Math. Ann. 321, 319–345 (2001) **Mathematische Annalen**

Digital Object Identifier (DOI) 10.1007/s002080100231

The $H^\infty-$calculus and sums of closed operators

N. J. Kalton · L. Weis

Received: 3 July 2000 / Revised version: 31 January 2001 /
Published online: 23 July 2001 –

Abstract. We develop a very general operator-valued functional calculus for operators with an $H^\infty-$calculus. We then apply this to the joint functional calculus of two sectorial operators when one has an H^∞calculus. Using this we prove theorem of Dore-Venni type on sums of sectorial operators and apply our results to the problem of L_p-maximal regularity. Our main assumption is the R-boundedness of certain sets of operators, and therefore methods from the geometry of Banach spaces are essential here. In the final section we exploit the special Banach space structure of L_1-spaces and $C(K)-$spaces, to obtain some more detailed results in this setting.

Mathematics Subject Classification (2000): 47A60, 47D06

1. Introduction

In recent years the notion of an $H^\infty-$calculus for sectorial operators on a Banach space has played an important role in spectral theory for unbounded operators and its applications to differential operators and evolution equations. We recall that a sectorial operator of type $0 \le \omega < \pi$ satisfies a "parabolic" estimate of the type

$$\|\zeta R(\zeta, A)\| \le C_\sigma \qquad |\arg \zeta| \ge \sigma \tag{1.1}$$

for every $\omega < \sigma < \pi$. This estimate allows a definition of $f(A)$ as a bounded operator for functions f which are bounded and analytic on the sector $\Sigma_\sigma = \{\zeta : |\arg \zeta| < \sigma\}$ and which obey a condition of the type $|f(\zeta)| \le C(|\zeta|/(1+|\zeta|^2)^\epsilon)$ for some $\epsilon > 0$. This is described in detail in [14], [29] and [36] and we give a somewhat different approach in Section 2 below. If we then have an estimate

$$\|f(A)\| \le C\|f\|_{H^\infty(\Sigma_\sigma)}$$

N.J. Kalton
Department of Mathematics, University of Missouri-Columbia, Columbia, MO 65211, USA (e-mail: nigel@math.missouri.edu)

L. Weis
Mathematisches Institut I, Universität Karlsruhe, 76128 Karlsruhe, Germany (e-mail: Lutz.Weis@math.uni-Karlsruhe.de)

The first author acknowledges support from NSF grant DMS-9800027 and from DFG grant Ka 97/126.

it is possible to extend the definition of $f(A)$ to any $f \in H^\infty(\Sigma_\sigma)$ and we say that f has an $H^\infty(\Sigma_\sigma)-$calculus. It is, by now, well-known that many systems of parabolic differential operators, Schrödinger operators and pseudo-differential operators do have an $H^\infty-$calculus (cf. [25], [3], [2] and [20]) and this has proved useful in applications.

Of particular importance are two closely related problems:

- the maximal L_p-regularity of the Cauchy problem

$$y'(t) + Ay(t) = f(t), \qquad y(0) = 0$$

  for a sectorial operator of type $\omega < \frac{\pi}{2}$
- the question whether the sum $A + B$ with domain $\mathcal{D}(A) \cap \mathcal{D}(B)$ of two sectorial operators is closed.

In fact the first problem can be reduced to the second, and the latter problem is essentially the question whether one can construct a bounded operator $B(A + B)^{-1}$. This then is a special case of the problem of constructing a joint functional calculus of A, B. In the case of Hilbert spaces and some related situations it was shown in [21], [29] and [31] that one can construct an operator-valued functional calculus associated to an operator with $H^\infty-$calculus and this permits a solution; however, it was also shown that such an approach cannot work in general Banach spaces and additional conditions are therefore needed.

We now describe the main results of this paper. In Section 2 we describe a method of setting up the joint functional calculus of n sectorial operators and an operator-valued extension. In Section 3 we recall the notion of Rademacher-boundedness (or R-boundedness) of families of operators. This implicitly goes back to work of Bourgain [6] and has recently been studied in [4], [10], [11] and [42] in connection with vector-valued multiplier theorems. We also introduce some weaker notions and study their relationship to certain Banach space properties of the underlying space.

Using these ideas in Section 4 we prove a very general result on the existence of an operator-valued functional calculus for operators with an H^∞-calculus. Roughly speaking this permits us to replace boundedness of the range of the function by Rademacher-boundedness (or even the weaker concept of U-boundedness introduced in Section 3).

We then study the relationship between R-boundedness and the functional calculus for general sectorial operators. Of particular importance is the notion of R-sectoriality when the boundedness condition (1.1) is replaced by an R-boundedness condition. Using this in Theorem 6.3 we prove a general result on sums which can be regarded as an extension of the Dore-Venni Theorem [18]. We show that if A, B are sectorial operators such that A has an $H^\infty(\Sigma_\sigma)-$calculus and B is R-sectorial of type σ' when $\sigma + \sigma' < \pi$ then $A + B$ (with domain $\mathcal{D}(A) \cap \mathcal{D}(B)$) is closed. One practical advantage of our result is that it is easier to check R-sectoriality than the boundedness of imaginary powers (see [28], [43] and [11]). We also give applications to the joint $H^\infty-$functional calculus (cf. [31]) and show how Banach space properties such as UMD, analytic UMD and property (α) of Pisier relate to our results.

It might be added that our results on the $H^\infty-$calculus emphasize the fact that an $H^\infty-$calculus really induces an unconditional expansion of the identity on the underlying Banach space. We feel our development of the theory here is somewhat simpler than preceding approachs (even for Hilbert spaces).

Finally in Section 7 we use this observation to show how classical results on unconditional bases due to Lindenstrauss and Pełczyński [35] can be recast as results on operators with an $H^\infty-$calculus on L_1 and $C(K)-$spaces. In these cases we get very strong conclusions, but they are mitigated by the observation that there are in practice very few examples of such operators on spaces of this type.

2. An operator-valued functional calculus for sectorial operators

In this section we sketch a method of setting up an operator-valued functional calculus for a sectorial operator and a joint operator-valued functional calculus for finite collections of sectorial operators. For an alternative construction of the H^∞-calculus based on McIntosh's approach see [31].

