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The coarse Lipschitz geometry of $\ell_p \oplus \ell_q$

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Abstract We show that for $1 < p < \infty$ with $p \neq 2$ the space $L_p(0, 1)$ is not uniformly homeomorphic to $\ell_p \oplus \ell_2$. We also show that if $1 < p < 2 < q < \infty$ the space $\ell_p \oplus \ell_q$ has unique uniform structure, answering a question of Johnson, Lindenstrauss and Schechtman (Geom. Funct. Anal. 6:430–470, 1996).

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1 Introduction

In this note we study some problems raised ten years ago in [11]. In [11] it was shown that ℓ_p has unique uniform structure for $1 < p < \infty$ in the sense that if X is a Banach space uniformly homeomorphic to ℓ_p then X is linearly isomorphic to ℓ_p. It is not known if the same result can be proved for the function spaces $L_p = L_p(0, 1)$ for $1 < p < \infty$ with $p \neq 2$ (see [11, p. 465]) It is known, however ([2, Theorem 10.5], [8]) that any Banach space X uniformly homeomorphic to L_p $(1 < p < \infty,\ p \neq 2)$ must be a $\mathcal{L}_p$−space (i.e. a complemented subspace of L_p not isomorphic to a Hilbert space). The simplest candidate space is $\ell_p \oplus \ell_2$, and the question whether $\ell_p \oplus \ell_2$ has unique uniform structure was raised in [11, p. 465].

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Here we show that $\ell_p \oplus \ell_2$ is not uniformly homeomorphic to L_p; this question was raised originally by Bill Johnson. In fact we show that if $1 \leq p < 2$ it is not uniformly homeomorphic to any Banach space containing a copy of ℓ_r where $p < r < 2$. We also show that for $1 \leq p < \infty$ and $p \neq 2$, $\ell_p \oplus \ell_2$ is not uniformly homeomorphic to any Banach space containing $\left(\sum \ell_2\right)_{\ell_p}$.

A related problem is the question of the uniqueness of uniform structure of $\ell_p \oplus \ell_q$ when $1 < p < q < \infty$. In the case $p < q < 2$ and $2 < p < q$ this is established in [11]. We show uniqueness for the case $p < 2 < q$, answering Problem 1(e) of [11].

Our results depend on the use of two basic techniques. The first is the midpoint principle which is a classical technique in nonlinear theory [2, p. 229]. The second technique is new and depends on asymptotic smoothness ideas (see Sects. 4, 6). Roughly speaking midpoint principles can be applied to maps from ℓ_p to ℓ_q when $q < p$ (corresponding to the case when all linear operators are compact) and asymptotic smoothness techniques apply when $q > p$ (corresponding to the case when all linear operators are strictly singular).

2 Coarse Lipschitz maps and embeddings

Let M_1 and M_2 be unbounded metric spaces and suppose $f : M_1 \to M_2$ is any map. We define

$$\omega_f(t) = \sup\{d(f(x), f(y)) : d(x, y) \leq t\}, \quad t > 0.$$

Then f is said to be *uniformly continuous* if $\lim_{t\to 0} \omega_f(t) = 0$. We say that f is *coarsely continuous* if we have $\omega_f(t) < \infty$ for some $t > 0$.

We will also define

$$\mathrm{Lip}_s(f) = \sup_{t\geq s} \frac{\omega_f(t)}{t}, \quad s > 0$$

and then

$$\mathrm{Lip}(f) = \sup_{s>0} \mathrm{Lip}_s(f),\ \ \mathrm{Lip}_\infty(f) = \inf_{s>0} \mathrm{Lip}_s(f).$$

f is a Lipschitz map if $\mathrm{Lip}(f) < \infty$ (and then f is uniformly continuous) and is a coarse Lipschitz map if $\mathrm{Lip}_\infty(f) < \infty$ (and then f is coarsely continuous).

If f is bijective then f is a *uniform homeomorphism*, (respectively, *coarse homeomorphism, Lipschitz homeomorphism, coarse Lipschitz homeomorphism*) if f, f^{-1} are uniformly continuous, (respectively, coarsely continuous, Lipschitz, coarse Lipschitz). Coarse Lipschitz homeomorphisms are essentially the same as homeomorphisms between nets as discussed in [2,11].

If X and Y are Banach spaces, then any coarsely continuous map is automatically coarse Lipschitz since ω_f is subadditive. This leads us to the following well-known principle (known to specialists as the "Lipschitz for large distances" principle):

Proposition 2.1 *Suppose X and Y are uniformly homeomorphic or more generally coarsely homeomorphic Banach spaces. Then X and Y are coarse Lipschitz homeomorphic and there is a bijection $f : X \to Y$ such that* $0 < \mathrm{Lip}_\infty(f), \mathrm{Lip}_\infty(f^{-1}) < \infty$.

Proof We only need to prove the last statement in view of the remarks above. This follows from the following lemma. □

Lemma 2.2 *Let X, Y and Z be Banach spaces and suppose $f : X \to Y$ and $g : Y \to Z$ are coarse Lipschitz maps. Then $g \circ f$ is coarse Lipschitz and* $\mathrm{Lip}_\infty(g \circ f) \leq \mathrm{Lip}_\infty(f)\mathrm{Lip}_\infty(g)$.

