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The uniform structure of Banach spaces

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Abstract We explore the existence of uniformly continuous sections for quotient maps. Using this approach we are able to give a number of new examples in the theory of the uniform structure of Banach spaces. We show for example that there are two non-isomorphic separable $\mathcal{L}_1$-subspaces of ℓ_1 which are uniformly homeomorphic. We also prove the existence of two coarsely homeomorphic Banach spaces (i.e. with Lipschitz isomorphic nets) which are not uniformly homeomorphic (answering a question of Johnson, Lindenstrauss and Schechtman). We construct a closed subspace of L_1 whose unit ball is not an absolute uniform retract (answering a question of the author).

1 Introduction

It was first proved by Ribe [43] that there exist separable Banach spaces which are uniformly homeomorphic without being linearly isomorphic. Ribe's construction is quite delicate and his technique has been used in subsequent papers by Aharoni and Lindenstrauss [1] and Johnson, Lindenstrauss and Schechtman [16] to create many interesting examples (see [3]).

In [22] we took an alternate approach, using what we will term the method of sections. The basic idea is that if $\mathcal{S} = 0 \to Z \to Y \to X \to 0$ is a short exact sequence of Banach spaces such that there is a uniformly continuous section $\varphi : X \to Y$ then

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$Z \oplus X$ and Y are uniformly homeomorphic. Indeed the map $\psi(z, x) = z + \varphi(x)$ has an inverse $\psi^{-1}(y) = (y - \varphi \circ Qy, Qy)$ where $Q : Y \to X$ is the quotient map. In [22] we used this to show that for every separable Banach space X there is a separable Banach space Z so that $X \oplus Z$ is uniformly homeomorphic to a Schur space. In this paper we will explore the method of sections in more generality to give some interesting new examples.

Let us now describe the content of the paper. Sections 2 and 3 are preparatory. In Sect. 4 we examine in more detail some ideas initially introduced in [22]. We say that a metric space M is approximable if there is an equi-uniformly continuous family $\mathcal{F}$ of functions $f : M \to M$ each with relatively compact range so that for every compact set $K \subset M$ and $\epsilon > 0$ we can find $f \in \mathcal{F}$ with $d(f(x), x) < \epsilon$ for every $x \in K$. This may be regarded as a nonlinear version (in the uniform category) of the bounded approximation property (BAP) for Banach spaces. In [22] it was shown that if X is super-reflexive then its unit ball B_X is approximable and it was asked if B_X is approximable for every Banach space. Here we prove much more general results and examine the conditions under which a Banach space X is approximable (which implies that B_X is also approximable). It is shown that every separable Banach space with separable dual is automatically approximable and indeed we do not know whether every Banach space is approximable (this is closely related to some unsolved problems related to the linear approximation property [4]). Based on this we give a simple example of two subspaces of c_0 which are uniformly homeomorphic but not linearly isomorphic. We also show that if a separable Banach space X is approximable then $X \oplus \mathcal{UB}$ and $\mathcal{UB}$ are uniformly homeomorphic where $\mathcal{UB}$ is Pełczyński's universal basis space [41]. We also show that every subspace of a Banach space with a shrinking (UFDD) is isomorphic to a complemented subspace of a separable Banach space which is uniformly homeomorphic to a space with a (UFDD).

In Sect. 5, we use these ideas to show that if X is a subspace of a space with shrinking (UFDD) then there is a uniformly continuous retraction of X^{**} onto X. We remark that Benyamini and Lindenstrauss [3, p. 180] raise the question whether for every Banach space there is a Lipschitz retraction of X^{**} onto X. See also the discussion of this problem and its connection with extension problems in [25]. Recently the author was able to give a counterexample to this problem [26]; indeed there is a Banach space X so that there is no uniformly continuous retraction of the unit ball $B_{X^{**}}$ onto B_X. However this counterexample is non-separable and the problem remains open for separable Banach spaces.

In Sect. 6, we rework some results on so-called good partitions, introduced in [22]; this material is not so new but our approach is cleaner than the original. These allow us to give general conditions under which, given the short exact sequence $\mathcal{S}$, there is a section φ which is locally uniformly continuous. We require that $\mathcal{S}$ locally splits, i.e. the dual sequence $0 \to X^* \to Y^* \to Z^* \to 0$ splits, X is approximable and has a good partition.

In Sect. 7, we develop some criteria for the existence of uniformly continuous sections φ (and also coarsely continuous sections). These are finally applied in Sect. 8 where we use a simple device to pass from a short exact sequence $\mathcal{S}$ where a locally uniformly continuous section exists to a short exact sequence $\tilde{\mathcal{S}}$ where a global uniformly continuous section exists. Applying this to various choices of $\mathcal{S}$ gives a number of examples. We show that there are two uniformly homeomorphic but not linearly isomorphic $\mathcal{L}_1$-spaces (one is a Schur space and the other contains L_1). A similar example can be created with both spaces embeddable in ℓ_1. We show the existence of two coarsely homeomorphic spaces (i.e. with Lipschitz equivalent nets) which are not uniformly homeomorphic, answering a question of Johnson, Lindenstrauss and Schechtman [16]. We also find a closed subspace Z of L_1 with the property that there is a Lipschitz map defined on a subset of a Hilbert space into Z which has no uniformly continuous extension into Z; this almost answers a question of Ball [2]. The unit ball of Z is thus not an absolute uniform retract (AUR) answering a question in [22] (note if X is a subspace of L_p for $p > 1$ then B_X is necessarily an AUR).

In fact the ideas of this paper can be combined with the Ribe approach to yield further interesting examples, but we postpone this to a separate paper [27].

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2 Preliminaries from linear Banach space theory

Our notation for Banach spaces is fairly standard (see e.g. [33]). All Banach spaces will be real. If X is a Banach space B_X denotes its closed unit ball and ∂B_X the unit sphere $\{x : \|x\| = 1\}$.

We recall that a separable Banach space X has the *BAP* if there is a sequence of finite-rank operators $T_n : X \to X$ such that $\lim_{n\to\infty} T_n x = x$ for every $x \in X$. X has the *metric approximation property (MAP)* if we can also impose the condition that $\|T_n\| \le 1$ for all n. If X^* is separable and, additionally, $\lim_{n\to\infty} T_n^* x^* = x^*$ for every $x^* \in X^*$, we say that X has *shrinking* (BAP) or (MAP). X has a *finite-dimensional decomposition (FDD)* if there is a sequence of finite-rank operators $P_n : X \to X$ such that $P_m P_n = 0$ when $m \neq n$ and $x = \sum_{n=1}^\infty P_n x$ for every $x \in X$. If each P_n has rank one then X has a *basis*. The (FDD) is called *shrinking* if we also have $x^* = \sum_{n=1}^\infty P_n^* x^*$ for every $x^* \in X^*$. If, in addition $x = \sum_{n=1}^\infty P_n x$ unconditionally for every $x \in X$ then X has an *unconditional finite-dimensional decomposition (UFDD)*. Finally if $\|\sum_{k=1}^n \eta_k P_k\| \le 1$ for every $n \in \mathbb{N}$ and $\eta_k = \pm 1$ for $1 \le k \le n$ then we say that X has a *1-(UFDD)*.

It is a remarkable result of Pełczyński [41] that there is a unique separable Banach space $\mathcal{UB}$ (*the universal basis space*) with (BAP) and the property that every separable Banach space with (BAP) is isomorphic to a complemented subspace of $\mathcal{UB}$; $\mathcal{UB}$ has a basis $(x_n)_{n=1}^\infty$ with the property that every basic sequence is equivalent to a (complemented) subsequence of $(x_n)_{n=1}^\infty$.

Let $(\epsilon_j)_{j=1}^\infty$ denote a sequence of independent Rademachers (i.e. independent random variables with $\mathbb{P}(\epsilon_j = 1) = \mathbb{P}(\epsilon_j = -1) = 1/2$). X is said to have (Rademacher) type p where $1 < p \le 2$ if there is a constant C so that

$$\left(\mathbb{E}\|\sum_{j=1}^n \epsilon_j x_j\|^p\right)^{\frac{1}{p}} \le C\left(\sum_{j=1}^n \|x_j\|^p\right)^{\frac{1}{p}}, \quad x_1, \ldots, x_n \in X.$$

We shall need the concept of asymptotic uniform smoothness. Let X be a separable Banach space. We define the *modulus of asymptotic uniform smoothness* (introduced by Milman [36]) $\overline{\rho}(t) = \overline{\rho}_X(t)$ by

$$\overline{\rho}(t) = \sup_{x \in \partial B_X} \inf_E \sup\{\|x + ty\| - 1 : \ y \in \partial B_E\}$$

where E runs through all closed subspaces of finite codimension.

As shown in [15] if $\overline{\rho}(t) < t$ for some $0 < t \le 1$ then X^* is separable. On the other hand if $\overline{\rho}(t) = 0$ for some $t > 0$ then X is isomorphic to a subspace of c_0 (see [11,15]). We say that X is *asymptotically uniformly smooth* if $\lim_{t \to 0} \overline{\rho}(t)/t = 0$. If X is asymptotically uniformly smooth then we have an estimate that $\overline{\rho}(t) \le Ct^{\theta}$ for some $0 < \theta < 1$ (see [12,28]). We then say that X is asymptotically uniformly smooth with power type θ.

We will need the following proposition:

Proposition 2.1 *Let X be an asymptotically uniformly smooth Banach space. Suppose $(x_n)_{n=1}^{\infty}$ is a normalized weakly null sequence in X. Then there is a subsequence $(x_n)_{n \in \mathbb{M}}$ so that for any k there exists $r \in \mathbb{M}$ so that if $r < n_1 < n_2 < \cdots < n_k$ with $n_j \in \mathbb{M}$ for each j we have*

$$\|x_{n_1} + \cdots + x_{n_k}\| \le 8(\overline{\rho}^{-1}(k^{-1}))^{-1}.$$

Proof Using standard Ramsey theory we can pass to a subsequence defining a spreading model S; thus we assume that

$$\lim_{(n_1,\ldots,n_m)\to\infty} \left\| \sum_{j=1}^{m} a_j x_{n_j} \right\| = \left\| \sum_{j=1}^{m} a_j e_j \right\|_S$$

exists for all finite scalar sequences $(a_1, \ldots, a_m)$ and defines a seminorm on c_{00}. By this notation we mean that for any $\epsilon > 0$ and $(a_1, \ldots, a_m)$ there exists q so that if $q < n_1 < n_2 < \cdots < n_m$ then

$$\left| \left\| \sum_{j=1}^{m} a_j x_{n_j} \right\| - \left\| \sum_{j=1}^{m} a_j e_j \right\|_S \right| < \epsilon.$$

Let $\sigma_k = \|e_1 + \cdots + e_k\|_S$. Then

$$\left\| \frac{1}{3}\sigma_k e_1 + e_2 + \cdots + e_{k+1} \right\|_S \ge \frac{2}{3}\sigma_k$$

and

$$\left\| \frac{1}{3}\sigma_k e_1 + e_2 + \cdots + e_{k+1} \right\|_S \le \frac{1}{3}\sigma_k (1 + \overline{\rho}(3/\sigma_k))^k.$$

Hence

$$k\overline{\rho}(3/\sigma_k) \geq \log 2 \geq 1/2$$

and thus $\sigma_k \leq 6(\overline{\rho}^{-1}(1/k))^{-1}$. The result then follows. □

We will also use the language of short exact sequences. If $Q : Y \to X$ is a quotient map then Q induces a short exact sequence $\mathcal{S} = 0 \to E \to Y \to X \to 0$, where $E = \ker Q$. Then $\mathcal{S}$ (or Q) *splits* if there is a projection $P : Y \to E$ or equivalently a linear operator $L : X \to Y$ so that $QL = Id_X$. $\mathcal{S}$ (or Q) *locally splits* if the dual sequence $0 \to X^* \to Y^* \to E^* \to 0$ splits, or equivalently, if there is a constant $\lambda \geq 1$ so that for every finite-dimensional subspace F of X there is a linear operator $L_F : F \to Y$ with $\|L_F\| \leq \lambda$ and $QL_F = Id_F$. An alternative formulation is that E is *locally complemented* in Y, i.e. there exists a linear operator $L : Y \to E^{**}$ such that $L|_E = Id_E$.

We shall frequently deal with ℓ_1-sums of a sequence of Banach spaces $(X_n)_{n=1}^\infty$. We denote by $(\sum_{n=1}^\infty X_n)_{\ell_1}$ the space of sequences $(x_n)_{n=1}^\infty$ with $x_n \in X_n$ and

$$\|(x_n)_{n=1}^\infty\| = \sum_{n=1}^\infty \|x_n\| < \infty.$$

In the special case when $X_n = X$ is a constant sequence we will write $\ell_1(X) = (\sum_{n=1}^\infty X)_{\ell_1}$. Let us now recall in particular the space C_1 introduced by Johnson [13] and later studied by Johnson and Zippin [19,20]. Let $(G_n)_{n=1}^\infty$ be a sequence of finite-dimensional Banach spaces dense in all finite-dimensional Banach spaces for the Banach–Mazur distance. Then we define $C_1 = (\sum_{n=1}^\infty G_n)_{\ell_1}$; this space is unique up to almost isometry.

A Banach space X is called an $\mathcal{L}_1$-space if there is some λ so that for any finite-dimensional subspace $F \subset X$ there is a finite-dimensional subspace $G \subset X$ containing F with $d(G, \ell_1^n) \leq \lambda$ where $n = \dim G$. If X is separable this is equivalent to the fact that X^* is isomorphic to ℓ_∞. If we have a short exact sequence $\mathcal{S} = 0 \to E \to Y \to X \to 0$ and X is a $\mathcal{L}_1$-space then $\mathcal{S}$ always locally splits.

For any separable Banach space X there is a quotient map $Q : \ell_1 \to X$ and hence a short exact sequence $0 \to E \to \ell_1 \to X \to 0$. A fundamental result of Lindenstrauss and Rosenthal [32] asserts that, except in the trivial case $X \approx \ell_1$, up to an automorphism of ℓ_1, this quotient map and the corresponding short exact sequence is unique. Thus we have a well-defined map $X \to \kappa(X)$ (up to isomorphism) which assigns to X a Banach space $\kappa(X)$ which is isomorphic to a subspace of ℓ_1, via the definition $\kappa(X) = E = \ker Q$. We let $\kappa(\ell_1) = \ell_1$. The map $X \to \kappa(X)$ is not injective. If X is a $\mathcal{L}_1$-space then $\kappa(X)$ is also a $\mathcal{L}_1$-space. Lindenstrauss [30] showed that the map κ is injective on the class of separable infinite-dimensional $\mathcal{L}_1$-spaces, i.e. if X and Y are $\mathcal{L}_1$-spaces and $\kappa(X) \approx \kappa(Y)$ then $X \approx Y$.

Let us conclude the section with a result we will need later:

Proposition 2.2 *Let X be a separable infinite-dimensional $\mathcal{L}_1$-space. Then the following are equivalent:*

(i) *X is isomorphic to $(\sum_{n=1}^\infty X_n)_{\ell_1}$ where each X_n is isomorphic to ℓ_1.*

(ii) $\kappa(X)$ *is isomorphic to* $(\sum_{n=1}^{\infty} Y_n)_{\ell_1}$ *where each* Y_n *is isomorphic to* ℓ_1.