Let us first introduce some notation. Suppose $0 < \sigma < \pi$. Then we denote by Σ_σ the sector $\{z : |\arg z| < \sigma, |z| > 0\}$ and by Γ_σ the contour $\{|t|e^{i(\operatorname{sgn} t)\sigma} : -\infty < t < \infty\}$. We denote by $H^\infty(\Sigma_\sigma)$ the space of all bounded analytic functions on Σ_σ. We define $H_0^\infty(\Sigma_\sigma)$ to be the subspace of all $f \in H^\infty(\Sigma_\sigma)$ which obey an estimate of the form $|f(z)| \le C(|z|/(1+|z|^2))^\epsilon$ with $\epsilon > 0$. Let us extend this to dimension m. In $\mathbb{C}^m$ if $\sigma = (\sigma_1, \cdots, \sigma_m)$ where $0 < \sigma_k < \pi$ we define $\Sigma_\sigma = \prod_{k=1}^m \Sigma_{\sigma_k}$ and $\Gamma_\sigma = \prod_{k=1}^m \Gamma_{\sigma_k}$. If $\sigma, \nu \in \mathbb{R}^m$ we write $\sigma > \nu$ if $\sigma_k > \nu_k$ for $1 \le k \le m$. We denote by $H^\infty(\Sigma_\sigma)$ the space of all bounded analytic functions on Σ_σ. We define $H_0^\infty(\Sigma_\sigma)$ to be the subspace of all $f \in H^\infty(\Sigma_\sigma)$ which obey an estimate of the form $|f(z)| \le C\prod_{k=1}^m(|z_k|/(1+|z_k|^2))^\epsilon$ with $\epsilon > 0$ where $z = (z_1, \cdots, z_m)$.

Next we introduce some corresponding vector-valued spaces. Now suppose X is a Banach space and $\mathcal{A}$ is a subalgebra of $\mathcal{L}(X)$, which is closed for the strong-operator topology. If $\sigma = (\sigma_1, \cdots, \sigma_m)$ as above, we define $H^\infty(\Sigma_\sigma; \mathcal{A})$ the space of all bounded functions $F : \Sigma_\sigma \to \mathcal{A}$, so that for every $x \in X$ the map $z \to F(z)x$ is analytic (i.e. F is analytic for the strong-operator topology). We consider the scalar space $H^\infty(\Sigma_\sigma)$ as a subspace of $H^\infty(\Sigma_\sigma; \mathcal{A})$ via the identification $f \to fI$. We shall say that F_n *converges boundedly* to F in $H^\infty(\Sigma_\sigma; \mathcal{A})$ if $\sup_n \sup_{z\in\Sigma_\sigma} \|F_n(z)\| < \infty$ and $F_n(z)x \to F(z)x$ for every $z \in \Sigma_\sigma$, and $x \in X$. We define $H_0^\infty(\Sigma_\sigma; \mathcal{A})$ the subspace of all $F \in H^\infty(\Sigma_\sigma; \mathcal{A})$

which obey an estimate of the form $\|F(z)\| \le C \prod_{k=1}^{m}(|z_k|/(1+|z_k|^2))^{\epsilon}$ with $\epsilon > 0$ where $z = (z_1, \cdots, z_m)$.

We next consider the space of germs of such functions. Fix $0 \le \omega_k < \pi$ for $1 \le k \le m$. We consider the space $\mathcal{H}(\omega, \mathcal{A}) = \cup_{\sigma>\omega} H^{\infty}(\Sigma_\sigma; \mathcal{A})$ where (F, G) are identified if there exists $\sigma > \omega$ with $F(z) = G(z)$ for all $z \in \Sigma_\sigma$. $\mathcal{H}(\omega, \mathcal{A})$ is then an algebra. In $\mathcal{H}(\omega, \mathcal{A})$ we define a notion of sequential convergence τ by $F_n \to F$ if there exists $\sigma > \omega$ so that each $F_n, F \in H^{\infty}(\Sigma_\sigma; \mathcal{A})$, $\sup_n \sup_{z\in\Sigma_\sigma} \|F_n(z)\| < \infty$ and $F_n(z)x \to F(z)x$ for all $z \in \Sigma_\sigma$ and all $x \in X$.

Recall that a closed densely defined operator A on a Banach space X is a *sectorial operator of type* $0 \le \omega = \omega(A) < \pi$ if A is one-one with dense range, the resolvent $R(\lambda, A)$ is defined and bounded for $\lambda = re^{i\theta}$ where $r > 0$ and $\omega < |\theta| \le \pi$ and satisfies an estimate $\|\lambda R(\lambda, A)\| \le C_\sigma$ for $\omega < \sigma \le |\theta|$.

Suppose $(A_1, \cdots, A_m)$ is a family of sectorial operators where A_k is of type ω_k for $1 \le k \le m$, and let $\omega = (\omega_1, \cdots, \omega_m)$. Define the resolvent for $|\arg \lambda| > \omega$ by $R(\lambda, A_1, \cdots, A_m) = \prod_{k=1}^{m} R(\lambda_k, A_k)$. Let $\mathcal{A}$ be the closed subalgebra of $\mathcal{L}(X)$ of all operators T so that T commutes with $R(\lambda, A_k)$ for every k and every λ with $|\arg \lambda| > \omega_k$.

If $F \in \mathcal{H}(\omega, \mathcal{A})$ is of the form $F(z) = \prod_{k=1}^{m}(\lambda_k - z_k)^{-p_k} S$ where $p_k \in \mathbb{N} \cup \{0\}$ and $S \in \mathcal{A}$ we define $F(A_1, \cdots, A_m) = \prod_{k=1}^{m} R(\lambda_k, A_k)^{p_k} S$ and then this definition can be extended by linearity to the linear span of such functions, which we call the *rational functions*, denoted $\mathcal{R}(\omega, \mathcal{A})$, in $\mathcal{H}(\omega, \mathcal{A})$.

To extend this definition further we use the following device. Consider the algebra of all $(F, F(A_1, \cdots, A_m))$ for $F \in \mathcal{R}(\omega, \mathcal{A})$ as a subset of $\mathcal{H}(\omega, \mathcal{A}) \times \mathcal{A}$. Denote by τ^* the sequential convergence $(F_n, T_n) \to (F, T)$ if $F_n \to F(\tau)$ and $T_n \to T$ in the strong-operator topology. Let $\mathcal{B}$ be the τ^*-closure of this set (i.e. the smallest set which is closed under sequential convergence and contains it). Notice that this construction might involve taking infinitely many iterations of sequential limits, but our construction actually shows that two iterations suffice. It is clear that $\mathcal{B}$ is an algebra. Our next task is to show that if $F \in \mathcal{H}(\omega, \mathcal{A})$ there is at most one choice of $T \in \mathcal{A}$ so that $(F, T) \in \mathcal{B}$, this will enable us to define $F(A_1, \cdots, A_m)$ unambiguously.

Consider the function on $\mathbb{C}$

$$\varphi_n(z) = \frac{n}{n+z} - \frac{1}{1+nz} \tag{2.1}$$

and then define on $\mathbb{C}^m$, $\psi_n(z) = \prod_{k=1}^{m} \varphi_n(z_k)$ so that $\psi_n \in H_0^{\infty}(\Sigma_\sigma)$ for every $\sigma > 0$. Then

$$\psi_n(A_1, \cdots, A_m) = \prod_{k=1}^{m} \left(\frac{1}{n} R\left(-\frac{1}{n}, A_k\right) - nR(-n, A_k) \right) = V_n$$

is an approximate identity in the sense that $\sup \|V_n\| < \infty$ and $V_n x \to x$ for every $x \in X$.