Proof Let $\alpha = \mathrm{Lip}_\infty(f)$ and $\beta = \mathrm{Lip}_\infty(g)$. Thus if $\alpha' > \alpha, \beta' > \beta$ there are constants a, b such that

$$\|f(x_1) - f(x_2)\| \leq \alpha'\|x_1 - x_2\| + a, \quad x_1, x_2 \in X$$

and

$$\|g(y_1) - g(y_2)\| \leq \beta'\|y_1 - y_2\| + b, \quad y_1, y_2 \in Y.$$

Thus

$$\|g \circ f(x_1) - g \circ f(x_2)\| \leq \alpha'\beta'\|x_1 - x_2\| + \beta' a + b, \quad x_1, x_2 \in X.$$

This proves that $\mathrm{Lip}_\infty(g \circ f) \leq \alpha\beta$. □

If X and Y are Banach spaces and $f : X \to Y$ is a coarse Lipschitz homeomorphism onto $f(X)$ then we say that f is a coarse Lipschitz embedding. The basic idea of the arguments in this paper is to investigate spaces which can be coarse Lipschitz embedded into a direct sum $\ell_p \oplus \ell_q$ where $1 \leq p < q < \infty$. This will enable us to prove results on uniform homeomorphisms (or more generally coarse homeomorphisms), because if X is uniformly homeomorphic to $\ell_p \oplus \ell_q$ then any closed subspace Y of X coarse Lipschitz embeds into $\ell_p \oplus \ell_q$.

Let us also note here one very well-known result, which we restate in this language:

Proposition 2.3 *If there is a coarse Lipschitz embedding of a Banach space X into a Banach space Y then X is crudely finitely representable in Y.*

See Corollary 10.2 of [2] (these ideas go back to [8,16]).

3 The midpoint technique

Given a metric space X, two points $x, y \in X$, and $\delta > 0$, the approximate metric midpoint between x and y with error δ is the set:

$$\mathrm{Mid}(x, y, \delta) = \left\{ z \in X : \ \max\{d(x, z), d(y, z)\} \leq (1 + \delta)\frac{d(x, y)}{2} \right\}.$$

The use of metric midpoints in the study of nonlinear geometry was first introduced by Enflo in an unpublished paper where he showed that L_1 and ℓ_1 are not uniformly homeomorphic. It has since been used elsewhere, for example to complete the study of the uniform structure of ℓ_p for $1 < p < \infty$ (see e.g. [3,6,11]).

Proposition 3.1 *Let X be a Banach space and suppose M is a metric space. Let $f : X \to M$ be a coarse Lipschitz map. If $\mathrm{Lip}_\infty(f) > 0$ then for any $t, \epsilon > 0$ and any $0 < \delta < 1$ there exist $x, y \in X$ with $\|x - y\| > t$ and*

$$f(\mathrm{Mid}(x, y, \delta)) \subset \mathrm{Mid}(f(x), f(y), (1 + \epsilon)\delta).$$

Proof This is well-known (see e.g. [2, Lemma 10.11]) but we include the proof for completeness. Notice that since X is a Banach space, $\mathrm{Lip}_\infty(f) = \inf_{s>0} R_s(f)$ where

$$R_s(f) = \sup\left\{\frac{d(f(x), f(y))}{d(x, y)} : d(x, y) \geq s\right\}.$$

Suppose t, δ, ϵ are given. For any $\nu > 0$ we may find $s > t$ so that $R_s(f) < (1 + \nu)\mathrm{Lip}_\infty(f)$. Suppose x, y are chosen so that $\|x - y\| \geq 2s(1 - \delta)^{-1}$ and $d(f(x), f(y)) > (1-\nu)\mathrm{Lip}_\infty(f)\|x-y\|$. Let $u \in \mathrm{Mid}(x, y, \delta)$. Then $\|x-u\|, \|y-u\| > s$ and so

$$\max(d(f(x), f(u)), d(f(y), f(u))) \leq \frac{1}{2}(1 + \nu)(1 + \delta)\mathrm{Lip}_\infty(f)\|x - y\|.$$

But this implies

$$\max(d(f(x), f(u)), d(f(y), f(u))) \leq \frac{1}{2}\frac{1 + \nu}{1 - \nu}(1 + \delta)d(f(x), f(y)).$$

By appropriate choice of $\nu > 0$ we obtain the proposition. □

The next lemma is also very well-known.

Lemma 3.2 *Suppose $1 \leq p < \infty$ and let $(X_j)_{j=1}^\infty$ be a sequence of Banach spaces. Suppose $x, y \in \left(\sum_{j=1}^\infty X_j\right)_{\ell_p}$. Let $u = \frac{1}{2}(x + y)$ and $v = \frac{1}{2}(x - y)$. Then for $0 < \delta < 1$, there is a closed subspace E of the form $E = \{w = (w_j)_{j=1}^\infty : w_j = 0,\ 1 \leq j \leq N\}$ so that*

$$u + \delta^{1/p}\|v\|B_E \subset \mathrm{Mid}(x, y, \delta).$$

Proof For $p = 1$ this is trivial; we suppose $p > 1$. Write $v = (v_j)_{j=1}^\infty \in \left(\sum_{j=1}^\infty X_j\right)_{\ell_p}$. Suppose $0 < \nu < (((1+\delta)^p - 1)^{1/p} - \delta^{1/p})\|v\|$. Pick N so that $\sum_{j>N} \|v_j\|^p < \nu^p$. Let $E = [X_j]_{j>N}$. If $z \in E$ and $\|z\| < ((1 + \delta)^p - 1)^{1/p}\|v\| - \nu$ then $u + z \in \mathrm{Mid}(x, y, \delta)$. □

