant \sup\left\{\int |g|\,d|\nu| : \|g\| \leqslant 1\right\}\\ &= \|\nu\|.\end{aligned}$$

Now for $s \in S$, the map P_s: $C_0^*(S) \to C_0^*(S)$

$$P_s \mu = \mu\{s\}\,\delta_s$$

is an hermitian projection, since

$$\begin{aligned}\|e^{itP_s}\mu\| &= \|e^{it}\mu\{s\}\,\delta_s + (\mu - \mu\{s\}\,\delta_s)\|\\ &= \|\,|e^{it}\mu\{s\}\,\delta_s + (\mu - \mu\{s\}\,\delta_s)|\,\|\\ &= \|\,|\mu|\,\| = \|\mu\|.\end{aligned}$$

Hence $\delta_s \in h(C_0^*(S))$ for $s \in S$.

Conversely suppose $\mu \in h(C_0^*(S))$, and consider the space $L^1(|\mu|)$ which can be embedded in $C_0^*(S)$ by the map $f \to f.\,|\mu|$. This is an isometry for the $\|\;\|_1$-norm and a homeomorphism for the equivalent norm under consideration. We can re-norm $L^1(|\mu|)$ equivalently by

$$\|f\| = \|f.\,|\mu|\,\|.$$

Let $\mu = \theta\,|\mu|$, where $|\theta| = 1$, $|\mu|$ a.e. Then $\theta \in h(L^1(|\mu|))$, with the new norm. However, with this norm $L^1(|\mu|)$ is a Banach function space as described immediately preceding Theorem 6·4.

If $|g| \leqslant |f|$ then $\|g.\,|\mu|\,\| \leqslant \|f.\,|\mu|\,\|$ by the preceding results. If $0 \leqslant f_n \uparrow f$ with $f_n \in L^1(|\mu|)$ and $\sup\|f_n\| < \infty$ then clearly $f \in L^1(|\mu|)$ as the norm is equivalent to the L^1-norm; also $f_n.\,|\mu| \to f.\,|\mu|$ weak* in $C_0^*(S)$ and hence $\|f\| \leqslant \sup\|f_n\|$.

Therefore, there is a weight function $k(s)$ on S such that

$$\|f\|^2 = \int_S k(s)\,|f(s)|^2\,d|\mu|\,(s).$$

As this is equivalent to the L^1-norm, $L^1(|\mu|)$ is finite-dimensional ((6), p. 338) and hence μ is a finite linear combination of measures (δ_s: $s \in S$).

Thus $\hat{h}(C_0(S))$ is precisely the closed linear span of $\{\delta_s\colon s \in S\}$, and $\{\delta_s\colon s \in S\}$ is an orthonormal basis of $\hat{h}(C_0^*(S))$. We introduce an equivalence relation $\sim$ on s by $s \sim s'$ if and only if δ_s and $\delta_{s'}$ belong to the same Hilbert component of $C_0^*(S)$. It is clear then that $\nu \in h(C_0^*(S))$ if and only if it has the form specified in the theorem. Thus it remains only to show $\sup_{s \in S} \operatorname{card}\{s'\colon s' \sim s\} < \infty$. For this note

$$\begin{aligned}\Big\|\sum_{s' \sim s} \delta_{s'}\Big\|^2 &= \sum_{s' \sim s} \|\delta_{s'}\|^2\\ &\leqslant m(s)\,k,\end{aligned}$$

where $m(s) = \operatorname{card}\{s'\colon s' \sim s\}$ and $k = \sup_{s \in S}\|\delta_s\|$. Thus

$$\Big\|\sum_{s' \sim s} \delta_{s'}\Big\|_1 = m(s) \leqslant K\sqrt{k}\sqrt{(m(s))},$$

where K is a constant such that

$$\|\nu\|_1 \leqslant K\|\nu\| \quad \nu \in C_0^*(S).$$

In particular $\sup_s m(s) < \infty$.

Next we seek conditions under which $\mathscr{G} = \mathscr{M}\hat{\Gamma}$ is a maximal bounded group, i.e. the usual norm on $C_0(S)$ is maximal.

THEOREM 8·2. *Let S be a locally compact Hausdorff space and suppose either*

(i) *there is a dense subset of S of points possessing a neighbourhood homeomorphic to an open set in a Euclidean space;*

(ii) *S is infinite and possesses a dense set of isolated points.*

Then $\mathscr{G} = \mathscr{M}\hat{\Gamma}$ is a maximal bounded group and so the standard norm on $C_0(S)$ is maximal.