If $F \in \mathcal{H}(\omega, \mathcal{A})$ then if $F \in H^\infty(\Sigma_\sigma, \mathcal{A})$ we can define

$$L_n(F)x = \left(\frac{-1}{2\pi i}\right)^m \int_{\Gamma_\nu} \psi_n(\zeta)F(\zeta)R(\zeta, A_1, \cdots, A_m)x\, d\zeta, \tag{2.2}$$

as long as $\omega < \nu < \sigma$. (Note that we are using short-hand and this is really a multiple contour integral.) An application of Cauchy's Theorem shows that L_n is independent of the choice of ν. By the Lebesgue Dominated Convergence Theorem $L_n : \mathcal{H}(\omega, \mathcal{A}) \to \mathcal{A}$ is τ−continuous if $\mathcal{A}$ is equipped with the strong-operator topology.

If F is rational then we have by a standard contour integration,

$$L_n(F)x = F(A_1, \cdots, A_m)V_n x \qquad x \in X. \tag{2.3}$$

Now the map $(F, T) \to L_n(F) - T$ is continuous for τ^* and the strong-operator topology. We conclude that if $(F, T) \in \mathcal{B}$,

$$L_n(F)x = TV_n x \qquad x \in X.$$

Since $V_n x \to x$ for all $x \in X$, this shows that T is uniquely determined by F. Hence we can define $\mathcal{H}(A_1, \cdots, A_m; \mathcal{A})$ to be the set of $F \in \mathcal{H}(\omega, \mathcal{A})$ such that for some T we have $(F, T) \in \mathcal{B}$ and then we can define $T = F(A_1, \cdots, A_m)$ for $F \in \mathcal{H}(A_1, \cdots, A_m, \mathcal{A})$. The space $\mathcal{H}(A_1, \cdots, A_m; \mathcal{A})$ is an algebra and $F \to F(A_1, \cdots, A_m)$ is an algebra homomorphism. For $F \in \mathcal{H}(A_1, \cdots, A_m; \mathcal{A}) \cap H^\infty(\Sigma_\sigma; \mathcal{A})$ and $\sigma > \nu > \omega$ then (2.2) and (2.3) can be rewritten as:

$$F(A_1, \cdots, A_m)V_n x = \left(\frac{-1}{2\pi i}\right)^m \int_{\Gamma_\nu} \psi_n(\zeta)F(\zeta)R(\zeta, A_1, \cdots, A_m)x\, d\zeta. \tag{2.4}$$

If $F \in H_0^\infty(\Sigma_\sigma; \mathcal{A})$ then the integrals in (2.4) converge as $n \to \infty$. We can show by approximating the integral by Riemann sums that $F \in \mathcal{H}(A_1, \cdots, A_m; \mathcal{A})$ and then we have:

$$F(A_1, \cdots, A_m)x = \left(\frac{-1}{2\pi i}\right)^m \int_{\Gamma_\nu} F(\zeta)R(\zeta, A_1, \cdots, A_m)x\, d\zeta \qquad x \in X. \tag{2.5}$$

It now follows that if $F \in \mathcal{H}(\omega, \mathcal{A})$ then $(\psi_k F) \in \mathcal{H}(A_1, \cdots, A_m; \mathcal{A})$ for each $k \in \mathbb{N}$. Furthermore if $F_n \to F(\tau)$ we have $(\psi_k F_n)(A_1, \cdots, A_m) \to (\psi_k F)(A_1, \cdots, A_m)$ in the strong-operator topology for each fixed k. From this it follows that if $F_n \in \mathcal{H}(A_1, \cdots, A_m; \mathcal{A})$ and $\sup \|F_n(A_1, \cdots, A_m)\| < \infty$ then $F \in \mathcal{H}(A_1, \cdots, A_m)$ and $F_n(A_1, \cdots, A_m) \to F(A_1, \cdots, A_m)$ in the strong-operator topology (indeed we have convergence on each $V_n x$). In particular it follows that $F \in \mathcal{H}(A_1, \cdots, A_m; \mathcal{A})$ if and only if $\sup_n \|(\psi_n F)(A_1, \cdots, A_m)\| < \infty$.

If we consider the scalar functions in $\mathcal{H}(A_1,\cdots,A_m) \subset \mathcal{H}(A_1,\cdots,A_m;\mathcal{A})$ then we have defined the *joint functional calculus* for $(A_1,\cdots,A_m)$. We recall that a single operator A has *an* $H^\infty(\Sigma_\sigma)$*−calculus* if $H^\infty(\Sigma_\sigma) \subset \mathcal{H}(A)$. The collection $(A_1,\cdots,A_m)$ has *a joint* $H^\infty(\Sigma_\sigma)$*−calculus* if $H^\infty(\Sigma_\sigma) \subset \mathcal{H}(A_1,\cdots,A_m)$.

3. Rademacher-boundedness and related ideas

We recall ([10],[42]) that a family $\mathcal{F}$ of bounded operators on a Banach space X is called *Rademacher-bounded or R-bounded* with R-boundedness constant C if letting $(\epsilon_k)_{k=1}^\infty$ be a sequence of independent Rademachers on some probability space then for every $x_1,\cdots,x_n \in X$ and $T_1,\cdots,T_n \in \mathcal{F}$ we have:

$$\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k T_k x_k\right\|^2\right)^{\frac{1}{2}} \leq C\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k x_k\right\|^2\right)^{\frac{1}{2}}. \tag{3.1}$$

It is important to note that this definition and the associated constant C are unchanged if we require $T_1,\cdots,T_n$ to be distinct in (3.1) (see e.g. [10], Lemma 3.3). The same remark applies to each of the following definitions.

We will introduce two related weaker notions. Let us say that $\mathcal{F}$ is *weakly Rademacher-bounded or WR-bounded* with WR-boundedness constant C if for every $x_1,\cdots,x_n \in X,\ x_1^*,\cdots,x_n^* \in X^*$ and $T_1,\cdots,T_n \in \mathcal{F}$ we have:

$$\sum_{k=1}^n |\langle T_k x_k, x_k^*\rangle| \leq C\mathbb{E}\left(\left\|\sum_{k=1}^n \epsilon_k x_k\right\|^2\right)^{\frac{1}{2}} \mathbb{E}\left(\left\|\sum_{k=1}^n \epsilon_k x_k^*\right\|^2\right)^{\frac{1}{2}}. \tag{3.2}$$

Finally we say that $\mathcal{F}$ is *unconditionally bounded or U-bounded* with U-boundedness constant C if for every $x_1,\cdots,x_n \in X,\ x_1^*,\cdots,x_n^* \in X^*$ and $T_1,\cdots,T_n \in \mathcal{F}$ we have

$$\sum_{k=1}^n |\langle T_k x_k, x_k^*\rangle| \leq C \max_{\epsilon_k=\pm1}\left\|\sum_{k=1}^n \epsilon_k x_k\right\| \max_{\epsilon_k=\pm1}\left\|\sum_{k=1}^n \epsilon_k x_k^*\right\|. \tag{3.3}$$

The following Lemma is recorded for future reference:

Lemma 3.1 *Let $\mathcal{F}$ be a subset of $\mathcal{L}(X)$. Then for $\mathcal{F}$, R-bounded $\Rightarrow$ WR-bounded $\Rightarrow$ U-bounded. If X has nontrivial Rademacher type then WR-bounded $\Rightarrow$ R-bounded.*

We note that the only really non-trivial part of the Lemma is the last sentence and this follows easily from Pisier's characterization of spaces with non-trivial type as those in which the Rademacher projection is bounded [38].