Lemma 3.3 *Suppose* $1 \le p < \infty$ *and that* $x, y \in \ell_p$. *Let* $u = \frac{1}{2}(x+y)$ *and* $v = \frac{1}{2}(x-y)$. *Then for* $0 < \delta < 1$, *there is a compact set* K *so that*

$$\mathrm{Mid}(x, y, \delta) \subset K + 2\delta^{1/p}\|v\| B_{\ell_p}.$$

Proof Suppose $\nu > 0$. Let $v = (v_j)_{j=1}^{\infty} \in \ell_p$. Pick N so that $\sum_{j>N} |v_j|^p < \nu^p$. Let $E = [e_j]_{j>N}$. If $u + z \in \mathrm{Mid}(x, y, \delta)$ write $z = z' + z''$ where $z' \in [e_j]_{j \le N}$ and $z'' \in [e_j]_{j>N}$. Then $\|z'\| \le (1+\delta)\|v\|$ so that $u + z' \in K := u + (1+\delta)\|v\| B_{[e_j]_{j=1}^N}$. Now

$$\frac{1}{2}(\|v+z\|^p + \|v-z\|^p) \ge \|v\|^p - \nu^p + \|z''\|^p$$

so that

$$\|z''\|^p \le ((1+\delta)^p - 1)\|v\|^p + \nu^p.$$

Observe that $(1+\delta)^p - 1 < 2^p\delta$ for $0 < \delta < 1$. For an appropriate choice of ν we obtain the conclusion. □

It is clear that Lemma 3.3 has a simple generalization to finite direct sums:

Lemma 3.4 *Suppose* $1 \le p_1 < p_2, \ldots < p_n < \infty$ *and that* $x, y \in X = (\ell_{p_1} \oplus \ell_{p_2} \oplus \cdots \oplus \ell_{p_n})_{\ell_{p_n}}$. *Let* $u = \frac{1}{2}(x+y)$ *and* $v = \frac{1}{2}(x-y)$. *Then for* $0 < \delta < 1$, *there is a compact set* K *so that*

$$\mathrm{Mid}(x, y, \delta) \subset K + 2\delta^{1/p_n}\|v\| B_X.$$

Proposition 3.5 *Suppose* $(X_j)_{j=1}^{\infty}$ *is a sequence of Banach spaces and* $1 \le p < r < \infty$. *Suppose* $f : \left(\sum_{j=1}^{\infty} X_j\right)_{\ell_r} \to \ell_p$ *is any coarse Lipschitz map. Then for any* $t > 0$, $\delta > 0$ *there exist* $x \in \left(\sum_{j=1}^{\infty} X_j\right)_{\ell_r}$, $\tau > t$ *and a subspace* $E \subset \left(\sum_{j=1}^{\infty} X_j\right)_{\ell_r}$ *of the form* $E = \{w = (w_j)_{j=1}^{\infty} \in \left(\sum_{j=1}^{\infty} X_j\right)_{\ell_r} : w_1 = \ldots = w_N = 0\}$ *for some* N, *so that for some compact set* $K \subset \ell_p$ *we have* $f(x + \tau B_E) \subset K + \delta\tau B_{\ell_p}$.

Proof If $\mathrm{Lip}_{\infty}(f) = 0$ the conclusion is trivial: for any t we can find $\tau > t$ so that $\mathrm{Lip}_{\tau}(f) < \delta$ and then we may take $x = 0$ and $K = \{f(0)\}$. We therefore assume that $0 < \mathrm{Lip}_s(f) = C < \infty$ for some s. Let $1 > \nu > 0$ be chosen so that $4C\nu^{1/p - 1/r} < \delta$. We first use Proposition 3.1 to find $u, v \in \left(\sum_{j=1}^{\infty} X_j\right)_{\ell_r}$ so that $\|u - v\| > \max(s, 2t\nu^{-1/r})$ and $f(\mathrm{Mid}(u, v, \nu)) \subset \mathrm{Mid}(f(u), f(v), 2\nu)$. Let $x = \frac{1}{2}(u+v)$ and let $\tau = \nu^{1/r}\|\frac{1}{2}(u-v)\|$. Then by Lemma 3.2, for some closed subspace $E = \{w = (w_j)_{j=1}^{\infty} \in: \left(\sum_{j=1}^{\infty} X_j\right)_{\ell_r} : w_1 = \ldots = w_N = 0\}$ in $\left(\sum_{j=1}^{\infty} X_j\right)_{\ell_r}$

we have $x + \tau B_E \subset \mathrm{Mid}(u, v, \nu)$. However, by Lemma 3.3, $\mathrm{Mid}(f(u), f(v), 2\nu) \subset K + 2^{1/p}\nu^{1/p}\|f(u) - f(v)\|B_{\ell_p}$ for a suitable compact set $K \subset \ell_p$. Now

$$2^{1/p}\nu^{1/p}\|f(u) - f(v)\| \leq 4C\nu^{1/p-1/r}\tau < \delta\tau.$$

□

Notice that the conclusion of Proposition 3.5 holds for any equivalent norm on $\left(\sum_{j=1}^{\infty} X_j\right)_{\ell_r}$. We can also replace the range by a finite direct sum of ℓ_p's using Lemma 3.4:

Proposition 3.6 *Suppose* $1 \leq p_1, p_2, \ldots, p_n < r < \infty$ *and* $(X_j)_{j=1}^{\infty}$ *is a sequence of Banach spaces. Suppose* $f : \left(\sum_{j=1}^{\infty} X_j\right)_{\ell_r} \to Z := \left(\sum_{j=1}^{n} \ell_{p_j}\right)_{\ell_{p_n}}$ *is any coarse Lipschitz map. Then for any* $t > 0$, $\delta > 0$ *there exist* $x \in \left(\sum_{j=1}^{\infty} X_j\right)_{\ell_r}$, $\tau > t$ *and a subspace* $E \subset \left(\sum_{j=1}^{\infty} X_j\right)_{\ell_r}$ *of the form* $E = \{w = (w_j)_{j=1}^{\infty} \in \left(\sum_{j=1}^{\infty} X_j\right)_{\ell_r} : w_1 = \ldots = w_N = 0\}$ *for some* N, *so that for some compact set* $K \subset Z$ *we have* $f(x + \tau B_E) \subset K + \delta\tau B_Z$.