Remark Let us remark that Johnson and Lindenstrauss used spaces of this type to show the existence of a continuum of $\mathcal{L}_1$-spaces in [14].

Proof If (i) holds we define a quotient map $Q_n : \ell_1 \to X_n$ and induce a quotient $Q : \ell_1(\ell_1) \to X$ via $Q((\xi_n)_{n=1}^{\infty}) = (Q_n \xi_n)_{n=1}^{\infty}$. Then it is clear that $\kappa(X)$ is isomorphic to $\ker Q$ and hence to $(\sum_{n=1}^{\infty} \ker Q_n)_{\ell_1}$.

Conversely if (ii) holds, then we may regard each Y_n as a subspace of $\kappa(X) = \ker Q_0$ where Q_0 is a quotient map from ℓ_1 onto X. Let $Q_n : \ell_1 \to X_n = \ell_1/Y_n$ be the induced quotient maps and consider the quotient map $Q : \ell_1(\ell_1) \to (\sum_{n=1}^{\infty} X_n)_{\ell_1}$. Then $\ker Q$ is linearly isomorphic to $\kappa(X)$. Further the spaces Y_n are uniformly complemented in $\kappa(X)$ and hence are uniformly locally complemented in ℓ_1, i.e. there exists a $\lambda < \infty$ and operators $L_n : \ell_1 \to Y_n^{**}$ with $L_n|_{Y_n} = Id_{Y_n}$ and $\|L_n\| \le \lambda$. Thus $\ker Q$ is locally complemented in $\ell_1(\ell_1)$ and hence $(\sum_{n=1}^{\infty} X_n)_{\ell_1}$ is a $\mathcal{L}_1$-space. By the uniqueness result of Lindenstrauss [30] we then have $(\sum_{n=1}^{\infty} X_n)_{\ell_1} \approx X$. To conclude we note that if each Y_n is isomorphic to ℓ_1, then local complementation together with the fact that Y_n is complemented in Y_n^{**} gives that Y_n is complemented in ℓ_1 (though not necessarily with a uniform constant). Thus each X_n is isomorphic to ℓ_1. □

3 Preliminaries from nonlinear theory

We refer to [3,25] for background on nonlinear theory.

Let (M, d) and (M', d') be metric spaces. If $f : M \to M'$ is any mapping we define $\omega_f : [0, \infty) \to [0, \infty]$ by

$$\omega(f; t) = \omega_f(t) := \sup\{d'(f(x), f(y));\ d(x, y) \le t\}.$$

f is Lipschitz if $\omega_f(t) \le ct$ for some constant c and *contractive* if $c \le 1$. f is said to be *uniformly continuous* if $\lim_{t\to 0} \omega_f(t) = 0$ and *coarsely continuous* if $\omega_f(t) < \infty$ for every $t > 0$. We also say that f is *coarse Lipschitz* if for some $t_0 > 0$ (or, equivalently, every $t_0 > 0$) there is a constant $c = c(t_0)$ so that

$$\omega_f(t) \le ct, \quad t \ge t_0.$$

This is equivalent to the requirement that for some constants c and a we have

$$\omega_f(t) \le ct + a, \quad 0 < t < \infty.$$

The notions of coarsely continuous and coarse Lipschitz maps are only nontrivial if and only if the metric d' on M' is unbounded. See for example [25].

It will be useful to track the constants for a coarse Lipschitz map. We will say that a map $f : M \to M'$ is of *CL-type* (L, ϵ) if we have an estimate

$$\omega_f(t) \le Lt + \epsilon, \quad t \ge 0.$$

We say that a map $f : M \to M'$ is a *uniform homeomorphism* if f is a bijection and f and f^{-1} are both uniformly continuous. A bijection $f : M \to M'$ is a *coarse homeomorphism* if and only if f and f^{-1} are coarsely continuous. A bijection $f : M \to M'$ is a *coarse Lipschitz homeomorphism* if and only if f and f^{-1} are coarse Lipschitz. In this case we say that f is a *CL-homeomorphism* of type (L, ϵ) if both f and f^{-1} are of CL-type (L, ϵ).

M is said to be *metrically convex* if given $x, y \in M$ and $0 < \lambda < 1$ there exists $z \in M$ with $d(x, z) + d(z, y) = d(x, y)$ and $d(x, z) = \lambda d(x, y)$. Any convex subset of a Banach space is metrically convex. If M is metrically convex and $f : M \to M'$ is any map then ω_f is subadditive, i.e.

$$\omega_f(s + t) \leq \omega_f(s) + \omega_f(t), \quad s, t \geq 0.$$

In particular, if M is metrically convex and f is coarsely continuous then f is coarse Lipschitz. We will refer to a subadditive map $\omega : [0, \infty) \to [0, \infty)$ with $\lim_{t \to 0} \omega(t) = 0$ as a *gauge*.

If X and Y are Banach spaces these considerations lead to the fact if $f : X \to Y$ is a uniform homeomorphism or a coarse homeomorphism then f is a coarse Lipschitz homeomorphism [25]. If X is any Banach space we define a *net* N_X in X to be any subset such that for some $0 < a < b < \infty$ we have

$$\|x_1 - x_2\| \geq a, \qquad x_1, x_2 \in N_X, \quad x_1 \neq x_2$$

and

$$d(x, N_X) \leq b, \quad x \in X.$$

We then refer to N_X as an (a, b)-net. In every separable Banach space one may find a net which is actually an additive subgroup of X (Theorem 5.5 of [7]). It was shown by Lindenstrauss, Matouskova and Preiss [31] that, if X is infinite-dimensional, any two nets in X are Lipschitz isomorphic. It is trivial to check that if X and Y are coarsely homeomorphic if and only if X and Y have Lipschitz isomorphic nets. It may also be easily verified that if X and Y are separable infinite-dimensional Banach spaces then they are coarse Lipschitz isomorphic if and only if they have Lipschitz isomorphic nets.

If M is a metric space and E is a subset of M then a *retraction* $r : M \to E$ is a map such that $r|_E = Id_E$. r is a Lipschitz (respectively uniform, respectively coarse) retraction if r is Lipschitz (respectively uniformly continuous, respectively coarsely continuous). A metric space M' is a called a *Lipschitz (respectively uniform, respectively coarse) retract* of M if it is Lipschitz isomorphic (respectively uniformly homeomorphic, respectively coarsely homeomorphic) to a subset E of M on which there is a Lipschitz (respectively uniformly continuous, respectively coarsely continuous) retraction from M.

A metric space (M, d) is called an *absolute Lipschitz retract* (ALR) if whenever M is isometrically embedded in a metric space M' there is a Lipschitz retraction

$r : M' \to M$. M is called a 1-ALR if r can be chosen to be contractive; more generally M is a λ-ALR if r can be chosen with Lipschitz constant at most λ. (M, d) is called an AUR if whenever M is isometrically embedded in a metric space M' there is a uniformly continuous retraction $r : M' \to M$.

If ω is a fixed gauge we say that $M \in AR(\omega)$ if whenever M' is a metric space, E is a subset of M' and $f_0 : E \to M$ is a contractive map then f_0 has an extension $f : M' \to M$ with

$$\omega_f(t) \le \omega(t), \quad 0 < t < \infty.$$

Lemma 3.1 *If M is an AUR there is a gauge ω so that $M \in AR(\omega)$.*

Proof We may embed M isometrically in the Banach space $\ell_\infty(I)$ for some set I. Of course $\ell_\infty(I)$ is a 1-ALR. By assumption there is a uniformly continuous retraction $r : \ell_\infty(I) \to M$. Then ω_r is a gauge. Now if M' is another metric space, E is a subset of M' and $f_0 : E \to M$ is any contractive map, then there is a contractive map $\phi : M' \to \ell_\infty(I)$ extending f_0. Let $f = r \circ \phi$ and then $\omega_f \le \omega_r$. □

Lemma 3.2 *Suppose ω is a gauge and $M \in AR(\omega)$. Suppose M' is a metric space, E is a subset of M' and $f_0 : E \to M$ is a uniformly continuous map. If ω' is any gauge such that $\omega_{f_0} \le \omega'$ then there is an extension $f : M' \to M$ of f_0 with $\omega_f \le \omega \circ \omega'$.*

Proof Consider M' with the metric $\omega' \circ d'$. Then $f_0 : (E, \omega' \circ d') \to M$ is contractive and using the definition we get an extension with the required properties. □

We will be most interested in the case when $M = B_X$ is the closed unit ball of a Banach space X. As we have already noticed, the space $\ell_\infty(I)$ is a 1-ALR and the same is true for its unit ball. c_0 and B_{c_0} are 2-ALR's; in fact if K is compact metric $C(K)$ and $B_{C(K)}$ are both 2-ALR's [24,29]. On the other hand B_X is an AUR whenever X is uniformly convex; this result goes back to [29] (see also [3, p. 28]). In fact $B_{\ell_2} \in AR(\omega)$ where $\omega(t) = \sqrt{2t}$ by a result of Minty [38] (see [3, p. 21]). If X is a Banach space with an unconditional basis and nontrivial cotype then B_X is uniformly homeomorphic to B_{ℓ_2} and hence is also an AUR by a result of Odell and Schlumprecht [40]. A similar result holds for X a separable Banach lattice with cotype [5].

If M has a base point (labelled 0), we refer to M as a *pointed metric space* and we define $\mathrm{Lip}(M)$ as the Banach space of all real-valued Lipschitz maps $f : M \to \mathbb{R}$ such that $f(0) = 0$ with the usual norm,

$$\|f\|_{\mathrm{Lip}} = \sup\left\{\frac{|f(x) - f(x')|}{d(x, x')} : x, x' \in M,\ d(x, x') > 0\right\}.$$

If $M = X$ is a Banach space or $M = B_X$, the base point is always the origin. The Arens–Eells space $\text{Æ}(M)$ is defined as the closed linear span of the point evaluations $\delta_s(f) = f(s)$ in $\mathrm{Lip}(M)^*$. The map $\delta : s \to \delta_s$ is then an isometry of M into $\text{Æ}(M)$. We refer to [10,44] for further details (in [10] the terminology *Lipschitz-free space* and the notation $\mathcal{F}(M)$ was used). If X is a Banach space there is a canonical quotient map $\beta : \text{Æ}(X) \to X$ and δ is an isometric section for β i.e. $\beta \circ \delta = I_X$.

4 Approximable metric spaces

In this section we will develop and improve some ideas originating in [22].

Let us say that a complete metric space M is *approximable* if there is a gauge ω so that for every finite set $E \subset M$ and every $\epsilon > 0$ we can find a uniformly continuous map $\psi : M \to M$ such that $d(x, \psi(x)) < \epsilon$ for every $x \in E$, $\psi(M)$ is relatively compact and $\omega_\psi \leq \omega$. For a specific choice of ω, we write $M \in \mathrm{App}(\omega)$. Note that in the above definition we may replace E by a compact set.

If M is separable then it is easy to see that M is *approximable* if and only if there is an equi-uniformly continuous sequence of maps $\psi_n : M \to M$ with relatively compact range such that $\lim_{n\to\infty} d(x, \psi_n(x)) = 0$ for each $x \in M$.

This definition was introduced in [22] for the special case when $M = B_X$ is the unit ball of a Banach space. In this case the terminology of [22] was that X has *(ucap)*. However, in retrospect, it seems it is interesting to consider this property also when $M = X$ so we will not use this terminology here.

We say that M is *Lipschitz approximable (with constant L)* if $M \in \mathrm{App}(\omega)$ when $\omega(t) = Lt$ for some constant L. Note that a Banach space X is Lipschitz approximable if and only if it has the (BAP) [10]. A concept which will be important here is that M is *almost Lipschitz approximable (with constant L)* if there is a constant L such that for every $\epsilon > 0$ we can find a gauge ω with $\omega(t) \leq Lt + \epsilon$ so that $M \in \mathrm{App}(\omega)$. We will say later that there are Banach spaces failing (BAP) which are almost Lipschitz approximable.

Lemma 4.1 *Let M be a complete separable metric space. Then M is almost Lipschitz approximable if and only if there is a constant L so that for every finite set E and $\epsilon > 0$ we can find a uniformly continuous map $\psi : M \to M$ with $\psi(M)$ relatively compact, $\omega_\psi(t) \leq Lt + \epsilon$, and*

$$d(x, \psi(x)) < \epsilon, \quad x \in E.$$

Proof If $\epsilon > 0$, it is easy to create a sequence $\psi_n : M \to M$ of uniformly continuous maps with $\psi_n(M)$ relatively compact, $\omega_{\psi_n}(t) \leq Lt + \epsilon/n$ and $\lim_{n\to\infty} \psi_n(x) = x$ for $x \in M$. Then $\omega(t) = \sup_n \omega_{\psi_n}(t)$ is a gauge i.e. $\lim_{t\to 0} \omega(t) = 0$ with $\omega(t) \leq Lt + \epsilon$ and $M \in \mathrm{App}(\omega)$. □

Lemma 4.2 *Let M be a metric space and let Y be a Banach space. Suppose $\psi : M \to Y$ be a uniformly continuous map with range contained in a compact set K. Let F be a compact convex set such that $\sup_{y\in K} d(y, F) < \epsilon$ for some positive ϵ. Then there is a uniformly continuous map $\psi' : M \to F$ with finite-dimensional relatively compact range such that $\|\psi(x) - \psi'(x)\| < \epsilon$ for $x \in M$ and $\omega_{\psi'}(t) < \omega_\psi(t) + 2\epsilon$ for $t > 0$.*

Proof Let $d = \sup_{y\in K} d(y, F)$ and suppose $\nu > 0$ is such that $\nu + d < \epsilon$. We pick a finite ν-net $(y_1, \ldots, y_n)$ for K and form a partition of unity $(\varphi_j)_{j=1}^n$ for K such that $\varphi_j(y) > 0$ implies $\|y - y_j\| < \nu$. Pick $z_j \in F$ with $\|y_j - z_j\| \leq d$. Then define

$$\psi'(x) = \sum_{j=1}^{n} \varphi_j(\psi(x)) z_j.$$

Since each φ_j is uniformly continuous on K, ψ' is uniformly continuous. Furthermore

$$\|\psi'(x) - \psi(x)\| \le d + \|\sum_{j=1}^{n} \varphi_j(\psi(x))(y_j - \psi(x))\|$$
$$\le d + \nu < \epsilon.$$

Finally

$$\|\psi'(x_1) - \psi'(x_2)\| < 2(d + \nu) + \|\psi(x) - \psi(x')\|.$$

□

The following result is proved in [22] but the proof here is easier:

Proposition 4.3 *Suppose M is a closed convex subset of a separable Banach space. If M is approximable then there is an equi-uniformly continuous sequence of maps $\psi'_n : M \to M$ each with finite-dimensional range such that $\lim_{n\to\infty} \psi'_n(x) = x$ for $x \in M$. If $M \in App(\omega)$ and $\epsilon > 0$ we can assume that $\omega_{\psi'_n} < \omega + \epsilon$ for each n.*

Proof Let (E_n) be an increasing sequence of finite subsets of M whose union is dense in M. Suppose $\epsilon > 0$ and $M \in \mathrm{App}(\omega)$. Choose $\psi_n : M \to M$ to be uniformly continuous maps with relatively compact range so that $\|\psi_n(x) - x\| < 1/n$ for $x \in E_n$ and $\omega_{\psi_n} \le \omega$. Let K_n be a compact set containing the range of ψ_n. Pick an $\epsilon/2n$-net for K_n and then let F_n be the linear span of this net. We use Lemma 4.2 to produce uniformly continuous functions $\psi'_n : M \to M \cap F_n$ with $\|\psi'_n(x) - \psi_n(x)\| \le \epsilon/2n$ for $x \in M$ and $\omega_{\psi'_n}(t) \le \omega(t) + \epsilon/n$. Clearly $\omega'(t) = \sup \omega_{\psi'_n}(t)$ is a gauge and $\omega' \le \omega + \epsilon$. □

We now characterize approximable Banach spaces. If X is any Banach space we denote by N_X a net for X which contains 0 (so that N_X can be considered as a pointed metric space).