We shall also need some related Banach space concepts. Suppose $(\epsilon_k)_{k=1}^\infty$ and $(\eta_k)_{k=1}^\infty$ are two mutually independent sequences of Rademachers. We say that X has property (α) (see [37] and [29]) if there is a constant C so that for any $(x_{jk})_{j,k=1}^n \subset X$ and any $(\alpha_{jk})_{j,k=1}^n \subset \mathbb{C}$ we have

(3.4)
$$\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \alpha_{jk}\epsilon_j\eta_k x_{jk}\right\|^2\right)^{\frac{1}{2}} \le C\max_{j,k}|\alpha_{jk}|\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\eta_k x_{jk}\right\|^2\right)^{\frac{1}{2}}.$$

We say that X has property (A) [29] if there is a constant C such that for any $(x_{jk})_{j,k=1}^n \subset X$ and for any $(x_{jk}^*)_{j,k=1}^n \subset X^*$ we have:

(3.5)
$$\sum_{j=1}^n\sum_{k=1}^n |\langle x_{jk}, x_{jk}^*\rangle| \le C\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\eta_k x_{jk}\right\|^2\right)^{\frac{1}{2}}\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\eta_k x_{jk}^*\right\|^2\right)^{\frac{1}{2}}.$$

Clearly (α) implies (A) and the converse holds if X has nontrivial Rademacher type; this is a fairly simple deduction from the boundedness of the Rademacher projection [38]. Any subspace of a Banach lattice with nontrivial cotype has property (α) while any Banach lattice has property (A). It is also observed in [29] that L_1/H_1 has (α). The Schatten ideals $\mathcal{C}_p$ when $1 \le p \le \infty$ and $p \ne 2$ fail to have (A).

We shall say that X has property (Δ) if there is a constant C so that for any $(x_{jk})_{j,k=1}^n \in X$

$$\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^j \epsilon_j\eta_k x_{jk}\right\|^2\right)^{\frac{1}{2}} \le C\left(\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\eta_k x_{jk}\right\|^2\right)^{\frac{1}{2}}. \tag{3.6}$$

It is clear that (Δ) is a weaker property than (α). It is in fact shared by all spaces with (UMD) and even analytic UMD. We recall ([5]) that X has *analytic UMD* if every L_1-bounded analytic martingale has unconditional martingale differences.

Proposition 3.2 *Suppose X has analytic UMD. Then X has property (Δ).*

Proof. Let $(\tilde{\epsilon}_k)_{k=1}^\infty$ and $(\tilde{\eta}_k)_{k=1}^\infty$ be two mutually independent sequences of Steinhaus variables (i.e. each is complex-valued and uniformly distributed on the unit circle). By applying the unconditionality of the Rademachers and the Khintchine-Kahane inequality it is sufficient to show the existence of a constant C so that for any $(x_{jk})_{j,k=1}^n$ we have:

$$\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^j \tilde{\epsilon}_j\tilde{\eta}_k x_{jk}\right\| \le C\mathbb{E}\left\|\sum_{j=1}^n\sum_{k=1}^n \tilde{\epsilon}_j\tilde{\eta}_k x_{jk}\right\|.$$

To see this we define f_j for $1 \le j \le 2n-1$ by $f_{2r-1} = \sum_{j\le r}\sum_{k\le r}\tilde{\epsilon}_j\tilde{\eta}_k x_{jk}$ and $f_{2r} = \sum_{j\le r+1}\sum_{k\le r}\tilde{\epsilon}_j\tilde{\eta}_k x_{jk}$. Let $f_0 = 0$. Then (f_j) is an analytic martingale and so for a suitable constant C depending only on X we have:

$$\mathbb{E}\left\|\sum_{r=0}^{n-1}(f_{2r+1} - f_{2r})\right\| \le C\mathbb{E}\,\|f_{2n-1}\|.$$

This yields the desired inequality. □

Since any space with (UMD) has analytic (UMD) this shows that (UMD)-spaces have (Δ); actually a direct proof using Rademacher in place of Steinhaus variables in the above argument is possible for this case. Thus the Schatten classes $\mathcal{C}_p$ have property (Δ) as long as $1 < p < \infty$. However Haagerup and Pisier [23] show that $\mathcal{C}_1$ (which has cotype 2) fails analytic UMD and their argument actually shows it fails property (Δ). This implies that $C(K)$-spaces of infinite dimension also fail (Δ) since $\mathcal{C}_1$ is finitely representable in any such space.

We now come to an important result relating the above properties to Rademacher-boundedness. Some similar results are shown in [10].

Theorem 3.3 *Suppose $(U_k)_{k=1}^\infty$ and $(V_k)_{k=1}^\infty$ are two sequences of operators in $\mathcal{L}(X)$ satisfying*

$$\sup_n \sup_{\epsilon_k=\pm1}\left\|\sum_{k=1}^n \epsilon_k U_k\right\| \le M < \infty$$

and

$$\sup_n \sup_{\epsilon_k=\pm1}\left\|\sum_{k=1}^n \epsilon_k V_k\right\| \le M < \infty.$$

Suppose further $\mathcal{F} \subset \mathcal{L}(X)$ is a family of operators which is R-bounded with constant R. Then:

1. *The sequence $(U_n)_{n=1}^\infty$ is R-bounded with constant M.*
2. *If X has property (α) the collection $\{\sum_{k=1}^n \alpha_k U_k T_k V_k \,:\, n \in \mathbb{N},\ |\alpha_k| \le 1,\ T_k \in \mathcal{F}\}$ is R-bounded with constant CRM^2 where C depends only on X. In particular the family $\{\sum_{k=1}^n \alpha_k U_k V_k \,:\, n \in \mathbb{N},\ |\alpha_1|, \cdots, |\alpha_n| \le 1\}$ is R-bounded with constant CM^2 where C depends only on X.*
3. *If X has property (A) then the family $\{\sum_{k=1}^n \alpha_k U_k V_k \,:\, n \in \mathbb{N},\ |\alpha_1|, \cdots, |\alpha_n| \le 1\}$ is WR-bounded with constant CM^2 where C depends only on X.*

4. *If X has property (Δ) then the set $\{\sum_{k=1}^n U_k V_k : n \in \mathbb{N}\}$ is R-bounded with constant CM^2 where C depends only on X.*

Proof. (1) We use the remark that it is enough to establish (3.1) for distinct operators $T_1, \cdots, T_n$. If $x_1, \cdots, x_n \in X$ and $\alpha_k = \pm 1$ then

$$\mathbb{E}\left(\left(\sum_{k=1}^n \epsilon_k U_k\right)\left(\sum_{k=1}^n \epsilon_k \alpha_k x_k\right)\right) = \sum_{k=1}^n \alpha_k U_k x_k$$

and hence

$$\sup_{\alpha_k=\pm 1}\left\|\sum_{k=1}^\infty \alpha_k U_k x_k\right\| \le M(\mathbb{E}\left(\left\|\sum_{k=1}^n \epsilon_k x_k\right\|^2\right)^{\frac{1}{2}}.$$

This proves (1) and indeed a rather stronger result.