4 Asymptotic smoothness arguments

Let $\mathbb{M}$ be any infinite subset of $\mathbb{N}$ and suppose $m \in \mathbb{N}$. Let $G_k(\mathbb{M})$ be the set of all k-subsets of $\mathbb{M}$. We now define a distance on $G_k(\mathbb{M})$ by setting

$$d((n_1, \ldots, n_k), (m_1, \ldots, m_k)) = |\{j : n_j \neq m_j\}|$$

where $n_1 < n_2 < \cdots < n_k$ and $m_1 < m_2 < \cdots < m_k$. We note that the diameter of this metric space, $\mathrm{diam} G_k(\mathbb{M})$, is k. The aim of this section is to show a quantitative version of the fact that $G_k(\mathbb{M})$ cannot be well Lipschitz embedded into ℓ_p when $1 < p < \infty$.

We remark that it is also possible for our purposes to use the (different) distance

$$d'(A, B) = \frac{1}{2}|A \Delta B|$$

where $A = \{n_1, \ldots, n_k\}$ and $B = \{m_1, \ldots, m_k\}$ in this context, but this form of the distance does not allow the generalizations we consider in the last section.

Lemma 4.1 *Suppose* X *is a Banach space and* $f : G_k(\mathbb{M}) \to X$ *is any map with the property that for some compact set* K *and some* $\delta > 0$ *we have* $f(G_k(\mathbb{M})) \subset K + \delta B_X$. *Then for any* $\epsilon > 0$ *there is an infinite subset* $\mathbb{M}'$ *of* $\mathbb{M}$ *so that* $\mathrm{diam}\, f(G_k(\mathbb{M}')) \leq 2\delta + \epsilon$.

Proof We can write $f = g + h$ where $g : G_k(\mathbb{M}) \to K$ and $h : G_k(\mathbb{M}) \to \delta B_X$. By standard Ramsey theory we can then find $\mathbb{M}'$ so that $\mathrm{diam}\ g(G_k(\mathbb{M}')) < \epsilon$ and the lemma follows. □

Theorem 4.2 *Suppose* $1 < p < \infty$. *Let* X *be a reflexive Banach space with the property that if* $x \in X$ *and* $(x_n)_{n=1}^{\infty}$ *is a weakly null sequence in* X *we have*

$$\limsup \|x + x_n\|^p \leq \|x\|^p + \limsup \|x_n\|^p. \tag{4.1}$$

Then if $\mathbb{M}$ *is an infinite subset of* $\mathbb{N}$, $\epsilon > 0$ *and* $f : G_k(\mathbb{M}) \to X$ *is a bounded map, there exists an infinite subset* $\mathbb{M}'$ *of* $\mathbb{M}$ *so that*

$$\operatorname{diam} f(G_k(\mathbb{M}')) < 2\mathrm{Lip}(f)k^{1/p} + \epsilon.$$

Proof We show by induction on k that given any bounded $f : G_k(\mathbb{M}) \to X$ and $\epsilon > 0$, there exist an infinite $\mathbb{M}' \subset \mathbb{M}$ and $u \in X$ so that

$$\|f(n_1, \ldots, n_k) - u\| < \mathrm{Lip}(f)k^{1/p} + \epsilon/2, \quad \{n_1, \ldots, n_k\} \in G_k(\mathbb{M}').$$

In the case $k = 1$ we pass to a subsequence $\mathbb{M}_0$ of $\mathbb{M}$ so that

$$\lim_{n \in \mathbb{M}_0} f(n) = u$$

exists weakly. Then

$$\lim_{n \in \mathbb{M}_0} \|f(n) - u\| \leq \lim_{n \in \mathbb{M}_0} \lim_{m \in \mathbb{M}_0} \|f(n) - f(m)\| \leq \mathrm{Lip}(f).$$

We obtain the inductive statement by discarding finitely many points from $\mathbb{M}_0$.

Now assume the theorem is proved for $k - 1$ and $f : G_k(\mathbb{M}) \to X$ is bounded. We may then find an infinite subset $\mathbb{M}_0$ so that

$$\lim_{n_k \in \mathbb{M}_0} f(n_1, \ldots, n_k) = \tilde{f}(n_1, \ldots, n_{k-1})$$

exists weakly for $(n_1, \ldots, n_{k-1}) \in \mathbb{G}_{k-1}(\mathbb{M})$. The map $\tilde{f} : G_{k-1}(\mathbb{M}) \to X$ satisfies $\mathrm{Lip}(\tilde{f}) \leq \mathrm{Lip}(f)$. By the inductive hypothesis we can find an infinite $\mathbb{M}_1 \subset \mathbb{M}_0$ and $u \in X$ so that for all $(n_1, \ldots, n_{k-1}) \in G_{k-1}(\mathbb{M}_1)$,

$$\|\tilde{f}(n_1, \ldots, n_{k-1}) - u\| < \mathrm{Lip}(\tilde{f})(k-1)^{1/p} + \epsilon/4.$$

Thus, writing $\overline{n} = (n_1, \ldots, n_{k-1})$,

$$\limsup_{n_k \in \mathbb{M}_1} \|f(\overline{n}, n_k) - u\|^p \leq (\mathrm{Lip}(\tilde{f})(k-1)^{1/p} + \epsilon/4)^p + \limsup_{n_k \in \mathbb{M}_1} \|f(\overline{n}, n_k) - \tilde{f}(\overline{n})\|^p.$$

Now

$$\limsup_{n_k\in\mathbb{M}_1} \| f(\overline{n}, n_k) - \tilde{f}(\overline{n})\|^p \le \limsup_{n_k\in\mathbb{M}_1} \limsup_{n'_k\in\mathbb{M}_1} \| f(\overline{n}, n_k) - f(\overline{n}, n'_k)\|^p \le (\mathrm{Lip}(f))^p.$$