Theorem 4.4 *Let X be a separable Banach space. Then the following are equivalent:*

(i) *X is approximable.*
(ii) *X is almost Lipschitz approximable.*
(iii) *B_X is almost Lipschitz approximable.*
(iv) *Æ(N_X) has (BAP).*
(v) *N_X is Lipschitz approximable.*

Proof (i) $\implies$ (ii). If $X \in \mathrm{App}(\omega)$ then it is clear by dilating that $X \in App(\omega_n)$ where $\omega_n(t) = \frac{1}{n}\omega(nt)$. Since $\omega(t) \le Lt + 1$ for some constant L we obtain (ii).

(ii) $\implies$ (iii) follows from the fact that there is a Lipschitz retraction of X onto B_X.

(iii) $\implies$ (i). Suppose B_X is almost Lipschitz approximable with constant L. If E is a finite subset of X then for a suitable constant $\lambda > 0$ we have $\lambda E \subset B_X$. If $\epsilon > 0$ we may find a uniformly continuous map $\psi : B_X \to B_X$ with relatively compact range such that $\|\psi(\lambda x)-\lambda x\| < \lambda\epsilon$ for $x \in E$ and $\omega_\psi(t) \le Lt + \lambda\epsilon$. Let $r : X \to B_X$ be the natural Lipschitz retraction. Let $\varphi : X \to X$ be given by $\varphi(x) = \lambda^{-1}\psi(r(\lambda x))$. Then $\omega_\varphi(t) \le 2Lt + \epsilon$ and $\|\varphi(x) - x\| < \epsilon$ for $x \in E$.

(ii) $\Longrightarrow$ (v). Suppose N_X is an (a, b)-net. Let L be a constant such that for every compact subset K of X and every $\epsilon > 0$ there is a (uniformly continuous) map $f : X \to X$ with CL-type (L, ϵ) and relatively compact range such that $\|f(x)-x\| < \epsilon$ for $x \in K$.

Let F be a finite subset of N_X. Then choosing $\epsilon < a/4$ we can find a map $f : X \to X$ with relatively compact range and CL-type (L, ϵ) so that $\|f(x)-x\| < \epsilon$ for $x \in F$. We can then define a map $\psi : f(X) \to N_X$ with finite range so that

$$\|\psi(x) - x\| \le 2d(f(x), N_X), \quad x \in f(X).$$

Let $g = \psi \circ f : N_X \to N_X$. Then g has finite range and $g(x) = x$ for $x \in F$. Furthermore if $x_1, x_2 \in N_X$ we have

$$\begin{aligned}\|g(x_1) - g(x_2)\| &\le 4b + \|f(x_1) - f(x_2)\| \\ &\le 4b + \epsilon + L\|x_1 - x_2\| \\ &\le (L + 1 + 4b/a)\|x_1 - x_2\|.\end{aligned}$$

(v) $\Longrightarrow$ (ii). Again suppose N_X is an (a, b)-net. Then there is a constant L so that if F is a finite subset of N_X there is a Lipschitz map $f : N_X \to N_X$ with Lipschitz constant at most L, finite range and such that $f(x) = x$ for $x \in F$.

Now suppose K is a finite subset of X and $\epsilon > 0$. Then we can find a finite subset F of N_X so that if $\theta = 12bL/\epsilon$,

$$d(\theta x, F) < 2b, \quad x \in K.$$

Define a map $f : N_X \to N_X$ with finite range G and Lipschitz constant at most L so that $f(x) = x$ for $x \in F$.

We next embed X isometrically into the space ℓ_∞. Then G is contained in a finite-dimensional subspace Z of ℓ_∞ which is at most 2-isomorphic to a space ℓ_∞^m for some finite m. Thus we can extend f to a Lipschitz map $\tilde{f} : X \to Z$ with $\mathrm{Lip}(\tilde{f}) \le 2L$. The range of $\tilde{f}$ is then relatively compact.

Let $h(x) = \theta^{-1}\tilde{f}(\theta x)$. If $x \in X$ we have $d(\theta x, N_X) < 2b$ and hence $d(\tilde{f}(\theta x), G) < 4Lb$. Hence

$$\sup_{x\in X} d(h(x), \theta^{-1}G) < \epsilon/3.$$

Now we can apply Lemma 4.2. There is a uniformly continuous function $g : X \to \theta^{-1}\mathrm{co}\, G$ so that

$$\|g(x) - h(x)\| < \epsilon/3, \quad x \in X$$

and

$$\|g(x_1) - g(x_2)\| \leq 2L\|x_1 - x_2\| + \epsilon/2, \quad x_1, x_2 \in X.$$

Note that if $x \in K$ there exists $x' \in F$ with $\|\theta x - x'\| < 2b$ and hence $\|\tilde{f}(\theta x) - x'\| < 4bL$. Thus

$$\|g(x) - \theta^{-1}x'\| < \epsilon/2$$

and so

$$\|h(x) - x\| \leq \|h(x) - g(x)\| + \|g(x) - \theta^{-1}x'\| + \|\theta^{-1}x' - x\| < \epsilon.$$

To conclude we can apply Lemma 4.1 to deduce that X is almost Lipschitz approximable.

(v) $\implies$ (iv). There is a constant L so that if F is a finite subset of N_X there is a finite set $G \supset F$ and a Lipschitz map $f : N_X \to G$ with constant at most L and $f(x) = x$ for $x \in F$. Then f induces a linear map $T_f : \text{Æ}(N_X) \to \text{Æ}(G) \subset \text{Æ}(N_X)$ such that $T_f(\delta_x) = \delta_{f(x)}$ for $x \in N_X$. Thus $T_f|_{\text{Æ}(F)} = I_{\text{Æ}(F)}$ and $\|T_f\| \leq L$. Since $\cup\text{Æ}(F)$ over all finite sets F is dense in $\text{Æ}(N_X)$ we have that $\text{Æ}(N_X)$ has (BAP).

(iv) $\implies$ (v). Assume (iv) and that N_X is an (a, b)-net. Then there is a constant L so that if F is a finite subset of N_X there is a finite set $G \subset N_X$ and a linear operator $T : \text{Æ}(N_X) \to \text{Æ}(G)$ with $\|T\| \leq L$ and $T\mu = \mu$ for $\mu \in \text{Æ}(F)$. Let $\beta : \text{Æ}(N_X) \to X$ denote the barycentric map. We can define a map $f : N_X \to G$ by

$$\|\beta(T\delta_x) - f(x)\| \leq 2d(\beta(T\delta_x), G), \quad x \in N_X.$$

Then $f(x) = x$ for $x \in F$ and if $x, x' \in N_X$,

$$\|f(x) - f(x')\| \leq 4b + \|\beta(T\delta_x - T\delta_{x'})\| \leq 4b + L\|x - x'\| \leq (L + 4b/a)\|x - x'\|.$$

□

It is, of course, clear that any Banach space with (BAP) is approximable. However we will show that any Banach space with separable dual is also approximable even if it fails (BAP). We prove first a technical Lemma which includes some features we will need later.

Lemma 4.5 *Let X be a separable Banach space and suppose Y is a Banach space containing X. Denote by $Q : Y \to Y/X$ the quotient map. Suppose that there is a sequence of finite-rank linear operators $T_n : X \to Y$ such that*

$$\lim_{n\to\infty} T_n x = x, \quad x \in X,$$

and

$$\lim_{n\to\infty} \|QT_n\| = 0.$$

Let $L = \sup_n \|T_n\|$ and let $r_n(x) = x/\min(2^n, \|x\|)$ for $x \in X$. Then if $(\epsilon_n)_{n=1}^\infty$ is a sequence of positive reals with $\lim_{n\to\infty} \epsilon_n = 0$, we can find a subsequence $(S_n)_{n=1}^\infty$ of $(T_n)_{n=1}^\infty$ and an equi-uniformly continuous sequence of maps $f_n : X \to X$ each with finite-dimensional relatively compact range such that

$$\|S_n(r_n x) - f_n(x)\| < \epsilon_n, \tag{4.1}$$

and

$$\omega_{f_n}(t) \leq 2Lt + \epsilon_n, \quad t > 0. \tag{4.2}$$

In particular X is approximable.

Proof We choose $(S_n)_{n=1}^\infty$ so that $\|QS_n\| < \epsilon_n/2^{n+1}$. Let K be the range of $S_n r_n$. Then for $y \in K$ we have

$$d(y, X) \leq 2^n \|QS_n\| < \epsilon_n/2.$$

Note that $S_n r_n$ has Lipschitz constant at most $2L$. Applying Lemma 4.2 we can find a uniformly continuous map $f_n : X \to X$ with a finite-dimensional relatively compact range so that

$$\|f_n(x) - S_n(r_n x)\| < \epsilon_n$$

and

$$\omega_{f_n}(t) \leq 2Lt + \epsilon_n.$$

It follows that $\lim_{n\to\infty} f_n(x) = x$ for $x \in X$ and that $(f_n)_{n=1}^\infty$ is equi-uniformly continuous. □

Theorem 4.6 *Let X be a Banach space with separable dual. Then X and X^* are both approximable.*

Proof For the case of X we use the fact that X is a subspace of a space Y with a shrinking basis [45]; in particular Y has shrinking (MAP). Let $Q : Y \to Y/X$ be the quotient map. Let $S_n : Y \to Y$ be a sequence of norm-one operators such that $S_n y \to y$ for $y \in Y$ and $S_n^* y^* \to y^*$ for $y^* \in Y^*$. Consider $QS_n : X \to Y/X$. For any $y^* \in X^\perp = (Y/X)^*$ we have

$$\lim_{n\to\infty} \|S_n^* Q^* y^*\| = 0$$

so that QS_n is a weakly null sequence in the space $\mathcal{K}(X, Y/X)$ [21, Corollary 3]. Hence there is a sequence of convex combinations T_n of $(S_k)_{k\geq n}$ such that

$$\lim_{n\to\infty} \|QT_n\|_{X\to Y/X} = 0$$

and we can apply Lemma 4.5.

For the case of X^* we use the result of [6] that X is also the quotient of a space Y with a shrinking basis. As before Y has (shrinking) (MAP). Let $Q : Y \to X$ be quotient map. Let $S_n : Y \to Y$ be a sequence of norm-one operators such that $S_n y \to y$ for $y \in Y$ and $S_n^* y^* \to y^*$ for $y^* \in Y^*$. In this case if Z is the kernel of the quotient map Q we may argue that (QS_n) is a weakly null sequence in $\mathcal{K}(Z, X)$. Indeed for $x^* \in X^*$ we have

$$\lim_{n\to\infty} \|S_n^* Q^* x^* - Q^* x^*\|_{Y^*} = 0$$

and so

$$\lim_{n\to\infty} \|S_n^* Q^* x^*|_Z\| = 0.$$

Again applying the result of [21] we have the existence of convex combinations (T_n) so that $\|QT_n\|_{Z\to X} \to 0$. Thus, if we identify X^* with the subspace $Z^\perp$ in Y^* we can apply Lemma 4.5 again. □

In view of this theorem it is natural to ask:

Problem 1 *Is every (separable) Banach space X approximable?*

In view of Theorem 4.4 above this is equivalent to asking if $Æ(N_X)$ always has the (BAP). In fact this is equivalent to the statement that $Æ(M)$ has (BAP) whenever M is uniformly discrete, i.e. if $\inf_{x\neq y} d(x, y) > 0$. For if M is uniformly discrete, then we may take $X = Æ(M)$ and a net N_X which contains $\{\delta_x : x \in M\}$. Then for some constant L, we have that if F is a finite subset of M there is a finite subset G of N_X and a Lipschitz map $f : N_X \to G$ with Lipschitz constant at most L such that $f(\delta_x) = \delta_x$ for $x \in F$. Now there is a linear map $T : Æ(M) \to Æ(M)$ so that $T(\delta_x) = f(x)$. Thus T has finite-dimensional range, $\|T\| \le L$ and $T\mu = \mu$ for $\mu \in Æ(F)$. Note that by Proposition 4.4 of [22] $Æ(M)$ always has the approximation property, but it is not clear that it has the (BAP).

Notice here a connection with a classical problem on the approximation property. It is unknown whether ℓ_1 has (MAP) in every equivalent norm (see [4, Problem 3.12]). Moreover Problem 3.8 of [4] asks if every dual space with (AP) has (MAP) which implies a positive answer to this question. Now the argument of Proposition 4.4 of [22] shows that, if the answer is positive, then we can conclude that $Æ(M)$ must have (BAP), whenever M is uniformly discrete.

We now apply the notion of approximability to the theory of uniform homeomorphisms.

Theorem 4.7 *Let X be an approximable separable Banach space. Then there is a closed subspace E of c_0 with an (FDD) and a subspace Y of $X \oplus c_0$ with an (FDD) so that Y is uniformly homeomorphic to $X \oplus E$.*

Proof By Proposition 4.3 there is an equi-uniformly continuous sequence of maps $\psi_n : X \to X$ with finite-dimensional range such that $\lim_{n\to\infty} \psi_n(x) = x$ for $x \in X$.

We let $\psi_0(x) = 0$ for all x. We may assume the existence of a gauge ω with $\omega_{\psi_k} \leq \omega$ for all k.

For $n \geq 1$, let F_n be the linear span of $\cup_{k=1}^n \psi_k(X)$. Let Y be the space of all sequences $(x_k)_{k=1}^\infty$ such that $x_k \in F_k$ and $\lim_{k\to\infty} x_k$ exists in X, under the norm

$$\|(x_k)_{k=1}^\infty\| = \sup_{k\geq 1} \|x_k\|.$$

Then Y has an (FDD) with coordinate projections

$$P_1(x_k)_{k=1}^\infty = (x_1, x_1, \dots)$$

and

$$P_n(x_k)_{k=1}^\infty = (0, \dots, 0, x_n - x_{n-1}, x_n - x_{n-1}, \dots)$$

where the first non-zero entry is in the nth slot.

There is a natural quotient map $Q : Y \to X$ given by $Q((x_k)_{k=1}^\infty) = \lim_{k\to\infty} x_k$. Let $E = \ker Q$. Then $E = (\sum_{k=1}^\infty F_k)_{c_0}$ has an (FDD) and embeds into c_0. By Sobczyk's theorem there is a linear operator $S : Y \to c_0$ so that $S|_E$ is an isomorphic embedding. Then $V : Y \to X \oplus c_0$ given by $Vy = (Qy, Sy)$ is a linear embedding and so Y is linearly isomorphic to a subspace of $X \oplus c_0$.