(2) Let $S_j = \sum_{k=1}^\infty \alpha_{jk} U_k T_{jk} V_k$ where $T_{jk} \in \mathcal{F}$ and (α_{jk}) is a finitely nonzero collection of complex numbers with $|\alpha_{jk}| \le 1$ and fix $x_1, \cdots, x_n \in X$. We first note that for all $y_1, \cdots, y_n \in Y$

$$\begin{aligned}\left\|\sum_{k=1}^n U_k y_k\right\| &= \left\|\mathbb{E}\left(\left(\sum_{k=1}^n \epsilon_k U_k\right)\left(\sum_{k=1}^n \epsilon_k y_k\right)\right)\right\| \\ &\le M\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k y_k\right\|^2\right)^{\frac{1}{2}}.\end{aligned}$$

We will also use the fact (Lemma 3.13 of [10]) that there is a constant C depending only on X so that for $(x_{jk})_{j,k=1}^n \in X$ we have from property (α),

$$\left(\mathbb{E}_\epsilon \mathbb{E}_\eta \left\|\sum_{j=1}^n\sum_{k=1}^n \alpha_{jk}\epsilon_j\eta_k T_{jk} x_{jk}\right\|^2\right)^{\frac{1}{2}} \le CR\left(\mathbb{E}_\epsilon \mathbb{E}_\eta \left\|\sum_{j=1}^n\sum_{k=1}^n \epsilon_j\eta_k x_{jk}\right\|^2\right)^{\frac{1}{2}}.$$

Hence, using $y_k = \sum_{j=1}^n \alpha_{jk}\epsilon_j T_{jk} V_k x_j$ in the first inequality,

$$\left(\mathbb{E}_\epsilon \left\|\sum_{j=1}^n \epsilon_j S_j x_j\right\|^2\right)^{\frac{1}{2}} = \left(\mathbb{E}_\epsilon \left\|\sum_{k=1}^\infty U_k \sum_{j=1}^n \alpha_{jk}\epsilon_j T_{jk} V_k x_j\right\|^2\right)^{\frac{1}{2}}$$

$$\leq M \left(\mathbb{E}_\epsilon \mathbb{E}_\eta \left\| \sum_{j=1}^{n} \sum_{k=1}^{\infty} \alpha_{jk} \epsilon_j \eta_k T_{jk} V_k x_j \right\|^2 \right)^{\frac{1}{2}}$$
$$\leq CRM \left(\mathbb{E}_\epsilon \mathbb{E}_\eta \left\| \sum_{j=1}^{n} \sum_{k=1}^{\infty} \epsilon_j \eta_k V_k x_j \right\|^2 \right)^{\frac{1}{2}}$$
$$\leq CRM^2 \left(\mathbb{E}_\epsilon \left\| \sum_{j=1}^{n} \epsilon_j x_j \right\|^2 \right)^{\frac{1}{2}}.$$

This proves (2).

(3) Let $S_j = \sum_{k=1}^{\infty} \alpha_{jk} U_k V_k$ where (α_{jk}) is a finitely nonzero matrix with $|\alpha_{jk}| \leq 1$. In this case if $x_1, \cdots, x_n \in X$ and $x_1^*, \cdots, x_n^* \in X^*$ we note that:

$$\sum_{j=1}^{n} |\langle S_j x_j, x_j^* \rangle| \leq \sum_{j=1}^{n} \sum_{k=1}^{\infty} |\langle V_k x_j, U_k^* x_j^* \rangle|$$
$$\leq C \left(\mathbb{E}_\epsilon \mathbb{E}_\eta \left\| \sum_{j=1}^{n} \sum_{k=1}^{\infty} \epsilon_j \eta_k V_k x_j \right\|^2 \right)^{\frac{1}{2}} \left(\mathbb{E}_\epsilon \mathbb{E}_\eta \left\| \sum_{j=1}^{n} \sum_{k=1}^{\infty} \epsilon_j \eta_k U_k^* x_j^* \right\|^2 \right)^{\frac{1}{2}}$$
$$\leq CM^2 \left(\mathbb{E} \left\| \sum_{j=1}^{n} \epsilon_j x_j \right\|^2 \right)^{\frac{1}{2}} \left(\mathbb{E} \left\| \sum_{j=1}^{n} \epsilon_j x_j^* \right\|^2 \right)^{\frac{1}{2}}$$

(4) We use the proof of (2). This time we again use the fact it suffices to consider the operators without repetition. So we consider $S_j = \sum_{k=1}^{j} U_k V_k$ and repeat the proof of (2) with $\alpha_{jk} = 1$ if $k \leq j$ and 0 otherwise and replace each T_{jk} by the identity. Using (3.6) in place of (3.4) gives the desired conclusion. □

We conclude this section with a useful Lemma. In fact in the case of R-boundedness, this result is found in [42].

Lemma 3.4 *Suppose* $0 < \sigma < \pi$ *and* $F \in H^\infty(\Sigma_\sigma; \mathcal{L}(X))$. *Suppose* $0 < \sigma_0 < \nu < \sigma$ *and for some* $M < \infty$ *and* $a > 1$, *and for each* $t \in \mathbb{R}$ *the set* $\{F(a^k t e^{\pm i\nu})\}_{k \in \mathbb{Z}}$ *is U-bounded (respectively, WR-bounded; respectively, R-bounded) with constant bounded by M (independent of* t*). Then the family* $\{F(z) : z \in \Sigma_{\sigma_0}\}$ *is U-bounded, (respectively, WR-bounded; respectively, R-bounded).*

Proof. We give the proof in the U-boundedness case, the others being similar. We first make the observation that it suffices to consider the case when $\nu = \frac{\pi}{2}$

as one can make the transformation $z = w^{2\nu/\pi}$. In this case we have the formula

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(it)\Re(z-it)^{-1}dt.$$

We write

$$F_1(z) = \frac{1}{\pi}\int_0^{\infty} F(it)\Re(z-it)^{-1}dt$$

and

$$F_2(z) = \frac{1}{\pi}\int_0^{\infty} F(-it)\Re(z+it)^{-1}dt$$

so that $F(z) = F_1(z) + F_2(z)$.

Note that for a suitable constant C we have an estimate $0 \le \Re(z \pm it)^{-1} \le C|z|| \min(t^{-2}, |z|^{-2})$ whenever $z \in \Sigma_{\sigma_0}$.