Thus it follows that

$$\limsup_{n_k\in\mathbb{M}_1} \| f(n_1, \dots, n_k) - u\| \le \mathrm{Lip}(f)k^{1/p} + \epsilon/4, \quad (n_1, \dots, n_k) \in G_k(\mathbb{M}_1).$$

Now we pass to a further infinite subset $\mathbb{M}'$ so that

$$\big|\| f(\overline{n}) - u\| - \| f(\overline{n}') - u\|\big| < \epsilon/4, \quad \overline{n}, \overline{n}' \in G_k(\mathbb{M}')$$

and hence

$$\| f(n_1, \dots, n_k) - u\| < \mathrm{Lip}(f)k^{1/p} + \epsilon/2, \quad (n_1, \dots, n_k) \in G_k(\mathbb{M}').$$

This completes the inductive argument and the theorem follows immediately. □

The assumption that X is reflexive in Theorem 4.2 is important. Note that the non-reflexive space c_0 satisfies the condition (4.1) for any p (and indeed for $p = \infty$) but every separable metric space can be Lipschitz embedded into c_0 by a result of Aharoni [1].

5 The main results

Theorem 5.1 *Suppose* $1 \le p_1 < p_2 < \dots < p_n < \infty$. *If* $r \notin \{p_1, \dots, p_n\}$ $(1 \le r < \infty)$ *then* ℓ_r *does not coarse Lipschitz embed into* $\ell_{p_1} \oplus \dots \oplus \ell_{p_n}$.

Proof Let $f : \ell_r \to \ell_{p_1} \oplus \dots \oplus \ell_{p_n}$ be a coarse embedding. It follows from Proposition 2.3 that $p_1 < r \le 2$, although, strictly speaking our proof works without this fact. We will only consider the case when $r < p_n$ (the case $r > p_n$ is easier and follows from applying only the midpoint principle).

Suppose $p_m < r < p_{m+1}$ and let $X = (\ell_{p_1} \oplus \dots \oplus \ell_{p_m})_{\ell_{p_m}}$ and $Y = (\ell_{p_{m+1}} \oplus \dots \oplus \ell_{p_n})_{\ell_\infty}$. Consider f as map into $X \oplus_\infty Y$ and assume that

$$\|x - y\| \le \| f(x) - f(y)\| \le C\|x - y\|, \quad \|x - y\| \ge 1.$$

Let $f(x) = (g(x), h(x))$. Suppose $k \in \mathbb{N}$ and $\delta > 0$. Then, by Proposition 3.6, we can find $\tau > k$, $x \in \ell_r$ and $N \in \mathbb{N}$ so that if $E = [e_j]_{j>N}$ then $g(x + \tau B_E) \subset K + \delta\tau B_X$ for some compact subset K of X.

Let $\mathbb{M} = \{n \in \mathbb{N} : n > N\}$. Define $\varphi : G_k(\mathbb{M}) \to \ell_r$ by

$$\varphi(n_1, \dots, n_k) = x + \tau k^{-1/r}(e_{n_1} + \dots + e_{n_k}).$$

Then $g \circ \varphi(G_k(\mathbb{M})) \subset K + \delta\tau B_X$. Thus by Lemma 4.1 we can find an infinite subset $\mathbb{M}_0$ of $\mathbb{M}$ so that

$$\operatorname{diam}(g \circ \varphi(G_k(\mathbb{M}_0))) \le 3\delta\tau.$$

On the other hand, $\operatorname{Lip}(h \circ \varphi) \le 2^{1/r} C\tau k^{-1/r}$ and so by Theorem 4.2 we can pass to an infinite subset $\mathbb{M}'$ of $\mathbb{M}_0$ so that

$$\operatorname{diam}(h \circ \varphi(G_k(\mathbb{M}'))) \le 3 \cdot 2^{1/r} C\tau k^{1/p_{m+1} - 1/r}.$$

Thus

$$\operatorname{diam}(f \circ \varphi(G_k(\mathbb{M}'))) \le 3 \cdot 2^{1/r} \tau (C k^{1/p_{m+1} - 1/r} + \delta).$$

However

$$\operatorname{diam}(\varphi(G_k(\mathbb{M}'))) > \tau$$

and this implies that

$$\operatorname{diam}(f \circ \varphi(G_k(\mathbb{M}'))) > \tau.$$

Hence

$$1 < 3 \cdot 2^{1/r} (C k^{1/p_{m+1} - 1/r} + \delta).$$

Since $\delta > 0$ and $k \in \mathbb{N}$ are arbitrary this is a contradiction. □

The following result answers a question of Bill Johnson for the case $1 < p < 2$. We will prove the same result in the case $2 < p < \infty$ below by a slightly different technique (Theorem 5.6).

Theorem 5.2 *If $1 \le p < 2$, L_p is not coarsely (or uniformly) homeomorphic to $\ell_p \oplus \ell_2$.*

Proof ℓ_r isometrically embeds into L_p if $p < r < 2$. The result follows from Theorem 5.1. □

The following theorem solves a problem in [11] where the same result is established for $1 < p < q < 2$ or $2 < p < q < \infty$.

Theorem 5.3 *If $1 < p < 2 < q < \infty$ and X is uniformly homeomorphic to $\ell_p \oplus \ell_q$ then X is linearly isomorphic to $\ell_p \oplus \ell_q$.*

Proof Classical nonlinear theory ([8,11] or [2]) allows us to deduce that X is linearly isomorphic to a complemented subspace of $L_p \oplus L_q$. Let $J : X \to L_p \oplus L_q$ be a linear embedding and let $(x, y) \to Sx + Ty$ be a left-inverse of J where $S : L_p \to X$ and $T : L_q \to X$ are bounded operators.