Finally define the map $\psi : X \to Y$ by

$$\psi(x) = (\psi_k(x))_{k=1}^\infty.$$

Then $Q \circ \psi(x) = x$ and

$$\|\psi(x) - \psi(x')\| = \sup_{k\geq 1} \|\psi_k(x) - \psi_k(x')\| \leq \omega(\|x - x'\|)$$

so that ψ is uniformly continuous. Thus Y is uniformly homeomorphic to $X \oplus E$. □

Theorem 4.8 *There are two closed subspaces Z_1 and Z_2 of c_0 which are uniformly homeomorphic but not linearly isomorphic.*

Proof By Theorems 4.6 and 4.7 we take X to be a closed subspace of c_0 which fails the approximation property (see [33, p. 37]) and then $Z_1 = X \oplus E$ and $Z_2 = Y$. Thus Z_2 has an (FDD) while Z_1 fails the approximation property. □

Remark The space Z_2 was constructed first by Johnson and Schechtman (see [9,18]); note that Z_2^* fails the approximation property.

In [22] it was observed that for a separable Banach space X, B_X is approximable if and only if B_X is a uniform retract of a Banach space with a basis. We now consider the analogous result for the case when X is approximable.

Theorem 4.9 *Let X be a separable Banach space. Then the following conditions on X are equivalent:*

(i) *X is approximable.*
(ii) *X is a uniform retract of a Banach space with a basis.*
(iii) *If $\mathcal{UB}$ denotes the universal basis space then $\mathcal{UB}$ and $\mathcal{UB} \oplus X$ are uniformly homeomorphic.*

Proof It is clear that (iii) $\implies$ (ii) $\implies$ (i) and that we only need to prove (i) $\implies$ (iii). We note that Theorem 4.7 implies that there is a separable Banach space E so that $X \oplus E$ is uniformly homeomorphic to a space Y with an (FDD). will first show that there is a space Z so that $X \oplus Z$ is uniformly homeomorphic to a space Y with an (FDD).

Next observe that $c_0(X)$ is also approximable. Since X is approximable, there is an equi-uniformly continuous sequence of maps $\psi_n : X \to X$ with finite-dimensional range such that $\lim_{n\to\infty} \psi_n(x) = x$ for $x \in X$. Then the maps

$$\Psi_n(x_j)_{j=1}^n = (\psi_n(x_1), \psi_n(x_2), \ldots, \psi_n(x_n), 0, \ldots)$$

define a suitable approximating sequence for $c_0(X)$. Thus there is a Banach space Y with an (FDD) and a separable Banach space Z so that $c_0(X) \oplus Z$ is uniformly homeomorphic to Y. But then $X \oplus c_0(X) \oplus Z$ is uniformly homeomorphic to both Y and $X \oplus Y$. Hence X is uniformly homeomorphic to $X \oplus Y$. However Y is isomorphic to a complemented subspace of $\mathcal{UB}$ (this follows from [41,42]). By the uniqueness of $\mathcal{UB}$ it is linearly isomorphic to $c_0(\mathcal{UB})$ and it follows that $Y \oplus \mathcal{UB}$ is linearly isomorphic to $\mathcal{UB}$ by a standard application of the Pełczyński decomposition technique. Thus we obtain (iii). □

Of course if X has the (BAP) then $\mathcal{UB} \oplus X$ is linearly isomorphic to $\mathcal{UB}$, but this theorem applies to certain spaces failing the approximation property.

Corollary 4.10 *Let X be a Banach space with separable dual. Then $X \oplus \mathcal{UB}$, $X^* \oplus \mathcal{UB}$ and $\mathcal{UB}$ are all uniformly homeomorphic.*

It follows from Theorem 4.9 that Problem 1 is closely related to:

Problem 2 *Does there exist a separable Banach space X so that every separable Banach space is a uniform retract of X?*

We now give another application of Lemma 4.5. For this and for future reference, if X is a Banach space we define $WUC(X)$ to be the space of weakly unconditionally Cauchy series in X i.e. the sequences $x = (x_n)_{n=1}^\infty$ where $x_n \in X$ such that

$$\|x\|_{WUC(X)} = \sup \left\{ \left\| \sum_{k=1}^n \sigma_k x_k \right\| : |\sigma_k| \le 1,\ n \in \mathbb{N} \right\} < \infty.$$

We define $UC(X)$ to be the closed subspace of $WUC(X)$ of all sequences $(x_n)_{n=1}^\infty$ such that $\sum_{n=1}^\infty x_n$ converges unconditionally. Of course $WUC(X)$ is linearly isometric to the space $\mathcal{L}(c_0, X)$ of all linear operators from c_0 into X and $UC(X)$ is the subspace $\mathcal{K}(c_0, X)$ of compact operators.

Theorem 4.11 *Let X be a separable Banach space which embeds into a Banach space V with a shrinking (UFDD). Then there is a Banach space Z and a Banach space Y with a (UFDD) so that Y and $X \oplus Z$ are uniformly homeomorphic. In particular X is a uniform retract of a Banach space with (UFDD).*

Proof We can assume by renorming that V has a 1-(UFDD). In this case the convex combinations argument used in Theorem 4.6 yields a sequence of operators $T_n : X \to Y$ with $T_0 = 0$ so that $\lim_{n\to\infty} \|QT_n\| = 0$ and

$$\|\sum_{k=1}^{n} \sigma_k (T_k - T_{k-1})\| \leq 1, \qquad |\sigma_k| \leq 1, \quad n \in \mathbb{N}.$$

We suppose that $\epsilon_n > 0$ are chosen so that $\sum_{n=1}^{\infty} \epsilon_n = \varepsilon < \infty$. Then by Lemma 4.5 we can find a subsequence $(S_n)_{n=1}^{\infty}$ of $(T_n)_{n=1}^{\infty}$ and uniformly continuous maps $f_n : X \to X$ with relatively compact and finite-dimensional range, so that

$$\|S_n(r_n x) - f_n(x)\| < \epsilon_n, \quad x \in X$$

and

$$\omega_{f_n}(t) \leq 2t + \epsilon_n, \quad t > 0.$$

Let F_n be a finite-dimensional subspace of X containing the range of $f_n - f_{n-1}$. We can define a space Y with a 1-(UFDD) as the subspace of $UC(X)$ of all sequences $(x_n)_{n=1}^{\infty}$ with $x_n \in F_n$ for all n. Define the map $Q : Y \to X$ by $Q(x_n)_{n=1}^{\infty} = \sum_{n=1}^{\infty} x_n$.

Let $f_0(x) = 0$ and $S_0 = 0$. For $x \in X$ let $a_n = \|r_n(x)\|/\|x\|$. Then for $|\sigma_k| \leq 1$ for $1 \leq k \leq n$ we have

$$\begin{aligned}
\left\|\sum_{k=1}^{n} \sigma_k (f_k(x) - f_{k-1}(x))\right\| &\leq 2\varepsilon + \left\|\sum_{k=1}^{n} \sigma_k (a_k S_k - a_{k-1} S_{k-1}) x\right\| \\
&\leq 2\varepsilon + \left\|\sum_{k=1}^{n} \left(a_k + \sum_{j=k+1}^{n} \sigma_j (a_j - a_{j-1})\right) (S_k - S_{k-1}) x\right\| \\
&\leq 2\varepsilon + \|x\|.
\end{aligned}$$

We define $\varphi : X \to Y$ by $\varphi(x) = (f_n(x) - f_{n-1}(x))_{n=1}^{\infty}$. This is well-defined by the preceding calculation and $Q\varphi(x) = x$. It follows that Q is onto and indeed is a quotient map. We will show that φ is a uniformly continuous section for Q. Let us suppose that $x, x' \in X$ with $\|x - x'\| \leq t$, and that $m \in \mathbb{N}$. Suppose $n \in \mathbb{N}$ and that $|\sigma_k| = 1$ for $1 \leq k \leq n$. If $n \leq m$ we have

$$\left\|\sum_{k=1}^{n} \sigma_k (f_k(x) - f_{k-1}(x) - f_k(x') - f_{k-1}(x'))\right\| \leq 2 \sum_{k=1}^{m} \omega_{f_k}(t).$$

If $n > m$ we have

$$\begin{aligned}&\left\|\sum_{k=1}^{n}\sigma_k(f_k(x)-f_{k-1}(x)-f_k(x')-f_{k-1}(x'))\right\|\\&\quad\le 2\sum_{k=1}^{m}\omega_{f_k}(t)+\left\|\sum_{k=m+1}^{n}\sigma_k(f_k(x)-f_{k-1}(x)-f_k(x')-f_{k-1}(x'))\right\|.\end{aligned}$$

The latter term is estimated by

$$\begin{aligned}&\left\|\sum_{k=m+1}^{n}\sigma_k(f_k(x)-f_{k-1}(x)-f_k(x')-f_{k-1}(x'))\right\|\\&\quad\le 2\sum_{k=m}^{\infty}\epsilon_k+\left\|\sum_{k=m+1}^{n}\sigma_k(S_k(x-x')-S_{k-1}(x-x'))\right\|\\&\quad\le 2\sum_{k=m}^{\infty}\epsilon_k+\|x-x'\|.\end{aligned}$$

Hence

$$\|\varphi(x)-\varphi(x')\|\le 2\sum_{k=m}^{\infty}\epsilon_k+2\sum_{k=1}^{m}\omega_{f_k}(t)+\|x-x'\|.$$

Hence

$$\omega_\varphi(t)\le 2\sum_{k=m}^{\infty}\epsilon_k+2\sum_{k=1}^{m}\omega_{f_k}(t)+t$$

and so

$$\limsup_{t\to 0}\omega_\varphi(t)\le 2\sum_{k=m}^{\infty}\epsilon_k.$$

Since m is arbitrary this means that φ is uniformly continuous.

It follows that Y is uniformly homeomorphic to $X\oplus Z$ where $Z=\ker Q$. □

Remark This result is a little unsatisfactory in that although X has separable dual, Y contains a copy of ℓ_1. We will prove a stronger result which implies that Y can be chosen with separable dual in a forthcoming paper [27].

Let us conclude by observing that in the sequel it will be more important to us to consider the case when B_X is approximable. In view of Theorem 4.4, if X is approximable then B_X is certainly approximable. Therefore Theorem 4.6 is a significant improvement on the result of [22] (Corollary 9.4) which asserts that B_X is approximable if X is super-reflexive. The following problem was also raised in [22].

Problem 3 *Is B_X approximable for every separable Banach space?*

We conclude with a small positive result.

Proposition 4.12 *Suppose X is a separable Banach space such that B_X is an AUR. Then B_X is approximable.*

Proof We embed X isometrically into $\mathcal{C}[0,1]$. Let S_n be the partial sum operators associated to the standard Schauder basis of $\mathcal{C}[0,1]$. Let $r : \mathcal{C}[0,1] \to B_X$ be a uniform retraction. Then $r \circ S_n(x) \to x$ for every $x \in B_X$ and $(r \circ S_n)_{n=1}^{\infty}$ is equi-uniformly continuous. □

5 Lipschitz and uniform retractions

In this section we give an application of the preceding ideas. As remarked in the introduction it is unknown whether for every separable Banach space X there is a Lipschitz retraction of X^{**} onto X. It is known that there is a Lipschitz retraction of ℓ_∞ onto c_0 and from $\mathcal{C}(K)^{**}$ onto $\mathcal{C}(K)$ for any compact metric space K. These results are due to Lindenstrauss [29]; see also [24] for the best constants.

Our first result is a simple generalization of the argument of Lindenstrauss for the existence of a Lipschitz retraction of ℓ_∞ onto c_0.

Theorem 5.1 *Let X be an arbitrary Banach space. Then there is a Lipschitz retraction of $WUC(X)$ onto $UC(X)$.*

Proof We will let $\psi(t) = \min(1, t-1)$ for $t \geq 1$. For $x = (x_n)_{n=1}^{\infty} \in WUC(X)$ we define

$$\pi_n(x) = \sup_{m \geq n} \sup_{\sigma_j = \pm 1} \left\| \sum_{j=n}^{m} \sigma_j x_j \right\|.$$

Note that $\pi_1(x) = \|x\|_{WUC(X)}$ and $(\pi_n(x))_{n=1}^{\infty}$ is a decreasing sequence of semi-norms. Let $\pi_\infty(x) = \lim_{n\to\infty} \pi_n(x) = d(x, UC(X))$. We then define a map $F : WUC(X) \to WUC(X)$ by $F(x) = (f_n(x))_{n=1}^{\infty}$ where

$$f_n(x) = \begin{cases} x_n, & x \in UC(X) \\ \psi(\pi_n(x)/\pi_\infty(x))x_n, & x \in WUC(X) \setminus UC(X). \end{cases}$$

It is easy to see that F maps into $UC(X)$ since if $\pi_\infty(x) > 0$ we have

$$\lim_{n\to\infty} \psi(\pi_n(x)/\pi_\infty(x)) = 0.$$

In order to check that F is Lipschitz we observe first that F is trivially Lipschitz on $UC(X)$. Suppose $x \in UC(X)$ and $y \in WUC(X) \setminus UC(X)$. Then if n is the first index such that $\pi_n(y) < 2\pi_\infty(y)$,

$$\begin{aligned}\|F(x)-F(y)\| &\le \|x-y\| + \|y-F(y)\| \\ &\le \|x-y\| + \sup_{m\ge n}\sup_{\sigma_j=\pm1}\left\|\sum_{j=n}^{m}\sigma_j(1-\psi(\pi_n(x)/\pi_\infty(x))y_j\right\| \\ &\le \|x-y\| + \sup_{m\ge n}\sup_{\sigma_j=\pm1}\left\|\sum_{j=n}^{m}\sigma_j y_j\right\| \\ &\le \|x-y\| + \pi_n(y) \\ &\le \|x-y\| + 2\pi_\infty(y) \\ &\le 3\|x-y\|.\end{aligned}$$

On the other hand suppose $x, y \in WUC(X)$ with

$$\max(\pi_\infty(x)/\pi_\infty(y), \pi_\infty(y)/\pi_\infty(x)) < 2.$$

Let n be the first index for which either $\pi_n(x) < 2\pi_\infty(x)$ or $\pi_n(y) < 2\pi_\infty(y)$; we shall assume, for convenience that the latter case occurs. Then we have

$$\begin{aligned}&\psi\left(\frac{\pi_j(x)}{\pi_\infty(x)}\right)x_j - \psi\left(\frac{\pi_j(y)}{\pi_\infty(y)}\right)y_j \\ &\quad= \psi\left(\frac{\pi_j(x)}{\pi_\infty(x)}\right)(x_j-y_j) + \left(\psi\left(\frac{\pi_j(x)}{\pi_\infty(x)}\right) - \psi\left(\frac{\pi_j(y)}{\pi_\infty(y)}\right)\right)y_j.\end{aligned}$$

If $j < n$ we have

$$\psi\left(\frac{\pi_j(x)}{\pi_\infty(x)}\right) - \psi\left(\frac{\pi_j(y)}{\pi_\infty(y)}\right) = 0.$$

If $j \ge n$ we have

$$\begin{aligned}\left|\psi\left(\frac{\pi_j(x)}{\pi_\infty(x)}\right) - \psi\left(\frac{\pi_j(y)}{\pi_\infty(y)}\right)\right| &\le \left|\frac{\pi_j(x)}{\pi_\infty(x)} - \frac{\pi_j(y)}{\pi_\infty(y)}\right| \\ &\le \left|\frac{\pi_j(x)-\pi_j(y)}{\pi_\infty(x)}\right| + \left|\frac{\pi_j(y)(\pi_\infty(x)-\pi_\infty(y))}{\pi_\infty(x)\pi_\infty(y)}\right| \\ &\le \frac{\pi_j(x-y)+2\pi_\infty(x-y)}{\pi_\infty(x)} \\ &\le 3\frac{\|x-y\|}{\pi_\infty(x)}\end{aligned}$$

Hence

$$\begin{aligned}\|F(x)-F(y)\| &\le \max_{j\ge 1}\psi\left(\frac{\pi_j(x)}{\pi_\infty(x)}\right)\|x-y\| + 3\frac{\|x-y\|}{\pi_\infty(x)}\pi_n(y)\\ &\le \|x-y\| + 6\frac{\pi_\infty(y)}{\pi_\infty(x)}\|x-y\|\\ &\le 13\|x-y\|.\end{aligned}$$

It follows that we have a local Lipschitz constant of at most 13 everywhere and hence F is Lipschitz. □

Theorem 5.2 *Let X be a separable Banach space, such that either:*

(i) *X has a (UFDD), or*
(ii) *X is a separable order-continuous Banach lattice.*

Then there is a Lipschitz retraction of X^{**} onto X.