Now suppose $x_1, \cdots, x_n \in X,\ x_1^*, \cdots, x_n^* \in X^*$. Suppose $z_1, \cdots, z_n \in \Sigma_{\sigma_0}$. Let us suppose that $m_j \in \mathbb{Z}$ are chosen so that $a^{m_j} \le |z_j| \le a^{m_j+1}$. We have for $z \in \Sigma_{\sigma_0}$:

$$\begin{aligned}\sum_{j=1}^n |\langle F_1(z_j)x_j, x_j^*\rangle| &\le \frac{1}{\pi}\sum_{j=1}^n \int_0^\infty |\langle F(ia^{m_j}t)x_j, x_j^*\rangle|\Re(z_j - ia^{m_j}t)^{-1}a^{m_j}dt \\ &\le \frac{aC}{\pi}\sum_{j=1}^n \int_0^\infty |\langle F(ia^{m_j}t)x_j, x_j^*\rangle| \min(1, t^{-2})dt \\ &\le C' \max_{\epsilon_j=\pm 1}\left\|\sum_{j=1}^n \epsilon_j x_j\right\| \max_{\epsilon_j=\pm 1}\left\|\sum_{j=1}^n \epsilon_j x_j^*\right\|\end{aligned}$$

for a suitable constant C'. A similar argument can be done for F_2. □

4. Functional calculus for operator-valued functions

Let us suppose A is sectorial of type ω and $\sigma > \omega$. We let $\mathcal{A}$ denote as in Section 2 the algebra of all bounded operators which commute with A.

Before we prove our basic estimate for an operator-valued functional calculus, we will describe in Lemma 4.1 and Proposition 4.3 the connection between the $H^\infty-$calculus and unconditional expansions in the underlying Banach space.

Lemma 4.1 *Suppose that A admits an $H^\infty \times(\Sigma_\sigma)$-calculus, and that $f \in H_0^\infty(\Sigma_\sigma)$. Then there is a constant C so that for any $t > 0$ and any finitely nonzero sequence $(\alpha_k)_{k\in\mathbb{Z}}$ we have:*

$$\left\|\sum_{k\in\mathbb{Z}} \alpha_k f(2^k tA)\right\| \le C \max_{k\in\mathbb{Z}} |\alpha_k|.$$

Furthermore for every $x \in X$ and $t > 0$ the series $\sum_{k\in\mathbb{Z}} f(2^k tA)x$ converges unconditionally in X.

Proof. We can assume $\max_{k\in\mathbb{Z}} |\alpha_k| \le 1$. For suitable constants C, C' and $\epsilon > 0$ we have

$$\begin{aligned}\Big\| \sum_{k\in\mathbb{Z}} \alpha_k f(2^k tA) \Big\| &\le C \sup_{z\in\Sigma_\sigma} \sum_{k\in\mathbb{Z}} |f(2^k z)| \\ &\le CC' \sup_{z\in\Sigma_\sigma} \sum_{k\in\mathbb{Z}} \left(\frac{2^k|z|}{1+2^{2k}|z|^2} \right)^\epsilon\end{aligned}$$

and the last quantity is finite.

For the last part observe that for any bounded sequence $(\alpha_k)_{k\in\mathbb{Z}}$ and $t > 0$, the series $\sum_{k\in Z} \alpha_k f(2^k tA)x$ must converge to $g(A)x$ where $g(z) = \sum_{k\in Z} \alpha_k f(2^k tz) \in H^\infty(\Sigma_\sigma)$. □

Proposition 4.2 *Suppose $F \in H_0^\infty(\Sigma_\sigma; \mathcal{A})$. Then for any $\omega < \nu < \sigma$, $0 < s < 1$, and any $x \in X$,*

$$F(A)x = \frac{-1}{2\pi i} \int_{\Gamma_\nu} \zeta^{-s} F(\zeta) A^s R(\zeta, A)x\, d\zeta. \tag{4.1}$$

Proof. First note that $A^s R(\lambda, A)$ is a bounded operator for $\lambda \in \Gamma_\nu$ which is given by the integral

$$A^s R(\lambda, A)x = \frac{-1}{2\pi i} \int_{\Gamma_{\nu'}} \zeta^s (\lambda - \zeta)^{-1} R(\zeta, A)x\, d\zeta,$$

if $\omega < \nu' < \nu$. This gives an estimate $\|A^s R(\lambda, A)\| \le C_s |\lambda|^{s-1}$ and shows that the integral in (4.1) converges as a Bochner integral. It is clear that we only need establish the formula if $x = \varphi_n(A)y$ (see (2.1)) for some $y \in X$. To do this we compute

$$\begin{aligned} F(A)\varphi_n(A)x &= F(A)\varphi_n^2(A)y \\ &= (A^s\varphi_n(A))(F(A)A^{-s}\varphi_n(A))y \\ &= \frac{-1}{2\pi i}(A^s\varphi_n(A)) \int_{\Gamma_\nu} \zeta^{-s}\varphi_n(\zeta)F(\zeta)R(\zeta, A)y\, d\zeta \\ &= \frac{-1}{2\pi i} \int_{\Gamma_\nu} \zeta^{-s}\varphi_n(\zeta)F(\zeta)(A^s\varphi_n(A))R(\zeta, A)y\, d\zeta \\ &= \frac{-1}{2\pi i} \int_{\Gamma_\nu} \zeta^{-s}\varphi_n(\zeta)F(\zeta)(A^s R(\zeta, A))x\, d\zeta \end{aligned}$$

Now using the Dominated Convergence Theorem we obtain (4.1). □

We fix now some notation which is used frequently in this paper. Let us rewrite (4.1) by using the parameterization $\zeta = |t|e^{i(\operatorname{sgn} t)\nu}$ for $-\infty < t < \infty$. To represent the resolvent we often use the function

$$h_s^\rho(z) = z^s(e^{i\rho} - z)^{-1}\,. \tag{4.2}$$

Then for $F \in H_0^\infty(\Sigma_\sigma; \mathcal{A})$ where $\sigma > \nu > \omega$,

$$\begin{aligned} F(A)x &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{i(1-s)(\operatorname{sgn} t)\nu}|t|^{-s}F(|t|e^{i(\operatorname{sgn} t)\nu})A^s R(|t|e^{i(\operatorname{sgn} t)\nu}, A)x\,dt \\ &= \frac{e^{i(1-s)\nu}}{2\pi i}\int_0^\infty F(te^{i\nu})h_s^\nu(t^{-1}A)x\,\frac{dt}{t} \\ &\quad + \frac{e^{-i(1-s)\nu}}{2\pi i}\int_0^\infty F(te^{-i\nu})h_s^{-\nu}(t^{-1}A)x\,\frac{dt}{t} \end{aligned}$$

This can then be reformulated as:

$$F(A)x = \frac{1}{2\pi i}\int_1^2 (M_+(t) + M_-(t))\,\frac{dt}{t} \tag{4.3}$$

where

$$M_\pm(t) = e^{\pm i(1-s)\nu}\sum_{k\in\mathbb{Z}} F(2^{-k}t^{-1}e^{\pm i\nu})h_s^{\pm\nu}(2^k tA)x. \tag{4.4}$$

We first make an essentially trivial deduction characterizing the H^∞-calculus.