By Theorem 5.1 the space ℓ_2 does not embed into X. Therefore by a result of Johnson [9] T factors through ℓ_q. This implies that X is linearly isomorphic to a complemented subspace of $L_p \oplus \ell_q$. The spaces L_p and ℓ_q are totally incomparable so that by the results of Edelstein and Wojtaszczyk ([4] or [18]) X is linearly isomorphic to $E \oplus F$ where E is a $\mathcal{L}_p$-space and F is complemented in ℓ_q. Since E contains no copy of ℓ_2, E is isomorphic to ℓ_p or is finite-dimensional (see [12]). And F is isomorphic to ℓ_q or is finite-dimensional as well (see [15]). Since ℓ_p and ℓ_q have unique uniform structure [11] both E and F are infinite-dimensional. □

Of course this argument works equally well for arbitrary finite sums (combining the arguments of [11]).

Theorem 5.4 *If* $1 < p_1, \dots, p_n < \infty$ *with* $2 \notin \{p_1, \dots, p_n\}$ *and* X *is uniformly homeomorphic to* $\ell_{p_1} \oplus \cdots \oplus \ell_{p_n}$ *then* X *is linearly isomorphic to* $\ell_{p_1} \oplus \cdots \oplus \ell_{p_n}$.

Unfortunately we do not know if $\ell_p \oplus \ell_2$ has unique uniform structure when $p \neq 2$. If $p > 2$ we do not know if the Rosenthal space X_p [17] is uniformly homeomorphic to $\ell_p \oplus \ell_2$; since each embeds linearly into the other, our techniques do not appear fine enough to distinguish these spaces.

Theorem 5.5 *If* $2 < p < \infty$, *there is no coarse Lipschitz embedding of* $\left(\sum \ell_2\right)_{\ell_p}$ *into* $\ell_p \oplus \ell_2$.

Proof As in Theorem 5.1 we suppose $f : \left(\sum \ell_2\right)_{\ell_p} \to \ell_2 \oplus_\infty \ell_p$ is a coarse Lipschitz embedding so that

$$\|x - y\| \le \|f(x) - f(y)\| \le C\|x - y\|, \quad \|x - y\| \ge 1.$$

Write $f(x) = (g(x), h(x))$. For each i let $(e_{ij})_{j=1}^{\infty}$ be the canonical basis of the ith. co-ordinate space ℓ_2.

We proceed similarly to Theorem 5.1. Suppose $k \in \mathbb{N}$ and $\delta > 0$. Then, since $p > 2$, by Proposition 3.5, we can find $\tau > k$, $x \in \left(\sum \ell_2\right)_{\ell_p}$ and $N \in \mathbb{N}$ so that if $E = [e_{ij}]_{i>N, j\ge 1}$ then $g(x + \tau B_E) \subset K + \delta\tau B_{\ell_2}$ for some compact subset K of ℓ_2.

Define $\varphi : G_k(\mathbb{N}) \to \left(\sum \ell_2\right)_{\ell_p}$ by

$$\varphi(n_1, \dots, n_k) = x + \tau k^{-1/2}(e_{N+1,n_1} + \cdots + e_{N+1,n_k}).$$

Then $g \circ \varphi(G_k(\mathbb{N})) \subset K + \delta\tau B_{\ell_2}$. Thus by Lemma 4.1 we can find an infinite subset $\mathbb{M}_0$ of $\mathbb{N}$ so that

$$\operatorname{diam}(g \circ \varphi(G_k(\mathbb{M}_0))) \le 3\delta\tau.$$

On the other hand, $\operatorname{Lip}(h \circ \varphi) \le C\sqrt{2}\tau k^{-1/2}$ and so by Theorem 4.2 we can pass to an infinite subset $\mathbb{M}$ of $\mathbb{M}_0$ so that

$$\operatorname{diam}(h \circ \varphi(G_k(\mathbb{M}))) \le 3\sqrt{2}C\tau k^{1/p-1/2}.$$

Thus

$$\operatorname{diam}(f \circ \varphi(G_k(\mathbb{M}))) \leq 3\sqrt{2}\tau(Ck^{1/p-1/2} + \delta).$$

However

$$\operatorname{diam}(\varphi(G_k(\mathbb{M}))) > \tau$$

and this implies that

$$\operatorname{diam}(f \circ \varphi(G_k(\mathbb{M}))) > \tau.$$

Hence

$$1 < 3\sqrt{2}(Ck^{1/p-1/2} + \delta).$$

Since $\delta > 0$ and $k \in \mathbb{N}$ are arbitrary this is a contradiction. □

Haydon, Odell and Schlumprecht recently proved [7] that a subspace of L_p, $p > 2$, that does not contain an isomorphic copy of $\left(\sum \ell_2\right)_{\ell_p}$ is isomorphically contained in $\ell_p \oplus \ell_2$. It thus follows from Theorem 5.5 that a Banach space coarsely homeomorphic to $\ell_p \oplus \ell_2$ is a $\mathcal{L}_p$ space which linearly embeds into $\ell_p \oplus \ell_2$. A result of Johnson and Odell [13] then implies that it is isomorphic to a complemented subspace of X_p. We thank the referee for pointing this out to us.

The other half of Theorem 5.2 is now immediate:

Theorem 5.6 *If* $2 < p < \infty$, L_p *is not coarsely* (*or uniformly*) *homeomorphic to* $\ell_p \oplus \ell_2$.

Finally we show that Theorem 5.5 holds in the case $1 \leq p < 2$.

Theorem 5.7 *If* $1 \leq p < 2$, *there is no coarse Lipschitz embedding of* $\left(\sum \ell_2\right)_{\ell_p}$ *into* $\ell_p \oplus \ell_2$.