Proof (i) Let P_n be the finite-rank projections associated to the (UFDD) so that for each $x \in X$ we have $x = \sum_{n=1}^{\infty} P_n x$ unconditionally. Then we define a linear map $T : X^{**} \to WUC(X)$ by $Tx^{**} = (P_n^{**}x^{**})_{n=1}^{\infty}$. If F is the retraction given by Theorem 5.1 and $S : UC(X) \to X$ is the natural summation operator we have that $S \circ F \circ T$ is a Lipschitz retract of X^{**} onto X.

(ii) Let u be a weak order-unit for X. We define a map $\mathcal{T} : X^{**} \to WUC(X)$ by

$$\mathcal{T}(x^{**}) = (\mathcal{T}_n(x^{**}))_{n=1}^{\infty}$$

where

$$\mathcal{T}_n(x^{**}) = x_+^{**} \wedge nu - x_+^{**} \wedge (n-1)u - x_-^{**} \wedge nu + x_-^{**} \wedge (n-1)u.$$

Since X is order-continuous it is an order ideal in X^{**} so each coordinate belongs to X. Furthermore it follows by considering a functional representation of X^{**} that if $\sigma_j = \pm 1$ for $1 \le j \le n$, then for any x^{**}, y^{**} we have

$$\left|\sum_{j=1}^{n} \sigma_j(\mathcal{T}_j(x^{**}) - \mathcal{T}_j(y^{**}))\right| \le |x^{**} - y^{**}|.$$

It follows (taking $y^{**} = 0$) that $\mathcal{T}$ maps into $WUC(X)$ and that $\mathcal{T}$ is Lipschitz with constant one. We then obtain a retraction by considering $S \circ F \circ \mathcal{T}$. □

Corollary 5.3 *If X is isomorphic to a complemented subspace of a space with (UFDD), then X is Lipschitz complemented in its bidual.*

Theorem 5.4 (i) *Let X be a separable Banach space with (BAP) and suppose Y is Lipschitz isomorphic to X. Then there is a Lipschitz retraction of Y^{**} onto Y if and only if there is a Lipschitz retraction of X^{**} onto X.*

(ii) *Let X be a separable Banach space which is approximable, and suppose Y is uniformly homeomorphic to X. Then there is a uniform retraction of Y^{**} onto Y if and only if there is a uniform retraction of X^{**} onto X.*

Proof (i) In this case Y also has (BAP) [10]. It therefore suffices to show that if there is a Lipschitz retraction of X^{**} onto X that the same follows for Y. Let $h : Y \to X$ be a Lipschitz isomorphism and suppose (T_n) is a bounded sequence of finite-rank operators $T_n : Y \to Y$ so that $\lim_{n\to\infty} T_n y = y$ for $y \in Y$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Denote r the retraction and define a map $g : Y^{**} \to Y$ by

$$g(y^{**}) = h^{-1} \circ r(w^* - \lim_{n\in\mathcal{U}} h(T_n^{**} y^{**})).$$

It is easy to check that g is the required retraction.

(ii) Here Y must also be approximable and so as in case (i) we only need consider one direction. Let $f_n : Y \to Y$ be an equi-uniformly continuous sequence of maps with relatively compact range so that $\lim_{n\to\infty} f_n(y) = y$ for $y \in Y$. We first extend these maps to Y^{**}. Let D be the set of all pairs (F, δ) where F is a finite-dimensional subspace of X^{**} and $\delta > 0$. Let $\mathcal{V}$ be an ultrafilter on D containing all the subsets $D(G, \nu) = \{(F, \delta) : \ G \subset F, \ \delta < \nu\}$ for G a finite-dimensional subspace of Y^{**} and $\nu > 0$. For each (F, δ) by the Principle of Local Reflexivity there is a linear operator $T_F : F \to Y$ so that $T_F y = y$ for $y \in F \cap X$ and $\|T_F\| < 1 + \delta$. Define

$$\phi_{n,F,\delta}(y^{**}) = \begin{cases} 0 & y^{**} \notin F \\ f_n(T_F y^{**}), & y^{**} \in F \end{cases}.$$

Then

$$\lim_{(F,\delta)\in\mathcal{V}} \phi_{n,F,\delta}(y^{**}) = \tilde{f}_n(y^{**})$$

exists in Y (in norm) and $\tilde{f}_n : Y^{**} \to Y$ is easily verified to be a equi-uniformly continuous sequence of maps with relatively compact range, extending the maps $(f_n)_{n=1}^{\infty}$. If $h : Y \to X$ is a uniform homeomorphism, we define $g : Y^{**} \to Y$ in a similar fashion to (i), i.e.

$$g(y^{**}) = h^{-1} \circ r(w^* - \lim_{n\in\mathcal{U}} h(\tilde{f}_n(y^{**})).$$

Then g is a uniform retraction. □

Remark We do not know whether the approximation conditions in (i) and (ii) of Theorem 5.4 are necessary.

Theorem 5.5 *If X is isomorphic to a subspace of a Banach space with a shrinking (UFDD) then there is a uniform retraction of X^{**} onto X.*

Proof This follows immediately by combining Theorem 5.4, Theorem 5.2 and its Corollary and Theorem 4.11. □

There are some natural problems here:

Problem 4 *If X is a separable Banach space is there always a uniformly continuous (or even Lipschitz) retraction of X^{**} onto X?*

Problem 5 *If X is a separable Banach space is there always a uniformly continuous retraction of $B_{X^{**}}$ onto B_X?*

As remarked in the introduction these problems have a negative answer for non-separable spaces [26]. We suspect both problems have a negative solution in general.

6 The existence of local sections for quotient maps

In this section we will describe some more ideas from [22]. Although many results appear in [22] we will give a more streamlined approach, improving the presentation.

We will now consider conditions on a quotient map $Q : Y \to X$ so that there is a uniformly continuous section relative to B_X, i.e. a uniformly continuous map $\psi : B_X \to Y$ so that $Q \circ \psi(x) = x$ for $x \in B_X$. Note that if such a map exists we can always assume that it is homogeneous i.e. $\psi(\alpha x) = \alpha\psi(x)$ if $x, \alpha x \in B_X$. Indeed we can define

$$\psi'(x) = \begin{cases} 0 & x = 0 \\ \frac{\|x\|}{2}(\psi(x/\|x\|) - \psi(-x/\|x\|)), & x \neq 0. \end{cases}$$

Clearly ψ' is also a uniformly continuous section on every bounded set and is homogeneous. Thus we will say that Q or, equivalently, the short exact sequence

$$0 \to \ker Q \to Y \to X \to 0,$$

has a *locally uniformly continuous section.*

Suppose M, M' are complete metric spaces. A map $f : M \to M'$ is called *perfect* if whenever $(x_n)_{n=1}^{\infty}$ is a sequence in M such that $(f(x_n))_{n=1}^{\infty}$ is convergent in M' then there is a subsequence $(x_n)_{n\in\mathbb{M}}$ which is convergent in M. We shall say that f is *uniformly perfect* if given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ with the property that if $(x_n)_{n=1}^{\infty}$ is a sequence in M then $\sup_{m,n\in\mathbb{N}} d'(f(x_m), f(x_n)) < \delta$ implies the existence of a subsequence $\mathbb{M}$ with $\sup_{m,n\in\mathbb{M}} d(x_m, x_n) < \epsilon$. Using elementary Ramsey theory, this can be equivalently stated in the form

$$\lim_{m\to\infty}\lim_{n\to\infty} d'(f(x_m), f(x_n)) < \delta \implies \lim_{m\to\infty}\lim_{n\to\infty} d(x_m, x_n) < \epsilon, \tag{6.3}$$

whenever all limits exist.

It is clear that a uniformly perfect map is also perfect and that the composition of uniformly perfect maps remains uniformly perfect.

Let $\partial_+ B_{\ell_1}$ denote the set $\partial B_{\ell_1} \cap P$ where P is the closed positive cone $\{\xi \in \ell_1 : \xi_j \geq 0,\ 1 \leq j < \infty\}$. We shall say that a Banach space X has *a good partition* [22] if there is a map $f : \partial B_X \to \partial_+ B_{\ell_1}$ which is uniformly continuous and uniformly perfect.

As a simple illustration we prove:

Proposition 6.1 *Let $(E_n)_{n=1}^\infty$ be a sequence of finite-dimensional normed spaces and let $X = (\sum_{n=1}^\infty E_n)_{\ell_1}$. Then X has a good partition.*

Proof We define $f : \partial B_X \to \partial_+ B_{\ell_1}$ by $f(\xi) = (\|\xi(k)\|)_{k=1}^\infty$, where $\xi = (\xi(k))_{k=1}^\infty$. We use (6.3). We may clearly consider a sequence $(\xi_n)_{n=1}^\infty$ in X so that $\lim_{n\in\mathbb{N}} \xi_n(k) = \xi(k)$ exists for each $k \in \mathbb{N}$. Then

$$\lim_{m\to\infty}\lim_{n\to\infty} \|\xi_m - \xi_n\| = 2(1 - \|\xi\|) = \lim_{m\to\infty}\lim_{n\to\infty} \|f(\xi_m) - f(\xi_n)\|,$$

and the result follows. □

Lemma 6.2 *For any Banach space X there exists a uniformly continuous and uniformly perfect map $g : B_X \to \partial B_X$.*

Proof Pick any $x_0 \in X$ with $\|x_0\| = 2$. Consider the map $G : B_X \to \partial B_X$ defined by $g(x) = (x + x_0)/\|x + x_0\|$. G is uniformly continuous and uniformly perfect. For the latter claim suppose $(x_n)_{n\in\mathbb{N}}$ is any sequence such that

$$\lim_{m\to\infty}\lim_{n\to\infty} \|x_m - x_n\|, \quad \lim_{m\to\infty}\lim_{n\to\infty} \|g(x_m) - g(x_n)\|$$

both exist. We can suppose also that $\lim_{n\to\infty} \|x_0 + x_n\| = \beta$ exists. Then

$$\lim_{m\to\infty}\lim_{n\to\infty} \|g(x_m) - g(x_n)\| = \beta^{-1} \lim_{m\to\infty}\lim_{n\to\infty} \|x_m - x_n\|$$

and as $1 \le \beta \le 3$ this shows that g is uniformly perfect. □

Proposition 6.3 *The following are equivalent:*

(i) *X has a good partition.*
(ii) *There is a uniformly continuous, uniformly perfect map $h : B_X \to B_{\ell_1}$.*

Proof (i) ⇒ (ii). Suppose $f : \partial B_X \to \partial_+ B_{\ell_1}$ is uniformly continuous and uniformly perfect, and let $g : B_X \to \partial B_X$ be the uniformly continuous, uniformly perfect map given by Lemma 6.2. Thus $h = f \circ g : B_X \to \partial B_{\ell_1}$ gives (ii).

(ii) ⇒ (i). Let $f = \phi \circ g \circ h$ where $g : B_{\ell_1} \to \partial B_{\ell_1}$ is given by Lemma 6.2 and $\phi : \partial B_{\ell_1} \to \partial_+ B_{\ell_1}$ is given by Proposition 6.1. □

Lemma 6.4 *Let X be a separable Banach space and suppose $f : \partial B_X \to \partial_+ B_{\ell_1}$ is uniformly perfect. Suppose $f(x) = (a_n(x))_{n=1}^\infty$. Then, given $\epsilon > 0$ there exists $\nu > 0$ with the property that for each N we can find a finite subset $A_N \subset B_X$ such that*

$$\sum_{k=1}^N a_k(x) > 1 - \nu \implies d(x, A_N) < \epsilon.$$

Proof Let $\eta = \eta(\epsilon) > 0$ be chosen as in the definition of uniformly perfect, i.e. as (6.3). Let $\nu = \eta/3$. If the conclusion fails for some N we may choose an infinite sequence $(x_n)_{n=1}^{\infty}$ with $\lim_{m\to\infty}\lim_{n\to\infty}\|x_m - x_n\| \geq \epsilon$ (and such that the limits exist) but

$$\sum_{k=1}^{N} a_k(x_n) > 1 - \nu.$$

Passing to a subsequence we may assume that $\lim_{n\to\infty} a_k(x_n) = \alpha_k$ exist for each k and

$$\sum_{k=1}^{N} \alpha_k \geq 1 - \nu.$$

Hence

$$\lim_{m\to\infty}\lim_{n\to\infty}\|f(x_m) - f(x_n)\| \leq 2\nu < \eta$$

giving a contradiction. □

Proposition 6.5 *Let $Q : Y \to X$ be a quotient mapping. In order that there exists a uniformly continuous section $f : B_X \to Y$ it is necessary and sufficient that for some $0 < \lambda < 1$ there is a uniformly continuous map $\phi : \partial B_X \to Y$ with $\|Q(\phi(x)) - x\| \leq \lambda$ for $x \in \partial B_X$.*

Proof We may extend ϕ to B_X to be positively homogeneous and ϕ remains uniformly continuous. Define $g(x) = x - Q(\phi(x))$, so that g is also positively homogeneous. Let $g^0(x) = x$ and then $g^n = g \circ g \circ \cdots \circ g$ (n times). Then $\|g(x)\| \leq \lambda\|x\|$ and so $\|g^n(x)\| \leq \lambda^n\|x\|$ for $x \in B_X$. Let

$$f(x) = \sum_{n=0}^{\infty} \phi(g^n(x)).$$

The series converges uniformly in $x \in B_X$ and so f is uniformly continuous. Furthermore

$$Qf(x) = \sum_{n=0}^{\infty}(g^n(x) - g^{n+1}(x)) = x.$$

□

The following theorem is proved in [22]; however the proof of the crucial Lemma 10.3 of [22] was given incorrectly so we present the essential steps in the proofs again:

Theorem 6.6 *Let X be a separable Banach space. The following conditions on X are equivalent:*

(i) *B_X is approximable and has a good partition.*
(ii) *There is a sequence of homogeneous maps $\varphi_j : B_X \to X$ each with finite-dimensional range such that $x = \sum_{j=1}^{\infty} \varphi_j(x)$ for $x \in B_X$ and a map $\omega : [0, \infty) \to [0, \infty)$ with $\lim_{t\to 0} \omega(t) = \omega(0) = 0$ such that*

$$\sum_{j=1}^{\infty} \|\varphi_j(x) - \varphi_j(y)\| \le \omega(\|x - y\|), \quad x, y \in B_X.$$

(iii) *Whenever $Q : Y \to X$ is a quotient map such that $\mathcal{S} = 0 \to \ker Q \to Y \to X \to 0$ locally splits then $\mathcal{S}$ has a locally uniformly continuous section.*
(iv) *B_X is a uniform retract of B_{C_1}.*

Proof (i) $\implies$ (ii). We first need a preparatory lemma:

For $0 < \lambda < 1$ define $\Psi_\lambda : \partial_+ B_{\ell_1} \to \partial_+ B_{\ell_1}$ by

$$\Psi_\lambda(\xi) = \lambda^{-1}\left(\left(\sum_{j=1}^{n} \xi_j - 1 + \lambda\right)_+ - \left(\sum_{j=1}^{n-1} \xi_j - 1 + \lambda\right)_+\right)_{n\ge 1}.$$

(The second summand is interpreted as 0 when $n = 1$.)