Proposition 4.3 *Suppose $\nu > \omega$ and $0 < s < 1$. Consider the conditions:*

$$\sup_{t>0}\sup_N\sup_{\epsilon_k=\pm 1}\left\|\sum_{k=-N}^{N}\epsilon_k(2^k t)^{(1-s)}A^s R(2^k te^{\pm i\nu}, A)\right\| < \infty, \tag{4.5}$$

Then (4.5) is necessary for A to admit an $H^\infty(\Sigma_\sigma)-$calculus for some $\sigma < \nu$ and sufficient for A to admit an $H^\infty(\Sigma_\sigma)$-calculus for every $\sigma > \nu$.

Proof. Necessity follows immediately from Lemma 4.1 for the functions $h_s^{\pm\nu}$. Conversely, by (4.5), if $f \in H^\infty(\Sigma_\sigma)$ where $\sigma > \nu$ we obtain by (4.3) and (4.4)

$$\|(\varphi_n f)(A)\| \le C$$

independent of n. This implies that $f \in \mathcal{H}(A)$. ◻

Our main result is also easy from (4.3) and (4.4).

Theorem 4.4 *Suppose A admits an $H^\infty(\Sigma_\sigma)-$calculus and $F \in H^\infty(\Sigma_\rho; \mathcal{A})$ for some $\rho > \sigma$. Suppose further that the set $\{F(z) : z \in \Sigma_\rho\}$ is U-bounded. Then $F \in \mathcal{H}(A, \mathcal{A})$.*

Remarks. (1) Of course the Theorem holds if we assume the stronger property that $\{F(z) : z \in \Sigma_\rho\}$ is WR-bounded or R-bounded.

(2) For Hilbert spaces and certain operators on $L_2(X)$ such an operator-valued functional calculus is constructed in [29] Theorem 5.2 and [31]. These are cases when the U-boundedness condition is satisfied automatically. See also [21]. In [11] there are constructions based on transference results which work for generators of bounded c_0-groups on UMD-spaces and some other special cases.

Proof. As before we consider $\varphi_n F = F_n$ so that $F_n \in \mathcal{H}(A, \mathcal{A})$. It suffices to show $\sup \|F_n(A)\| < \infty$. Referring to (4.3) and (4.4) with some fixed $0 < s < 1$ and $\rho > \nu > \sigma$ for $x \in X,\ x^* \in X^*$ with $\|x\|, \|x^*\| \leq 1$, we obtain the estimate for $1 \leq t \leq 2$:

$$|\langle M_\pm(t)x, x^*\rangle| \leq \sum_{k\in\mathbb{Z}} |\langle F_n(2^{-k}t^{-1}e^{\pm i\nu})g(2^k tA)x, g(2^k tA)^* x^*\rangle|$$

where $g(z) = (h_s^{\pm\nu}(z))^{\frac{1}{2}}$. Suppose C is the U-boundedness constant of $\{F(z) : z \in \Sigma_\rho\}$. Then

$$|\langle M_\pm(t)x, x^*\rangle| \leq C \sup_{\epsilon_k=\pm 1} \sup_N \left\| \sum_{|k|\leq N} \epsilon_k g(2^k tA) \right\|^2 .$$

Hence by Lemma 4.1 we have

$$\sup_n \|F_n(A)\| < \infty\ . \ \square$$

Let us apply this to the case of two operators:

Theorem 4.5 *Suppose A, B are sectorial operators, such that A admits a $H^\infty(\Sigma_\sigma)-$calculus and $\omega(B) < \sigma'$. Suppose $f \in H^\infty(\Sigma_\rho \times \Sigma_{\sigma'})$ where $\sigma < \rho < \pi$ is such that $\{f(w, \cdot) : w \in \Sigma_\rho\}$ is contained in $\mathcal{H}(B)$. Suppose further the set $\{f(w, B) : w \in \Sigma_\rho\}$ is U-bounded. Then $f \in \mathcal{H}(A, B)$ (i.e. $f(A, B)$ is a bounded operator).*

Proof. We define $F(w) = f(w, B)$ and note that $F \in H^\infty(\Sigma_\rho; \mathcal{A})$; this follows easily from the integral representation (2.3). Our conditions and Theorem 4.4 ensure that $F \in \mathcal{H}(A; \mathcal{A})$. It is only necessary to check that this implies $f \in \mathcal{H}(A, B)$ and of course $F(A) = f(A, B)$. But this follows directly from (2.3), (2.4) and the remarks thereafter. $\square$

Example. Let us show by example that Theorem 4.5 is close to the best possible. Let B be a sectorial operator on X. Suppose $0 < \sigma < \pi$ and consider the space $L_2(\{-1, 1\}^{\Sigma_\sigma}; X)$ where $\{-1, 1\}^{\Sigma_\sigma}$ has the usual product measure. Denote by ϵ_z the co-ordinate maps for $z \in \Sigma_\sigma$. Let Rad X denote the closed linear span of the functions $\{\epsilon_z \otimes x : z \in \Sigma_\sigma,\ x \in X\}$. We define $\tilde{B} = I \otimes B$ on $L_2(X)$ and restrict it to the subspace Rad X which is invariant. We define A on Rad X by

$$A\left(\sum_{z\in\Sigma_\sigma} \epsilon_z x_z\right) = \sum_{z\in\Sigma_\sigma} z\epsilon_z x_z$$

with domain consisting of all $\sum \epsilon_z x_z \in L_2$ so that $\sum z\epsilon_z x_z \in L_2$.

Clearly A has an $H^\infty(\Sigma_\sigma)-$calculus and $f(A, \tilde{B})$, for some $f \in H^\infty(\Sigma_\sigma \times \Sigma_{\sigma'})$ with $\sigma' > \omega(B)$, is bounded if and only if the operators $f(z, B)$, $z \in \Sigma_\sigma$, exist in $B(X)$ and form a R-bounded set.

We remark that the reader who prefers separable spaces can easily modify this example when X is separable to replace Rad X by a separable subspace (just take a dense countable subset of Σ_σ).

5. R-boundedness and the functional calculus

We now consider strengthenings of the boundedness conditions in the definition of sectoriality. Let A be a sectorial operator and let $\omega(A)$ denote the infimum of all σ so that A is of type σ. We will say that A is *R-sectorial, (respectively WR-sectorial, respectively U-sectorial)* if there exists $0 < \sigma < \pi$ so that the family of operators $\{\lambda R(\lambda, A) : |\arg\lambda| > \sigma\}$ is R-bounded (respectively WR-bounded, respectively U-bounded). We then define $\omega_R(A)$, (respectively $\omega_{WR}(A)$, respectively $\omega_U(A)$) to be the infimum of all such σ. We will say A is *$H^\infty-$sectorial (respectively, $RH^\infty-$sectorial, respectively $WRH^\infty-$sectorial)* if there exists a $0 < \sigma < \pi$ so that A admits an $H^\infty(\Sigma\sigma)-$calculus (respectively, such that the set $\{f(A) : \|f\|_{H^\infty(\Sigma_\sigma)} \leq 1\}$ is R-bounded, respectively such that the set $\{f(A) : \|f\|_{H^\infty(\Sigma_\sigma)} \leq 1\}$ is WR-bounded). The infimum of all such σ is denoted $\omega_H(A)$ (respectively $\omega_{RH}(A)$, respectively $\omega_{WRH}(A)$).