Proof We suppose $f : \left(\sum \ell_2\right)_{\ell_p} \to \ell_p \oplus_\infty \ell_2$ is a coarse Lipschitz embedding such that

$$\|x - y\| \leq \|f(x) - f(y)\| \leq C\|x - y\|, \quad \|x - y\| \geq 1.$$

Let $f(x) = (g(x), h(x))$ as usual. Suppose $k \in \mathbb{N}$ and $\delta > 0$.

Consider the space $Y = [e_{ij}]_{i \leq k, j \geq 1}$. Thus Y is linearly isomorphic to ℓ_2 and we can apply Proposition 3.5. As observed earlier, Proposition 3.5 holds for any equivalent norm on the domain space; therefore we can apply to Y with the subspace norm inherited from $\left(\sum \ell_2\right)_{\ell_p}$. Thus there exist $y \in Y$, $\tau > k$, a compact subset K of ℓ_p and N so that if $E = [e_{ij}]_{i \leq k, j > N}$ then $g(y + \tau B_E) \subset K + \delta\tau B_{\ell_p}$. Let $\mathbb{M} = \{j : j > N\}$.

Define $\varphi : G_k(\mathbb{M}) \to y + \tau B_E$ by

$$\varphi(n_1, \ldots, n_k) = y + \tau k^{-1/p} \sum_{i=1}^{k} e_{i,n_i}.$$

Then $\mathrm{Lip}(\varphi) = \sqrt{2}\tau k^{-1/p}$.

From Lemma 4.1 we deduce that there is an infinite subset $\mathbb{M}_0$ of $\mathbb{M}$ so that $\mathrm{diam}(g \circ \varphi(G_k(\mathbb{M}_0))) < 3\delta\tau$. From Theorem 4.2 we obtain an infinite subset $\mathbb{M}'$ of $\mathbb{M}_0$ so that $\mathrm{diam}(h \circ \varphi(G_k(\mathbb{M}')) < 3\sqrt{2}C\tau k^{1/2-1/p}$.

Thus

$$\begin{aligned} 1 &\leq \tau^{-1}\mathrm{diam}(f \circ \varphi(G_k(\mathbb{M}'))) \\ &\leq 3(\sqrt{2}Ck^{1/2-1/p} + \delta). \end{aligned}$$

Since $k \in \mathbb{N}$ and $\delta > 0$ are arbitrary this is a contradiction. □

6 Concluding remarks

It is clear that Theorem 4.2 is capable of considerable generalization. The correct concept here is *asymptotic smoothness* introduced by Milman [14] and studied in [5,10]. Let X be a Banach space. If $t > 0$ we define $\overline{\rho}_X(t)$ (the modulus of asymptotic smoothness of X) by

$$\overline{\rho}_X(t) = \sup_{\|x\|=1} \inf_{\dim X/E < \infty} \sup_{y \in B_E} (\|x + ty\| - 1).$$

If $x \in X$ with $\|x\| = 1$ and $(x_n)_{n=1}^{\infty}$ is any weakly null sequence we have

$$\limsup \|x + x_n\| \leq 1 + \overline{\rho}_X(\limsup \|x_n\|).$$

If X has separable dual this formula may also be used to define $\overline{\rho}_X$. Notice that $\overline{\rho}_X$ is a convex function which satisfies the inequality

$$\overline{\rho}_X(t) \geq \max(t - 1, 0).$$

Thus $\overline{\rho}_X$ is an Orlicz function and we can define the Orlicz sequence space $\ell_{\overline{\rho}_X}$ which we equip with the Luxemburg norm:

$$\|\xi\|_{\ell_{\overline{\rho}_X}} = \inf \left\{ \lambda : \sum_{j=1}^{\infty} \overline{\rho}_X(|\xi_j|/\lambda) \leq 1 \right\}.$$

Now assume $k \in \mathbb{N}$ and that $a_1, \ldots, a_k > 0$. For any infinite subset $\mathbb{M}$ of $\mathbb{N}$ we define the metric space $G_k(\mathbb{M}; a_1, \ldots, a_k)$ to be space of all k-subsets $(n_1, n_2, \ldots, n_k)$

with $n_1 < n_2 < \ldots < n_k$ with the metric

$$d((n_1, \ldots, n_k), (m_1, \ldots, m_k)) = \sum_{n_j \neq m_j} a_j.$$

Theorem 6.1 *Let X be a reflexive Banach space with modulus of asymptotic smoothness $\overline{\rho}_X$. Then if $\mathbb{M}$ is an infinite subset of $\mathbb{N}$, $\epsilon > 0$ and $f : G_k(\mathbb{M}; a_1, \ldots, a_k) \to X$ is a bounded map, there exists an infinite subset $\mathbb{M}'$ of $\mathbb{M}$ so that*

$$\operatorname{diam} f(G_k(\mathbb{M}')) < 2e\text{Lip}(f)\|(a_1, \ldots, a_k)\|_{\ell_{\overline{\rho}_X}} + \epsilon.$$

Proof Let us define a norm on $\mathbb{R}^2$ by

$$N_2(\xi, \eta) = \begin{cases} |\xi|\overline{\rho}_X(|\eta|/|\xi|) + |\xi|, & \xi \neq 0 \\ |\eta|, & \xi = 0. \end{cases}$$

Thus we can give N_2 an alternative definition:

$$N_2(\xi, \eta) = \sup_{\|x\|=1} \inf_{\dim X/E < \infty} \sup_{y \in B_E} \|\xi x + \eta y\|$$

and from this formula it is clear that $N_2(0, 1) = N_2(1, 0) = 1$ and N_2 is an absolute norm, i.e.

$$N_2(\xi_1, \eta_1) \leq N_2(\xi_2, \eta_2) \quad \text{if } |\xi_1| \leq |\xi_2|, \ |\eta_1| \leq |\eta_2|.$$