Lemma 6.7 *Ψ_λ is Lipschitz.*

Proof Suppose $\xi, \eta \in \partial_+ B_{\ell_1}$. For convenience we extend ξ and η to be functions on $[0, \infty)$ by $\xi(t) = \xi_n$ for $n - 1 < t \le n$ and $\eta(t) = \eta_n$ for $n - 1 < t \le n$.

Let u, v be the minimal solutions of

$$\int_0^u \xi(t)\, dt = 1 - \lambda, \quad \int_0^v \eta(t)\, dt = 1 - \lambda.$$

Without loss of generality we may assume that $u \le v$. Then

$$\begin{aligned}\|\Psi_\lambda(\xi) - \Psi_\lambda(\eta)\|_1 &\le \lambda^{-1}\left(\int_v^{\infty} |\xi(t) - \eta(t)|\, dt + \int_u^v \xi(t)\, dt\right)\\ &\le \lambda^{-1}\left(\|\xi - \eta\| + \int_0^v (\xi(t) - \eta(t))\, dt\right)\\ &\le 2\lambda^{-1}\|\xi - \eta\|.\end{aligned}$$

□

Now we continue with the proof of (i) $\implies$ (ii) of Theorem 6.6. There is an equi-uniformly continuous sequence of functions $\psi_n : B_X \to B_X$ with finite-dimensional range so that $x = \lim_{n\to\infty} \psi_n(x)$ for $X \in B_X$. We let $\omega(t) = \sup_n \omega(\psi_n; t)$. We may assume $(E_n)_{n=1}^{\infty}$ is an increasing sequence of finite-dimensional subspaces so that $\psi_n(B_X) \subset E_n$. Consider the quotient map $Q : (\sum_{n=1}^{\infty} E_n)_{\ell_1} \to X$ given by $Q((\xi(j))_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \xi(j)$. Then the conclusion of (ii) restates the fact that there is a uniformly continuous section of Q on B_X. Thus we will establish the existence of such a section.

To prove this let $f : B_X \to \partial_+ B_{\ell_1}$ be a uniformly continuous and uniformly perfect map. Suppose $f(x) = (a_n(x))_{n=1}^{\infty}$. Pick $\epsilon > 0$ so that $\omega(\epsilon) + \epsilon < 1/4$. Then, by Lemma 6.7 there is a choice of $0 < \lambda < 1$ such that for every n we can find a finite subset A_n of X with the property that

$$\sum_{k=1}^{n} a_k(x) > 1 - \lambda \implies d(x, A_n) < \epsilon.$$

Let $U_n = \{x : \sum_{k=1}^{n} a_k(x) > 1 - \lambda\}$.

For each n we may find $m = m(n)$ so that

$$\|\psi_m(z) - z\| < \frac{1}{4}, \quad z \in A_n.$$

Then for $x \in U_n$ we have that for some $z \in A_n$, $\|x - z\| < \epsilon$ and so

$$\|\psi_m(x) - x\| \leq \omega(\epsilon) + \epsilon + 1/4 < 1/2, \quad x \in U_n.$$

Let us define $\rho_n : \partial B_X \to (\sum_{n=1}^{\infty} F_n)_{\ell_1}$ by $\rho_n(x) = (0, \ldots, 0, \psi_n(x), 0, \ldots)$ where the only nonzero term is in the nth. slot. Thus $\|Q\rho_n(x) - x\| < 1/2$ for $x \in U_n$.

Consider $f' = \Psi_\lambda \circ f$; this also defines a uniformly continuous map into $\partial_+ B_{\ell_1}$, by Lemma 6.7. Let $f'(x) = (b_n(x))_{n=1}^{\infty}$.

Now define

$$\phi(x) = \sum_{n=1}^{\infty} b_n(x)\rho_n(x), \quad x \in \partial B_X.$$

Then

$$\begin{aligned} \|\phi(x) - \phi(y)\| &\leq \sum_{n=1}^{\infty} |b_n(x) - b_n(y)| + \sum_{n=1}^{\infty} b_n(y)\|\rho_n(x) - \rho_n(y)\| \\ &\leq \omega_{f'}(\|x - y\|) + \omega(\|x - y\|), \end{aligned}$$

so that ϕ is uniformly continuous.

Furthermore

$$\|Q\phi(x) - x\| \leq \sum_{x \in U_n} b_n(x)\|Q\rho_n(x) - x\| < 1/2.$$

The conclusion now follows from Proposition 6.5.

(ii) ⇒ (iii). Let $\varphi_n(B_X) \subset E_n$ where E_n is a finite-dimensional subspace of X. For some $\mu \geq 1$ we can find linear sections $T_n : E_n \to Y$ of Q with $QT_nx = x$ for $x \in E_n$ and $\|T_n\| \leq \mu$. Define

$$\psi(x) = \sum_{n=1}^{\infty} T_n \circ \varphi_n(x).$$

Then ψ is a uniformly continuous section of Q on B_X.

(iii) ⇒ (iv). There is a quotient map $Q : C_1 \to X$ which locally splits. Hence there is a uniformly continuous section $\psi : B_X \to C_1$. Let r denote the Lipschitz retraction of X onto B_X. If $\psi(B_X) \subset \mu B_{C_1}$ where $\mu \geq 1$ then $r \circ Q$ is a uniformly continuous map of μB_{C_1} onto B_X and $r \circ Q \circ \psi$ is the identity on B_X.

(iv) ⇒ (i). If B_X is a uniform retract of B_{C_1} then it is clear that X has a good partition (by Proposition 6.1) and B_X is approximable (since C_1 has the (MAP)). □

7 The existence of globally uniformly continuous sections for quotient maps

We now consider the conditions which are necessary for the existence of a global uniformly continuous (or coarsely continuous) section of a quotient map.

We will start with a simple proposition on the existence of uniform and coarse sections.

Proposition 7.1 *Let Y be a Banach space and let Z be a closed subspace of Y. Consider the following conditions on Z:*

- (i) *There is a uniform section of the quotient map $Q : Y \to Y/Z$.*
- (i)′ *There is a coarse section of the quotient map $Q : Y \to Y/Z$.*
- (ii) *There is a uniform retraction of Y onto Z.*
- (ii)′ *There is a coarse retraction of Y onto Z.*
- (iii) *The short exact sequence $0 \to Z \to Y \to Y/Z \to 0$ locally splits.*

Then (i) $\implies$ (i)′, (ii) $\implies$ (ii)′, (i) $\implies$ (ii) $\implies$ (iii) *and* (i)′ $\implies$ (ii)′ $\implies$ (iii).

Proof Most of the implications are trivial. The only implication requiring a proof is $(ii)' \implies (iii)$. Suppose $\varphi : Y \to Z$ is a coarse retract. We will prove the existence of a bounded linear operator $L : Y \to Z^{**}$ so that $L(z) = z$ for $z \in Z$. This follows directly from the argument of Theorem 7.2 of [3, p. 171]. □

Let X and Y be Banach spaces. We define $\mathcal{H}(X, Y)$ to be the space of all maps $f : X \to Y$ which are positively homogeneous, i.e.

$$f(\alpha x) = \alpha f(x), \qquad x \in X, \quad \alpha \geq 0$$

and bounded, i.e.

$$\|f\| = \sup\{\|f(x)\| : \|x\| \leq 1\} < \infty.$$

It is clear that $\mathcal{H}(X, Y)$ is a Banach space with this norm containing the space $\mathcal{L}(X, Y)$ of all bounded linear operators. We will identify a subspace $\mathcal{HU}(X, Y)$ as the set of f such that the restriction of f to B_X (and hence to any bounded set) is uniformly continuous.

If $\epsilon > 0$, for $f \in \mathcal{H}(X, Y)$ we define $\|f\|_\epsilon$ to be the least constant L so that $L \geq \|f\|$ and

$$\|f(x) - f(x')\| \leq L \max\{\|x - x'\|, \epsilon\|x\|, \epsilon\|x'\|\}, \quad x, x' \in X.$$

It is easy to see that for each $\epsilon > 0$, $\|\cdot\|_\epsilon$ is a norm on $\mathcal{H}(X, Y)$ which is equivalent to the original norm; precisely

$$\|f\| \leq \|f\|_\epsilon \leq 2\epsilon^{-1}\|f\|, \quad f \in \mathcal{H}(X, Y).$$

Observe that $\|f\|_\epsilon$ increasing in ϵ and $\sup_{\epsilon>0} \|f\|_\epsilon < \infty$ if and only if f is a Lipschitz map.

We will need the following easy Lemma.

Lemma 7.2 *Let X be a Banach space and suppose $x, z \in X$ with $\|x\| \geq \|z\| > 0$. Then*

$$\left\| \frac{x}{\|x\|} - \frac{z}{\|z\|} \right\| \leq 2\frac{\|x - z\|}{\|x\|}$$

and

$$\|x - z\| \leq \|x\| - \|z\| + \|z\| \left\| \frac{x}{\|x\|} - \frac{z}{\|z\|} \right\| \leq 3\|x - z\|.$$

Proposition 7.3 *Suppose $f \in \mathcal{H}(X, Y)$ and $\varphi = f|_{\partial B_X}$. Then*

(i) *If φ is of CL-type (L, ϵ) where $L \geq 1$, $\epsilon > 0$ and $\|\varphi(x)\| \leq K$ for $x \in \partial B_X$ then we have $\|f\|_\epsilon \leq 2K + 4L$.*
(ii) *If $\|f\|_\epsilon = L$ then φ is of CL-type $(2L, 2L\epsilon)$.*

Proof (i) Suppose $\|x\| \geq \|z\| > 0$. Then, using Lemma 7.2

$$\begin{aligned}\|f(x) - f(z)\| &= \|\|x\|\varphi(x/\|x\|) - \|z\|\varphi(z/\|z\|)\| \\ &\leq K\|x - z\| + \|z\|\|\varphi(x/\|x\|) - \varphi(z/\|z\|)\| \\ &\leq K\|x - z\| + L\|z\| \left\| \frac{x}{\|x\|} - \frac{z}{\|z\|} \right\| + \epsilon\|z\| \\ &\leq (K + 2L)\|x - z\| + \epsilon\|z\| \\ &\leq (2K + 4L)\max\{\|x - z\|, \epsilon\|x\|\}.\end{aligned}$$

(ii) If $x, z \in \partial B_X$ then

$$\|\varphi(x) - \varphi(z)\| \le L \max\{\|x - z\|, \epsilon\} \le 2L(\|x - z\| + \epsilon).$$

□

Our interest in the norms $\|\cdot\|_\epsilon$ is based on the following critical construction. We will state and prove this Lemma in a slightly more general from than we need, because we have in mind further applications. For our purposes here (7.5) can be simplified by omitting the exponential terms, with a corresponding simplification of the proof.

Lemma 7.4 *Let X and Y be Banach spaces and suppose $t \to f_t$ is a map from* $[0, \infty)$ *into $\mathcal{H}(X, Y)$ with the property that for some constant K we have:*

$$\|f_t\|_{e^{-2t}} \le K, \quad t \ge 0, \tag{7.4}$$

and

$$\|f_t - f_s\| \le K(|t - s| + e^{-2t} + e^{-2s}), \quad t, s \ge 0. \tag{7.5}$$

Define $F : X \to Y$

$$F(x) = \begin{cases} 0 & x = 0 \\ f_0(x), & \|x\| \le 1 \\ f_{\log \|x\|}(x) & \|x\| > 1. \end{cases}$$

Then F is coarsely continuous.

If further for every $t \ge 0$, $f_t \in \mathcal{HU}(X, Y)$ and the map $t \to f_t$ is continuous then F is uniformly continuous.

Proof First note that $\|F(x)\| \le K\|x\|$ so that F is continuous at the origin. Now suppose $\|x\| \ge \|z\| > 0$. For convenience we define $f_t = f_0$ if $t < 0$. Then if $\|x\| \ge 1$,

$$\begin{aligned}\|f_{\log\|x\|}(x) - f_{\log\|x\|}(z)\| &\le \|(f_{\log\|x\|})\|_{1/\|x\|^2} \max\{\|x - z\|, \|x\|^{-1}\} \\ &\le K \max\{\|x - z\|, \|x\|^{-1}\}.\end{aligned}$$

If $\|x\| \le 1$ we have

$$\|f_{\log\|x\|}(x) - f_{\log\|x\|}(z)\| \le K \max\{\|x - z\|, \|x\|\},$$

so that in general

$$\|f_{\log\|x\|}(x) - f_{\log\|x\|}(z)\| \le K \max\{\|x - z\|, \min(\|x\|, \|x\|^{-1})\}.$$

We also note that

$$\|f_{\log\|z\|}(z) - f_{\log\|x\|}(z)\| \le K\|z\| \left(\log \frac{\|x\|}{\|z\|} + \min(1, \|x\|^{-2}) + \min(1, \|z\|^{-2}) \right).$$

Since

$$\|z\| \log \frac{\|x\|}{\|z\|} \le \|x\| - \|z\| \le \|x - z\|$$

we can combine to get a basic estimate

$$\|F(x) - F(z)\| \le 2K\|x - z\| + 2K \min\{\|x\|, \|x\|^{-1}\} + K \min\{\|z\|, \|z\|^{-1}\}. \tag{7.6}$$

In particular,

$$\|F(x) - F(z)\| \le 3K(\|x - z\| + 1)$$

and F is coarsely continuous.