Proof. Let $\|\cdot\|$ be any norm on $C_0(S)$ for which $\mathscr{G}$ is contained in the group of isometries. Let $\sim$ be the equivalence relation of Theorem 8·1. We observe that if $s \sim s'$ then since $T_{\gamma^{-1}}$ is an isometry for $\gamma \in \Gamma$, $T^*_{\gamma^{-1}}(\delta_s + \delta_{s'}) \in h(C_0^*(S))$. Thus $\delta_{\gamma(s)} + \delta_{\gamma(s')} \in h(C_0^*(S))$, i.e. $\gamma(s) \sim \gamma(s')$.

(i) Suppose s has a Euclidean neighbourhood U and $s \sim s'$ with $s \neq s'$. Then there is a closed neighbourhood B of s such that $B \subset \operatorname{int} U$, $s' \notin B$ and B is homeomorphic to a closed ball in R^n. Thus for any $s'' \in \operatorname{int} B$ there is a homeomorphism $\tau: B \to B$ such that $\tau(s) = s''$ and $\tau(\bar{s}) = \bar{s}$ for $\bar{s} \in \partial B$. Define $\gamma: S \to S$ by $\gamma(s) = \tau(s)$ $s \in B$ and $\gamma(s) = s$ for $s \notin B$. Then $\gamma \in \Gamma$ and $\gamma(s) = s''$, but $\gamma(s') = s'$. Hence $s'' \sim s' \sim s$. As this is true for all $s'' \in B$, we have $\operatorname{card}\{s' : s' \sim s\} = \infty$, a contradiction to 8·1. Thus $s \sim s'$ implies $s' = s$.

Let S_0 be the set of $s \in S$ which possess Euclidean neighbourhoods. Suppose $U: C_0(S) \to C_0(S)$ is an isometry. Then, for $s_0 \in S_0$, $U^*\delta_{s_0}$ belongs to a Hilbert component of $C_0^*(S)$ of dimension one. Thus as $\{\delta_s : s \in S\}$ is a maximal orthonormal system,

$$U^*\delta_{s_0} = \theta(s_0)\,\delta_{\gamma(s_0)},$$

where $|\theta(s_0)| = 1$ and γ is a map from S_0 to S. By the weak*-continuity of U^*, it quickly follows that

$$U^*\delta_s = \theta(s)\,\delta_{\gamma(s)},$$

where θ is continuous, $|\theta(s)| = 1$ and $\gamma: S \to S$ is continuous. As U is invertible, γ is a homeomorphism and hence $U \in \mathscr{M}\hat{\Gamma}$.

(ii) The proof is similar. Suppose s is isolated and $s \sim s'$ where $s' \neq s$. Let s'' be any other isolated point. Then there exists $\gamma \in \Gamma$ such that $\gamma(s) = s''$, $\gamma(s'') = s$ and $\gamma(\bar{s}) = \bar{s}$ for $\bar{s} \notin \{s, s''\}$. It follows that $s'' \sim s'$ and so again the equivalence class of s is infinite, a contradiction. The remainder of the proof is as in (i).

Remarks. This settles the problem of Rolewicz by showing that the norm on $C[0,1]$ is maximal. In fact $C_0(S)$ has maximal norm whenever S is a manifold with boundary. The only examples we know of spaces S for which $C(S)$ is non-maximal are those which contain a finite number of isolated points; these may be renormed by

$$\|f\| = \sup_{S - S_0}\|f\| + \sqrt{\left(\sum_{s \in S_0} |f(s)|^2\right)},$$

where S_0 is the set of isolated points.

9. *The space $C_{\mathsf{R},0}(S)$.*

In this final section, we use different techniques to study the maximality of the norm in the space $C_{\mathsf{R},0}(S)$ of continuous real-valued functions vanishing at infinity on a locally compact Hausdorff topological space S. In this case there are, in general, fewer isometries, due to the lack of multipliers. Indeed, throughout this section we shall suppose that S is a connected manifold without boundary, and in this case, every isometry of $C_{\mathbb{R},0}(S)$ takes the form $\pm T_\gamma$, where $\gamma \in \Gamma(S)$. We shall need the following well-known lemma.

LEMMA 9·1. *Suppose S is a connected manifold without boundary of dimension greater than one. Then if $(s_1, \ldots, s_n)$ and $(s'_1, \ldots, s'_n)$ are two sets of distinct points of S, there exists $\gamma \in \Gamma(S)$ with $\gamma(s_i) = s'_i$.*

Now suppose $\|\cdot\|$ is a norm on $C_{\mathbb{R},0}(S)$, equivalent to $\|\cdot\|_\infty$ and such that every T_γ, $\gamma \in \Gamma(S)$, is an isometry.