Then define by induction a norm N_k on $\mathbb{R}^k$ so that

$$N_k(\xi_1, \ldots, \xi_k) = N_2(N_{k-1}(\xi_1, \ldots, \xi_{k-1}), \xi_k).$$

We now prove by induction that given $\mathbb{M}$ an infinite subset of $\mathbb{N}$, $\epsilon > 0$ and $f : G_k(\mathbb{M}; a_1, \ldots, a_k) \to X$ a bounded map, there exists an infinite subset $\mathbb{M}'$ of $\mathbb{M}$ and $u \in X$ so that

$$\|f(n_1, \ldots, n_k) - u\| < N_k(a_1, \ldots, a_k)\text{Lip}(f) + \epsilon, \quad (n_1, \ldots, n_k) \in G_k(\mathbb{M}').$$

As in Theorem 4.2 this is elementary for $k = 1$. We pass to a subset $\mathbb{M}_0$ so that $\lim_{n \in \mathbb{M}_0} f(n) = u$ exists weakly. Then

$$\limsup_{n \in \mathbb{M}_0} \|f(n) - u\| \leq \limsup_{n \in \mathbb{M}_0} \|f(n) - f(m)\| \leq \text{Lip}(f)a_1.$$

Thus passing to some further infinite subset $\mathbb{M}'$ gives the result.

Assume the result is proved for $k-1$. Then we pass to a subset $\mathbb{M}_0$ so that for every $\overline{n} \in G_{k-1}(\mathbb{M})$ the limit $\tilde{f}(\overline{n}) = \lim_{n_k \in \mathbb{M}_0} f(\overline{n}, n_k)$ exists weakly.

By the inductive hypothesis we can pass to a further infinite subset $\mathbb{M}_1$ so that for some $u \in X$ we have

$$\|\tilde{f}(\overline{n}) - u\| \leq \operatorname{Lip}(\tilde{f}) N_{k-1}(a_1, \ldots, a_{k-1}) + \epsilon/2, \quad \overline{n} \in G_{k-1}(\mathbb{M}_1).$$

Note that $\operatorname{Lip}(\tilde{f}) \leq \operatorname{Lip}(f)$.

We pass to a further infinite subset $\mathbb{M}'$ of $\mathbb{M}_1$ so that

$$\Big| \|f(\overline{n}) - u\| - \|f(\overline{n}') - u\| \Big| < \epsilon/2, \quad \overline{n}, \overline{n}' \in G_k(\mathbb{M}'; a_1, \ldots, a_k).$$

Then note that for any $\overline{n} \in G_{k-1}(\mathbb{M}')$, we have

$$\limsup_{n_k \in \mathbb{M}'} \|f(\overline{n}, n_k) - u\| \leq N_2\left(\|\tilde{f}(\overline{n}) - u\|, \limsup_{n_k \in \mathbb{M}'} \|f(\overline{n}, n_k) - \tilde{f}(\overline{n})\| \right).$$

Now

$$\begin{aligned}
\limsup_{n_k \in \mathbb{M}'} \|f(\overline{n}, n_k) - \tilde{f}(\overline{n})\| &\leq \limsup_{n_k \in \mathbb{M}'} \limsup_{n_k' \in \mathbb{M}'} \|f(\overline{n}, n_k) - f(\overline{n}, n_k')\| \\
&\leq \operatorname{Lip}(f) a_k.
\end{aligned}$$

Hence

$$\begin{aligned}
\limsup_{n_k \in \mathbb{M}'} \|f(\overline{n}, n_k) - u\| &\leq N_2(N_{k-1}(a_1, \ldots, a_{k-1}) + \epsilon/2, a_k) \\
&\leq N_k(a_1, \ldots, a_k) + \epsilon/2.
\end{aligned}$$

Recalling the selection of the set $\mathbb{M}'$ this proves the inductive hypothesis.

To complete the proof we show that $N_k(\xi_1, \ldots, \xi_k) \leq e\|(\xi_1, \ldots, \xi_k)\|_{\ell_{\overline{\rho}_X}}$ for all $\xi_1, \ldots, \xi_k$.

Assume $\|(\xi_1, \xi_2, \ldots, \xi_k)\|_{\ell_{\overline{\rho}_X}} \leq 1$. We define $N_1(t) = |t|$. Assume that $N_k(\xi_1, \ldots, \xi_k) > 1$. Then there is a least r with $1 \leq r \leq k$ so that $N_r(\xi_1, \ldots, \xi_r) > 1$. Then we have

$$\begin{aligned}
N_j(\xi_1, \ldots, \xi_j) &= N_2(N_{j-1}(\xi_1, \ldots, \xi_{j-1}), \xi_j) \\
&\leq (1 + \overline{\rho}_X(|\xi_j|)) N_{j-1}(\xi_1, \ldots, \xi_{j-1}), \quad j > r.
\end{aligned}$$

If $r > 1$ we have

$$N_r(\xi_1, \ldots, \xi_r) \leq N_2(1, \xi_r) = 1 + \overline{\rho}_X(|\xi_r|).$$

If $r = 1$ then

$$N_1(\xi_1) = |\xi_1| \leq 1 + \overline{\rho}_X(|\xi_1|).$$

Thus

$$N_k(\xi_1, \ldots, \xi_k) \leq \prod_{j=r}^{k} (1 + \overline{\rho}_X(|\xi_j|)) \leq e.$$

□

We believe that Theorem 6.1 may well have other applications in nonlinear theory. The connections between asymptotic smoothness and uniform homeomorphisms is explored in [5], but the results obtained there depend on the fact that one has a uniform homeomorphism between two spaces and the Gorelik principle can be applied. Theorem 6.1 allows us to still get information when one only has a coarse Lipschitz embedding, as long as the spaces are reflexive.

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