Now suppose each f_t is uniformly continuous and the map $t \to f_t$ is continuous. Suppose $\epsilon > 0$. We first pick $\delta_0 > 0$ so that

$$3K\delta_0 < \epsilon/3.$$

Then we can fix $a > 1$ so that $3K/a < \epsilon/3$. It follows from (7.6) that if $\|x\| \ge \|z\| \ge a$, $\|x - z\| < \delta_0$ we have

$$\|F(x) - F(z)\| \le 2K\delta_0 + 3Ka^{-1} < \epsilon.$$

Let $b = \log(a + 1)$. We pick an integer $N > b$ so large that if $0 \le \sigma, \tau \le b$ and $|\sigma - \tau| \le \frac{b}{N}$ we have $\|f_\sigma - f_\tau\| < \epsilon/(3e^b)$. This is possible since $t \to f_t$ is a continuous function.

Finally pick $\delta_1 > 0$ with $\delta_1 < \min(\frac{b}{N}, \delta_0)$ so that if $u, v \in B_X$ with $\|u - v\| \le \delta_1$ we have

$$\|f_{kb/N}(u) - f_{kb/N}(v)\| < \epsilon/(3e^b), \quad 0 \le k \le N.$$

This is possible since each f_t is uniformly continuous on B_X.

Now assume $\|x - z\| < \delta_1$ with $\|x\| \ge \|z\|$ and $\|z\| \le a$ so that $\|x\| \le a + \delta_1 < a + 1$. If $\|x\| \le 1$ then

$$\|F(x) - F(z)\| = \|f_0(x) - f_0(z)\| \le \epsilon/(3e^b) < \epsilon/3.$$

On the other hand if $\|x\| > 1$ we may find $0 \le k \le N$ so that

$$\left| \log \|x\| - \frac{kb}{N} \right|, \quad \left| \log \|z\| - \frac{kb}{N} \right| \le \frac{b}{N}.$$

Hence

$$\|F(x) - f_{kb/N}(x)\|, \|F(z) - f_{kb/N}(z)\| < \epsilon/3.$$

Also

$$\|f_{kb/N}(x) - f_{kb/N}(z)\| \le \|x\|\epsilon/(3e^b) \le \epsilon/3.$$

Hence

$$\|F(x) - F(z)\| < \epsilon.$$

Combining these estimates shows for all x, z with $\|x - z\| < \delta_1$ we have $\|F(x) - F(z)\| < \epsilon$, and hence F is uniformly continuous. □

Proposition 7.5 *Let $Q : Y \to X$ be a quotient map.*
(i) *In order that there exists a global coarse section $\psi : X \to Y$ it is necessary and sufficient that there exist a constant L, a sequence $\epsilon_n > 0$ with $\lim_{n\to\infty} \epsilon_n = 0$ and a sequence of sections $\varphi_n : \partial B_X \to Y$ so that φ_n is of CL-type (L, ϵ_n).*
(ii) *In order that there exists a global uniformly continuous section $\psi : X \to Y$ it is necessary and sufficient that there exist a constant L, a sequence $\epsilon_n > 0$ with $\lim_{n\to\infty} \epsilon_n = 0$ and a sequence of uniformly continuous sections $\varphi_n : \partial B_X \to Y$ and so that φ_n is of CL-type (L, ϵ_n).*

Proof The proof for (i) and (ii) is similar. Assume we are in case (i). We apply Lemma 7.4. We can suppose $\varphi_n(x) = \varphi_n(-x)$ for $x \in \partial B_X$ (by replacing φ_n by $\frac{1}{2}(\varphi(x) - \varphi(-x))$). Then it follows that $\|\varphi_n(x)\| \le L$ for all $x \in \partial B_X$. Using Proposition 7.3 we may suppose that we have $f_n \in \mathcal{H}(X, Y)$ so that $Q \circ f_n = Id_X$ and $\|f_n\|_{e^{-2n-2}} \le 6L$ for $n = 1, 2, \ldots$. Define for $t \ge 0$,

$$f_t(x) = (n + 1 - t) f_n(x) + (t - n) f_{n+1}(x) \quad n \le t < n + 1.$$

Then $Q \circ f_t = Id_X$ and $\|f_t\|_{e^{-2t}} \le 6L$. Then we can apply Lemma 7.4. Case (ii) follows similarly from Lemma 7.4. □

8 Examples based on global sections

Let X be a separable Banach space and consider a quotient map $Q : \ell_1 \to X$. As described in §2, an old result of Lindenstrauss and Rosenthal shows that, if X is not isomorphic to ℓ_1 then up to automorphism the quotient map $Q : \ell_1 \to X$ is unique [32; 33, p. 108].

For each $n \in \mathbb{N}$ we may renorm ℓ_1 to define a Banach space Y_n by the norm

$$\|y\|_n = \max(2^{-n}\|y\|, \|Qy\|).$$

Then $Q_n = Q : Y_n \to X$ is also a quotient mapping. We will then construct a space $\mathcal{Z}_1(X) = (\sum_{n=1}^{\infty} Y_n)_{\ell_1}$.

We will need the following lemma:

Lemma 8.1 (i) *For all n, Y_n is linearly isomorphic to ℓ_1 and so $\mathcal{Z}_1(X)$ is a Schur space.*
(ii) *For each n there is a subspace V_n of $X \oplus_\infty \ell_1$ so that V_n is isometric to Y_n.*
(iii) *If X is a $\mathcal{L}_1$-space then the space $\mathcal{Z}_1(X)$ is a $\mathcal{L}_1$-space.*
(iv) *If X is isomorphic to a locally complemented subspace of ℓ_1 then the space $\mathcal{Z}_1(X)$ is isomorphic to a locally complemented subspace of ℓ_1.*

Proof (i) is trivial.

(ii) Define $S_n : Y_n \to X \oplus_\infty \ell_1$ by $S_n y = (Qy, 2^{-n}y)$.

(iii) and (iv). The hypotheses of (iii) and (iv) imply that X can be isomorphically embedded into a space W as a locally complemented subspace, where $W = L_1$ in (iii) and $W = \ell_1$ in (iv). We can suppose the embedding is isometric and denote the embedding $j : X \to W$. Thus $Q^* j^* : W^* \to \ell_\infty$ is a map whose kernel is $X^\perp$.

Now Y_n can be identified with a subspace E_n of $\ell_1 \oplus_\infty W$ of all pairs $(2^{-n}\xi, Q\xi)$ for $\xi \in \ell_1$. It will suffice to show the existence of a projection $P_n : \ell_\infty \oplus_1 W^* \to E_n^\perp$ with a uniform bound on $\|P_n\|$.

Let $V = \ker Q$. There is a bounded projection $P : \ell_\infty \to V^\perp = X^*$. There is also a bounded projection $P' : W^* \to X^\perp \subset W^*$. Observe that $Q^* j^* P = P$ and $Q^* j^* P' = 0$. Then

$$E_n^\perp = \{(\xi^*, w^*) : \; Q^* j^* w^* + 2^{-n}\xi^* = 0\}.$$

We define a projection $P_n : \ell_\infty \oplus_1 W^* \to E_n^\perp$ by

$$P_n(\xi^*, w^*) = (Q^* j^* w^*, 2^{-n} P\xi^* + P'w^*).$$

Then P_n is a projection and it is clear that $\sup \|P_n\| < \infty$. □

Our first result partially answers a question raised by Ball [2]. Let H be a Hilbert space and let E be a subset of H and suppose Y is a closed linear subspace of L_p where $1 < p \le 2$. Then Ball [2] showed that any Lipschitz map $f_0 : E \to Y$ where $1 < p \le 2$ has a Lipschitz extension $f : H \to Y$. At the same time he asked whether a similar result holds when $Y = L_1$. Later in [39] it was shown that similar results hold if E is a subset of L_q where $2 \le q < \infty$. In the linear setting results of Maurey [35] show that if E is a closed subspace of $X = L_q$ for $q \ge 2$ and Y is a closed subspace of L_1 then any linear operator $T_0 : E \to Y$ has an extension $T : X \to Y$; more generally one only needs X of type 2 and Y to be of cotype 2 (see e.g. [34]).

We need the following fact from [22]:

Proposition 8.2 *There is no locally uniformly continuous section of the quotient map $Q : \ell_1 \to \ell_2$. More generally if X has nontrivial type, there is no locally uniformly continuous section of the quotient map $Q : \ell_1 \to X$.*

Proof This is a special case of Theorem 7.6 of [22]. We remark that there is a misstatement of Theorem 2.7 of [22] (an inaccurate quotation from [37]; n should be replaced by $n-2$ for $n \geq 3$) and hence in the quantitative estimate in Lemma 7.4. Of course this does not affect the conclusion.

It also follows from Lemma 7.4 of [22] that there can be no locally uniformly continuous section of the quotient $Q : \ell_1 \to X$ if X contains uniformly complemented ℓ_2^n's; a result of Figiel and Tomczak-Jaegermann [8] implies that this will happen when X has nontrivial type. $\square$

Theorem 8.3 *There is a closed subspace Z of L_1, a subset A of B_{ℓ_2} and a Lipschitz map $f : A \to Z$ which has no uniformly continuous extension $f' : B_{\ell_2} \to Z$. In particular B_Z is not an AUR and is not uniformly homeomorphic to B_{ℓ_2}.*

Proof We will choose the space $Z = \mathcal{Z}_1(\ell_2)$ which embeds into L_1 by Lemma 8.1. For each n let A_n be a maximal subset of B_{ℓ_2} with the property that $0 \in A_n$ and $\|x - x'\| \geq 2^{-n}$ for every $x, x' \in A_n$. For each $x \in A_n$ pick $u(x) \in \ell_1$ so that $\|u(x)\| \leq 2$ and $Qu(x) = x$. Let $f_n : A_n \to Y_n = (\ell_1, \|\cdot\|_n)$ be the map $f_n(x) = u(x)$. Then if $x, x' \in A_n$ we have

$$\begin{aligned}\|f_n(x) - f_n(x')\| &= \max(2^{-n}\|u(x) - u(x')\|, \|x - x'\|) \\ &\leq \max(4.2^{-n}, \|x - x'\|) \leq 4\|x - x'\|.\end{aligned}$$

We then consider $H = \ell_2(\ell_2)$ and let $S_n : \ell_2 \to H$ be the embedding into the nth coordinate space. Let $\tilde{A}_n = S_n(A_n)$ and $A = \cup_{n=1}^{\infty} \tilde{A}_n$. We then define $f : A \to \mathcal{Z}_1(\ell_2)$ by $f(x) = j_n \circ f_n \circ S_n^{-1}(x)$ where $j_n : Y_n \to \mathcal{Z}_1(\ell_2)$ is the canonical embedding. If $x \in \tilde{A}_n$ and $x' \in \tilde{A}_m$ with $m \neq n$ we have

$$\begin{aligned}\|f(x) - f(x')\| &= \|f_n(S_n^{-1}x)\| + \|f_m(S_m^{-1}x')\| \\ &\leq 2\|x\| + 2\|x'\| \leq 2\sqrt{2}\|x - x'\|.\end{aligned}$$

Thus f is a Lipschitz map.

Now suppose $\tilde{f} : B_H \to Z$ is a uniformly continuous extension, and let $\omega = \omega_{\tilde{f}}$. Let $\tilde{f}(x) = (\tilde{f}_n(x))_{n=1}^{\infty}$ and then $\tilde{g}_n : B_{\ell_2} \to Y_n$ is defined by $\tilde{g}_n(x) = \tilde{f}_n(S_n x)$. Let us pick n so large that $2^{-n} + \omega(2^{-n}) < \frac{1}{2}$. If $x \in B_{\ell_2}$ there exists $x' \in A_n$ with $\|x - x'\| < 2^{-n}$. Hence

$$\|Q_n\tilde{g}_n(x) - Q_n\tilde{g}(x')\| \leq \omega(2^{-n}).$$

Hence

$$\|Q_n\tilde{g}_n(x) - x\| \leq \|Q_n\tilde{g}_n(x) - Q_n\tilde{g}(x')\| + \|x - x'\| \leq \tfrac{1}{2}.$$

Now we can apply Lemma 6.5 to deduce that there is a uniformly continuous section $\psi : B_{\ell_2} \to Y_n$ of Q_n. But this implies that there is a locally uniformly continuous section of $Q : \ell_1 \to \ell_2$ and contradicts Proposition 8.2. $\square$

Remark The fact that L_1 has a subspace Z so that B_Z is not an AUR and hence not uniformly homeomorphic to B_{ℓ_2} answers a question in [22]. We recall that if X is super-reflexive then B_X is always an AUR and that for every subspace X of L_p where $1 < p < \infty$, B_X is uniformly homeomorphic to B_{ℓ_2}. See [3, (p. 28 and p. 202)] for details.

The fact that there is no extension result for Lipschitz maps from subspaces of Hilbert spaces into Z does not quite answer Ball's question about extensions into L_1, but does answer an alternate question that Ball might have posed! Thus it indicates that there is no general result for range spaces of cotype 2 or for subspaces of L_1.

Proposition 8.4 *Let X be a separable Banach space and define the quotient map $\tilde{Q} : \mathcal{Z}_1(X) \to X$ by $\tilde{Q}(y_n)_{n=1}^{\infty} = \sum_{n=1}^{\infty} Q_n y_n$. Then*

(i) *$\tilde{Q}$ admits a coarse section.*
(ii) *If $Q : \ell_1 \to X$ admits a locally uniformly continuous section then $\tilde{Q}$ admits a uniform section.*

Suppose X is isomorphic to X^2. Then $\mathcal{Z}_1(X)$ is coarsely homeomorphic to $\mathcal{Z}_1(X) \oplus X$; if $Q : \ell_1 \to X$ admits a locally uniformly continuous section then $\mathcal{Z}_1(X)$ is uniformly homeomorphic to $\mathcal{Z}_1(X) \oplus X$.

Proof Suppose C is any constant. For each $x \in B_X$ pick $\psi(x) \in \ell_1$ with $\|\psi(x)\| \leq C$ and $Q\psi(x) = x$. Suppose $\omega_\psi = \omega$. Let $\psi_n(x) = \psi(x)$ regarded as a map into $Y_n = (\ell_1, \|\cdot\|_n)$. Then $\omega_{\psi_n}(t) \leq \max(2^{-n}\omega(t), t) \leq t + 2.2^{-n}$. If we define $\varphi_n : B_X \to \mathcal{Z}_1(X)$ by $\varphi_n(x) = S_n\psi_n(x)$ where $S_n : Y_n \to \mathcal{Z}_1(X)$ is the canonical embedding, we have $\tilde{Q}\varphi_n(x) = x$ for $x \in B_X$. Then $\omega_{\varphi_n} \leq t + 2.2^{-n}$.

Now by Proposition 7.5 there is a coarse section of $\tilde{Q}$. If ψ can be chosen to be uniformly continuous then each φ_n is uniformly continuous and the same result gives a uniform section.

It now follows that $\ker \tilde{Q} \oplus X$ is coarsely homeomorphic to $\mathcal{Z}_1(X)$ and these spaces are uniformly homeomorphic in case (ii) which yields the last part of the Proposition. □

Recently Johnson, Maurey and Schechtman [17] showed that the class of $\mathcal{L}_1$-spaces is preserved under uniform or even coarse homeomorphisms. It is known that ℓ_1 is not uniformly homeomorphic to L_1 (an unpublished result of Enflo, see [3]).