LEMMA 9·2. *Under the assumption of Lemma 9·1, if $\mu, \lambda \in C_{\mathbb{R},0}(S)^*$, $\mu^+(S) = \lambda^+(S)$ and $\mu^-(S) = \lambda^-(S)$ then $\|\mu\| = \|\lambda\|$.*

Proof. We have immediately from Lemma 9·1 that if (s_1, s_2) and (s'_1, s'_2) are two pairs of distinct points in S then for $\alpha, \beta \geqslant 0$,

$$\|\alpha\delta_{s_1} - \beta\delta_{s_2}\| = \|\alpha\delta_{s'_1} - \beta\delta_{s'_2}\|.$$

Define $\phi(\alpha, \beta) = \|\alpha\delta_{s_1} - \beta\delta_{s_2}\|$ for $\alpha, \beta \geqslant 0$. For general $\mu \in C_{\mathbb{R},0}(S)^*$, μ is in the weak*-closed-convex hull of $\{\mu^+(S)\,\delta_{s_1} - \mu^-(S)\,\delta_{s_2} : s_1, s_2 \in S\}$ and so

$$\|\mu\| \leqslant \phi(\mu^+(S), \mu^-(S)).$$

Conversely for given $\mu \in C^*_{\mathbb{R},0}(S)$ and $\epsilon > 0$ we choose disjoint compact sets F_1 and F_2 such that $\mu \geqslant 0$ on F_1, $\mu \leqslant 0$ on F_2 and $|\mu|\,(S \backslash (F_1 \cup F_2)) \leqslant \epsilon$. By compactness we may cover F_1 with a finite collection of open subsets $B_1, \ldots, B_k$ such that each $\bar{B}_i$ is homeomorphic to a closed ball in $\mathbb{R}^d$, where $d = \dim S$ and $\bar{B}_i \cap F_2 = \varnothing$. Similarly we may cover F_2 with $B_{k+1}, \ldots, B_m$ such that each $\bar{B}_i$, $i > k$, is homeomorphic to a closed ball in $\mathbb{R}^d$, and $\bar{B}_i \cap F_1 = \varnothing$, $i > k$. For each $i \leqslant m$, pick $s_i \in B_i$. Then there is a sequence $\gamma^{(i)}_n \in \Gamma(S)$ such that $\gamma^{(i)}_n(s) = s$ for $s \notin B_i$ and $\lim_{n\to\infty} \gamma^{(i)}_n(s) = s_i$.

In the weak*-topology for any $\lambda \in C^*_{\mathbb{R},0}(S)$

$$\lim_{n\to\infty} T^*_{\gamma^{(i)}_n}\lambda = \lambda(B_i)\,\delta_{s_i} + \lambda_{S\backslash B_i}$$

and hence, since the norm is weak*-lower-semi-continuous,

$$\|\lambda(B_i)\,\delta_{s_i} + \lambda_{S\backslash B_i}\| \leqslant \|\lambda\|.$$

Applying this to μ for $i = 1, 2, \ldots, m$ in turn we obtain

$$\left\| \sum_{i=1}^m \mu\left(B_i \backslash \bigcup_{j=1}^{i-1} B_j\right) \delta_{s_i} + \mu_{S\backslash \cup B_i} \right\| \leqslant \|\mu\|.$$

Since $\|\ \|$ is equivalent to $\|\ \|_1$,

$$\left\| \sum_{i=1}^m \mu\left(B_i \backslash \bigcup_{j=1}^{i-1} B_j\right) \delta_{s_i} \right\| \leqslant \|\mu\| + K\epsilon$$

for some constant K, independent of μ.

Next we use Lemma 9·1 to determine a sequence $\gamma_n \in \Gamma(S)$ such that

$$\gamma_n(s_i) \to s_1, \; 1 \leqslant i \leqslant k \quad \text{and} \quad \gamma_n(s_i) \to s_m, \; k+1 \leqslant i \leqslant m.$$

Then we obtain by weak*-lower-semi-continuity of the norm,

$$\left\| \mu\left(\bigcup_{i=1}^{k} B_i\right)\delta_{s_1} + \mu\left(\bigcup_{i=1}^{m} B_i \setminus \bigcup_{i=1}^{k} B_i\right)\delta_{s_m}\right\| \leqslant \|\mu\| + K\epsilon.$$

Now
$$\mu\left(\bigcup_{i=1}^{k} B_i\right) = \mu(F_1) + \mu\left(\bigcup_{i=1}^{k} B_i \setminus F_1\right)$$

$$\geqslant \mu^+(S) - 2\epsilon$$

and
$$\mu\left(\bigcup_{i=1}^{m} B_i \setminus \bigcup_{i=1}^{k} B_i\right) \leqslant -\mu^-(S) + 2\epsilon.$$

Hence
$$\|\mu^+(S)\,\delta_{s_1} - \mu^-(S)\,\delta_{s_m}\| \leqslant \|\mu\| + 5K\epsilon$$

and so
$$\|\mu\| \geqslant \phi(\mu^+(S), \mu^-(S)).$$

THEOREM 9·3. *Let S be a connected manifold without boundary of dimension greater than one. Then $C_{\mathsf{R},0}(S)$ has maximal norm.*

Proof. We again suppose $\|\cdot\|$ is a norm on $C_{\mathsf{R},0}(S)$ such that every T_γ, $\gamma \in \Gamma(S)$, is an isometry. Let K^* denote the unit ball of $(C^*_{\mathsf{R},0}(S), \|\cdot\|)$. Suppose $\lambda \in exK^*$; then λ is also an extreme point of the set $\{\mu : \mu^+(S) = \lambda^+(S), \mu^-(S) = \lambda^-(S)\}$ by Lemma 9·2. Hence $\lambda = \alpha\delta_{s_1} - \beta\delta_{s_2}$ for $\alpha, \beta \geqslant 0$ and $s_1 \neq s_2$. By Lemma 9·1 it follows that if $\alpha\delta_{s_1} - \beta\delta_{s_2}$ is an extreme point then so is $\alpha\delta_{s_1'} - \beta\delta_{s_2'}$ for any $s_1' \neq s_2'$.

Now let $U: C_{\mathsf{R},0}(S) \to C_{\mathsf{R},0}(S)$ be an isometric isomorphism. We note that $U^*(exK^*) = exK^*$. We shall show that $\operatorname{supp} U^*(\delta_s)$ contains at most two points for each $s \in S$.

First suppose exK^* contains a point $\alpha\delta_s - \beta\delta_{s'}$ for $\alpha \neq \beta$. Then for any $u \neq s$ $\operatorname{supp} U^*(\alpha\delta_s - \beta\delta_u)$ contains at most two points. Letting u approach s and using the weak*-continuity of U^* we see that $\operatorname{supp} U^*(\delta_s)$ contains at most two points.

If exK^* contains no such points, then every extreme point of K^* is of the form $\alpha(\delta_s - \delta_{s'})$. Hence S is not compact (since the function $e(s) \equiv 1$ cannot belong to $C_{\mathsf{R},0}(S)$, as $\int e\, d\mu = 0$ for $\mu \in exK^*$). Therefore, $U^*\delta_s = \lim_{u\to\infty} U^*(\delta_s - \delta_u)$ has again at most two points in its support.

Thus in general $\operatorname{supp} U^*(\delta_s)$ contains at most two points. We may assume that for some $\alpha, \beta > 0$, $\alpha\delta_s - \beta\delta_{s'}$ is an extreme point, since otherwise the norm is simply a multiple of $\|\;\|_\infty$.

The set S_0 of s such that $\operatorname{supp} U^*(\delta_s)$ has two points is open in S and hence is either empty or infinite. Suppose the latter, and suppose s and s' belong to S_0. Then $U^*(\alpha\delta_s - \beta\delta_{s'})$ also has at most two points in its support. It follows that

$$\operatorname{supp}(U^*\delta_s) \cap \operatorname{supp}(U^*\delta_{s'}) \neq \varnothing.$$

As the set $\{U^*\delta_s : s \in S\}$ is linearly independent, at most two points can have the same support. Therefore there exist s_1, s_2, s_3 and $s_4 \in S_0$ such that the sets $\operatorname{supp}(U^*\delta_{s_i})$ are distinct. If supp $(U^*\delta_{s_1}) = \{u_1, u_2\}$, then supp $(U^*\delta_{s_2})$ and supp $(U^*\delta_{s_3})$ must be $\{u_1, u_3\}$ and $\{u_2, u_3\}$, and then it is impossible to choose a two-point set A intersecting each of these three sets. Therefore $S_0 = \varnothing$.

Therefore $U^*\delta_s = \theta(s)\,\delta_{\gamma(s)}$, $s \in S$; by weak*-continuity γ and θ are continuous. Furthermore $\|\delta_s\| = \|\delta_{s'}\|$ for $s \neq s'$, and hence $|\theta(s)| = 1$ for $s \in S$. As S is connected $\theta(s) \equiv +1$ or $\theta(s) \equiv -1$, and $U^* = \pm T_\gamma$.

We are grateful to Dr A. D. Thomas for calling our attention to reference (10), in which it is shown that there is a compact connected subset P of R^2 such that P has no non-trivial automorphisms. Thus $C_{\mathsf{R}}(P)$ has only two isometries, $\pm I$, and clearly has non-maximal norm. It is not clear whether $C(P)$ can have maximal norm. A similar example has also been observed by A. Pełczyński (see (6)).

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