Theorem 8.5 *There exist two separable $\mathcal{L}_1$-spaces which are uniformly homeomorphic but not linearly isomorphic.*

Proof Suppose X is a separable $\mathcal{L}_1$-space. We start with the quotient $Q : \ell_1 \to X$. Note that X has a basis and embeds into L_1. The unit ball B_{L_1} is uniformly homeomorphic to B_{ℓ_2}; hence B_X is approximable and has a good partition. Hence there is a locally uniformly continuous section of the quotient Q. By Proposition 8.4 $\mathcal{Z}_1(X)$ is uniformly homeomorphic to $\ker \tilde{Q} \oplus X$. If X is isomorphic to X^2 we can simply reduce this to the fact that $\mathcal{Z}_1(X)$ is uniformly homeomorphic to $\mathcal{Z}_1(X) \oplus X$. Notice that $\mathcal{Z}_1(X)$ is a separable $\mathcal{L}_1$-space by Lemma 8.1 (iii).

Take $X = L_1$. Then $\mathcal{Z}_1(L_1)$ is uniformly homeomorphic to $\mathcal{Z}_1(L_1) \oplus L_1$ but these space cannot be linearly isomorphic because the former is a Schur space (Lemma 8.1 (i)). □

We can improve and refine this example a little to get two $\mathcal{L}_1$-spaces which embed into ℓ_1.

Theorem 8.6 *There exist two separable $\mathcal{L}_1$-spaces which are both subspaces of ℓ_1, and are uniformly homeomorphic but not linearly isomorphic.*

Proof To get this example we use the argument of Theorem 8.5 but take $X = \kappa(L_1)$. Then by considering the quotient map $\ell_1 \oplus \ell_1 \to L_1 \oplus L_1$ and using the Lindenstrauss–Rosenthal theorem [32] we conclude that $\kappa(L_1) \approx \kappa(L_1)^2$. Hence $\mathcal{Z}_1(\kappa(L_1))$ is uniformly homeomorphic to $\mathcal{Z}_1(\kappa(L_1)) \oplus \kappa(L_1)$.

By Lemma 8.1 the space $\mathcal{Z}_1(\kappa(L_1))$ is isomorphic to locally complemented subspaces of ℓ_1 and hence can be identified with $\kappa(W)$ for some separable $\mathcal{L}_1$-space W. But then by Proposition 2.2 the space W is a Schur space. However $W \oplus L_1$ is not a Schur space and $\kappa(W \oplus L_1) \approx \mathcal{Z}_1(\kappa(L_1)) \oplus \kappa(L_1)$. By the result of Lindenstrauss [30] this implies that $\mathcal{Z}_1(\kappa(L_1))$ is not linearly isomorphic to $\mathcal{Z}_1(\kappa(L_1)) \oplus \kappa(L_1)$. □

Remark In particular there are two non-isomorphic subspaces of ℓ_1 which are uniformly homeomorphic. We will give a quite different proof of a similar statement for ℓ_p for $1 < p < \infty$ and c_0 in [27].

Before continuing we prove a technical lemma on the embedding of asymptotically uniformly smooth spaces into ℓ_1-sums.

Lemma 8.7 *Let X be asymptotically uniformly smooth, and suppose $Z = (\sum_{n=1}^{\infty} Y_n)_{\ell_1}$ is an ℓ_1-sum of Banach spaces Y_n. Suppose $f : X \to Z$ is a coarsely continuous map and $f(x) = (f_k(x))_{k=1}^{\infty}$. Then given $r > 0$ and $\epsilon > 0$, there exists $w \in X$, $s > r$ $N \in \mathbb{N}$ and a closed finite-codimensional subspace X_0 of X so that*

$$\sum_{k=N+1}^{\infty} \|f_k(w+x)\| \le \epsilon s, \qquad \|x\| \le s, \quad x \in X_0.$$

Proof This is a standard mid-point argument. Let $\overline{\rho} = \overline{\rho}_X$ be the modulus of asymptotic smoothness for X. Suppose $K = \lim_{t\to\infty} \omega(f;t)/t$. If $K = 0$ the conclusion is trivial. We thus assume $K > 0$. Given $r, \epsilon > 0$, we choose η so that $\overline{\rho}(\eta) \le 2^{-2}K^{-1}\epsilon\eta$. We then choose $\tau > 0$ with $\omega(f;\tau) < (K + 2^{-3}\epsilon\eta)\tau$ and $2\tau\eta > r$. We can pick $u, v \in X$ with $\|u - v\| > 4\tau$ and so that $\|f(u) - f(v)\| > K(1 - 2^{-3}\epsilon^2)\|u - v\|$.

We can choose N sufficiently large so that

$$\sum_{k=N+1}^{\infty} \|f_k(u)\|, \quad \sum_{k=N+1}^{\infty} \|f_k(v)\| \le 2^{-3}\epsilon\eta\|u - v\|.$$

Now for any $x \in X$ we have

$$\begin{aligned}&(K + 2^{-3}\epsilon\eta)(\|u - x\| + \|x - v\|)\\ &\quad \ge \|f(u - x)\| + \|f(x - v)\|\end{aligned}$$

$$\geq \|f(u-v)\| + 2\sum_{k=N+1}^{\infty}\|f_k(x)\| - 2^{-2}\epsilon\eta\|u-v\|$$
$$\geq (K - 3.2^{-3}\epsilon\eta)\|u-v\| + 2\sum_{k=N+1}^{\infty}\|f_k(x)\|.$$

Hence

$$\sum_{k=N+1}^{\infty}\|f_k(x)\| \leq \frac{1}{2}K(\|u-x\| + \|v-x\| - \|u-v\|) + 2^{-2}\epsilon\eta\|u-v\|.$$

Let $s = \frac{1}{2}\eta\|u-v\|$ so that $s > r$. If $\|x\| \leq s$, then $2\|x\|/\|u-v\| \leq \eta$. Hence we may pick a closed subspace X_0 of finite-codimension so that if $x \in X_0$ and $\|x\| \leq s$ then $\|\frac{1}{2}(u-v) + x\| \leq \frac{1}{2}\|u-v\|(1 + (2K)^{-1}\epsilon\eta)$. Then for $x \in sB_{X_0}$,

$$\|u-(w+x)\| + \|v-(w+x)\| - \|u-v\| \leq (2K)^{-1}\epsilon\eta\|u-v\|.$$

Hence

$$\sum_{k=N+1}^{\infty}\|f_k(w+x)\| \leq \frac{1}{2}\epsilon\eta\|u-v\| = \epsilon s.$$

□

For the next result, we recall a concept introduced in [23]. For $r \geq 1$, we shall denote by $\mathcal{P}_r(\mathbb{N})$ the collection of r-subsets of $\mathbb{N}$. This is a graph if we say that distinct vertices $\{m_1, \ldots, m_r\}$ and $\{n_1, \ldots, n_r\}$ are adjacent if they interlace i.e. either

$$m_1 \leq n_1 \leq m_2 \leq \cdots \leq m_r \leq n_r$$

or

$$n_1 \leq m_1 \leq n_2 \leq \cdots \leq n_r \leq m_r.$$

Then $\mathcal{P}_r(\mathbb{N})$ becomes a metric space under the path metric in the graph. We say that a Banach space X has property $\mathcal{Q}$ if there is a constant $\mathcal{Q}_X > 0$ so that if $f : \mathcal{P}_r(\mathbb{N}) \to X$ is Lipschitz with constant L, then given $\lambda > 1$ there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that diam $f(\mathcal{P}_r(\mathbb{M})) \leq \lambda\mathcal{Q}_X^{-1}L$. As shown in [23], if B_X uniformly embeds into a reflexive space then X has property $\mathcal{Q}$.

Theorem 8.8 *Let X be a separable asymptotically uniformly smooth Banach space and suppose $Z = (\sum_{n=1}^{\infty} Y_n)_{\ell_1}$ is an ℓ_1-sum of Banach spaces Y_n with the property that each B_{Y_n} can each be uniformly embedded into a reflexive Banach space. If there is a coarse Lipschitz embedding $f : X \to Z$ which is also uniformly continuous then X is reflexive.*

Proof We first note that the unit ball of $V_N = Y_1 \oplus_1 \cdots \oplus_1 Y_N$ for each fixed N uniformly embeds into a reflexive space. Hence, using [23], V_N has property $\mathcal{Q}$ with constant $\mathcal{Q}_N > 0$. Let P_N denote the canonical projection of Z onto V_N.

We start by fixing some $x^{**} \in X^{**} \setminus X$, with $\|x^{**}\| < 1$.

Let us assume that f obeys an estimate

$$\|f(u) - f(v)\| \geq \|u - v\|, \quad \|u - v\| \geq 1.$$

For any $\epsilon > 0$ we may use Lemma 8.7 to produce $w \in X$, $s > 1/\epsilon$, a finite-codimensional subspace X_0 of X and $N \in \mathbb{N}$ so that

$$\sum_{k=N+1}^{\infty} \|f_k(w + x)\| \leq \epsilon s \quad x \in X_0, \|x\| \leq s.$$

We also let $\omega(t) = \omega(f; t)$ be its modulus of continuity.

Since X cannot contain a copy of ℓ_1 we can find a weakly Cauchy sequence $(x_n)_{n=1}^{\infty}$ in B_X which converges weak* to x^{**} and such that $x_m - x_n \in X_0$ for every $m \neq n$. By passing to a subsequence we can suppose $(x_n)_{n=1}^{\infty}$ is spreading i.e. that for every finite sequence of scalars $(a_j)_{j=1}^{k}$ the limit

$$\lim_{(n_1,\ldots,n_k)\to\infty} \left\| \sum_{j=1}^{k} a_j x_{n_j} \right\| = \left\| \sum_{j=1}^{k} a_j e_j \right\|$$

exists and defines a spreading model. Since X is asymptotically uniformly smooth we have an estimate

$$\left\| \sum_{j=1}^{2r} (-1)^j e_j \right\| \leq C r^{\theta}, \quad 1 \leq r < \infty$$

for some constant C and $0 < \theta < 1$.

Let us define a map $h_r : \mathcal{P}_r(\mathbb{N}) \to Z$ by

$$h_r(n_1, n_2, \ldots, n_r) = f(w + \frac{s}{r}(x_{n_1} + x_{n_2} + \cdots + x_{n_r})).$$

For a fixed r we may find q so that if $q < m_1 \leq n_1 \leq m_2 \leq n_2 \leq \cdots \leq m_r \leq n_r$ then

$$\left\| \sum_{j=1}^{2r} (x_{n_j} - x_{m_j}) \right\| \leq 2C r^{\theta}.$$

Thus there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that if $(m_1, \ldots, m_r), (n_1, \ldots, n_r) \in \mathcal{P}_r(\mathbb{M})$ we have

$$\|P_N h_r(n_1, \ldots, n_r) - P_N h_r(m_1, \ldots, m_r)\| \le 2\mathcal{Q}_N^{-1}\omega(2Cr^{\theta-1}s).$$

Hence

$$\|h_r(n_1, \ldots, n_r) - h_r(m_1, \ldots, m_r)\| \le 2\mathcal{Q}_N^{-1}\omega(2Cr^{\theta-1}s) + 2\epsilon s.$$

Since $\epsilon s > 1$, the properties of f gives us that

$$\frac{s}{r}\|(x_{n_1} + \cdots + x_{n_r}) - (x_{m_1} + \cdots + x_{m_r})\| \le 2\mathcal{Q}_N^{-1}\omega(2Cr^{\theta-1}s) + 2\epsilon s$$

or

$$\left\| \frac{1}{r}\sum_{j=1}^{r} x_{n_j} - \frac{1}{r}\sum_{j=1}^{r} x_{m_j} \right\| \le 2\mathcal{Q}_N^{-1}\omega(2Cr^{\theta-1}s)/s + 2\epsilon.$$

Taking limits as $n_1, \ldots, n_r \to \infty$ but $m_1, \ldots, m_r$ are fixed, gives the estimate

$$d(x^{**}, X) \le 2\mathcal{Q}_N^{-1}\omega(2Cr^{\theta-1}s)/s + 2\epsilon.$$

We can now let $r \to \infty$ to get (this is where we use $\lim_{t\to 0}\omega(t) = 0$)

$$d(x^{**}, X) \le 2\epsilon.$$

Thus we have $x^{**} \in X$. □

The next theorem answers a question of Johnson, Lindenstrauss and Schechtman [16].

Theorem 8.9 *There exist separable Banach spaces X and Y which are coarsely homeomorphic but not uniformly homeomorphic.*

Proof Consider the space $\mathcal{Z}_1(c_0)$ which is coarsely homeomorphic to $\mathcal{Z}_1(c_0) \oplus c_0$ by Proposition 8.4. However, $\mathcal{Z}_1(c_0)$ and $\mathcal{Z}_1(c_0) \oplus c_0$ cannot be uniformly homeomorphic since c_0 is asymptotically uniformly smooth and so does not coarse Lipschitz embed into $\mathcal{Z}_1(c_0)$ via a uniformly continuous map by Theorem 8.8. □

Remark Of course any non-reflexive asymptotically uniformly smooth space can be used in place of c_0.

The above theorem also implies that the existence of a coarse section for a quotient map $Q : Y \to X$ does not in general imply the existence of a uniform section. For a special type of example this can be proved in wider generality:

Theorem 8.10 *Suppose X is asymptotically uniformly smooth and has non-trivial type. Assume further X has a shrinking (FDD). Then there is no uniformly continuous section of $\tilde{Q} : \mathcal{Z}_1(X) \to X$.*

Proof We use Lemma 8.7. Since X has shrinking (FDD) it follows that there is a constant C so that every closed subspace E of finite codimension contains a further closed subspace E_0 of finite codimension which is the range of a projection of norm at most C.

Taking $0 < \epsilon < 1/C$ we can find $w \in X,\ s > 0,\ N \in \mathbb{N}$ and a subspace of finite codimension X_0 so that

$$\sum_{k=N+1}^{\infty} \|f_k(w+x)\| \le \epsilon s, \qquad x \in X_0, \quad \|x\| \le s.$$

In fact, we may choose X_0 to be the range of a projection R of norm at most C. Let V_N be the subspace of $\mathcal{Z}_1(X)$ consisting of all sequences $(\xi_n)_{n=1}^{\infty}$ so that $\xi_k = 0$ for $k > N$. Let P be the canonical projection of $\mathcal{Z}_1(X)$ onto V_N. Then

$$\|f(w+x) - Pf(w+x)\| \le \epsilon s, \qquad x \in X_0, \quad \|x\| = s.$$

Thus

$$\|\tilde{Q}Pf(w+x) - w - x\| \le \epsilon s, \qquad x \in X_0, \quad \|x\| = s.$$

If we define

$$\psi(x) = \frac{1}{s}(QPf(w+sx) - w)$$

then $\psi : \partial B_{X_0} \to V_N$ is uniformly continuous and

$$\|R\tilde{Q}\psi(x) - x\| \le C\epsilon < 1.$$

Renorm V_N with the equivalent norm $\|v\|_1 = \max(\|v\|, \|R\tilde{Q}v\|)$. Then $R\tilde{Q} : (V_N, \|\cdot\|_1) \to X_0$ is a quotient map. By Proposition 6.5 there is a uniformly continuous section $g : B_{X_0} \to V_N$. Since V_N is isomorphic to ℓ_1 and X_0 has non-trivial type this is a contradiction to Proposition 8.2. □

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