There are certain obvious and trivial relationships between these concepts. Clearly R-sectorial implies WR-sectorial implies U-sectorial and whenever these concepts are defined, $\omega_R(A) \geq \omega_{WR}(A) \geq \omega_U(A) \geq \omega(A)$. Similarly $RH^\infty-$sectorial implies $WRH^\infty-$sectorial implies $H^\infty-$sectorial and $\omega_{RH}(A) \geq \omega_{WRH}(A) \geq \omega_H(A) \geq \omega(A)$.

We now turn to less trivial observations:

Proposition 5.1 *Suppose A is $H^\infty-$sectorial and U-sectorial. Then $\omega_H(A) \leq \omega_U(A)$.*

Proof. Let us assume that $\{\lambda R(\lambda, A) : |\arg\lambda| \geq \nu\}$ is U-bounded with constant K where $\nu > \omega(A)$, and that $\sigma > \nu$. We will show that A admits an $H^\infty(\Sigma_\sigma)-$calculus. We use Proposition 4.3. Fix some $0 < s < 1$. We can assume that there exists $\rho > \sigma$ so that

$$\sup_N \sup_{\epsilon_k=\pm 1} \sup_{t>0} \left\| \sum_{|k|\le N} \epsilon_k h_s^{\pm\rho}(2^k tA) \right\| = M < \infty$$

and so that A admits an $H^\infty(\Sigma_\tau)-$calculus for some $\tau < \rho$.

Now suppose $x \in X$ and $x^* \in X^*$. Then for any N and $\epsilon_j = \pm 1$ we have

$$\left| \left\langle \sum_{|k|\le N} \epsilon_k h_s^\nu(2^k tA)x, x^* \right\rangle \right| \le M\|x\|\|x^*\| + \sum_{|k|\le N} |\langle (h_s^\nu(2^k tA) - h_s^\rho(2^k tA))x, x^*\rangle|.$$

By the resolvent equation,

$$h_s^\nu(2^k tA) - h_s^\rho(2^k tA) = (e^{i(\rho-\nu)} - 1)2^{-k}t^{-1}e^{i\nu}R(2^{-k}t^{-1}e^{i\nu}, A)h_s^\rho(2^k tA).$$

Since A has an $H^\infty(\Sigma_\tau)-$calculus we can define $g(z) = (h_s^\rho(z))^{\frac{1}{2}}$ and note that

$$\sup_N \sup_{\epsilon_k=\pm 1} \left\| \sum_{|k|\le N} \epsilon_k g(2^k tA) \right\| \le C$$

where C is independent of t. Thus, by the U-boundedness of $\{\lambda R(\lambda, A) : \arg\lambda = \nu\}$

$$\sum_{|k|\le N} |\langle 2^{-k}t^{-1}R(2^{-k}t^{-1}e^{i\nu}, A)g(2^k tA)x, g(2^k tA)^* x^*\rangle| \le KC^2\|x\|\|x^*\|.$$

It follows that

$$\left| \left\langle \sum_{|k|\le N} \epsilon_k h_s^\nu(2^k tA)x, x^* \right\rangle \right| \le (M + 2KC^2)\|x\|\|x^*\|$$

and this gives

$$\sup_N \sup_{\epsilon_k=\pm 1} \sup_{t>0} \left\| \sum_{|k|\le N} \epsilon_k h_s^\nu(2^k tA) \right\| \le M + 2KC^2 < \infty.$$

Combined with a similar estimate for $-\nu$ we obtain the result by using Proposition 4.3. □

In order to study an analytic semigroup with generator $(-A)$ it is of particular interest to know that $\omega_H(A) < \frac{\pi}{2}$. Therefore we use Proposition 5.1 to improve on a result of X.T. Duong ([19], see also [25]).

Corollary 5.2 *Let $(-A)$ generate an analytic contractive and positive semi-group on $L_p(\Omega,\mu)$ for some $1<p<\infty$. Then $\omega_H(A)<\frac{\pi}{2}$.*

Proof. It is shown in [25] that $\omega_H(A)<\pi$ and in [43], Section 5, that $\omega_R(A)<\frac{\pi}{2}$. Hence we can apply Proposition 5.1. □

We remark that it is an open problem (cf. [14]) whether $\omega_H(A)=\omega(A)$ whenever A is $H^\infty-$sectorial. The next Theorem gives some results in this direction.

Theorem 5.3 *Suppose A is an $H^\infty-$sectorial operator on a Banach space X. Then:*

1. *If X has property (α) then A is $RH^\infty-$sectorial and $\omega_H(A)=\omega_{RH}(A)$ $=\omega_R(A)=\omega_U(A)$.*
2. *If X has property (A) then A is $WRH^\infty-$sectorial and $\omega_H(A)=\omega_{WRH}(A)$ $=\omega_{WR}(A)=\omega_U(A)$.*
3. *If X has property (Δ) then A is R-sectorial and $\omega_H(A)=\omega_R(A)=\omega_U(A)$.*

Proof. (1) Assume that A admits an $H^\infty(\Sigma_\sigma)$-calculus. Suppose $\sigma<\nu<\pi$. Suppose $0<s<1$ and let $g_\pm(z)=(h_s^{\pm\nu}(z))^{\frac{1}{2}}$. We then can argue by Lemma 4.1 that

$$\sup_N \sup_{\epsilon_k=\pm1}\left\|\sum_{k=-N}^{N}\epsilon_k g_\pm(2^k tA)\right\|\le M<\infty$$

independent of t. Hence by Theorem 3.3 the family $\{\sum_{|k|\le N}\alpha_k h_s^{\pm\nu}(2^k tA)\}$ is R-bounded with constant bounded independent of t. Now by (4.3) and (4.4) it follows that if $\sigma'>\nu$ then $\{f(A):\|f\|_{H^\infty(\Sigma_{\sigma'})}\le 1\}$ is Rademacher-bounded. Indeed for $f_k\in H_0^\infty(\Sigma_{\sigma'})$ and $x_k\in X$ for $1\le k\le n$, we have

$$\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k f_k(A)x_k\right\|^2\right)^{\frac{1}{2}}\le$$

$$\le 4\max_{\pm}\sup_{t>0}\sup_{N\in\mathbb{N}}\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\left(\sum_{|j|\le N} f_n(e^{\pm i\nu}t^{-1}2^{-j})h_s^{\pm i\nu}(2^j tA)\right)x_k\right\|^2\right)^{\frac{1}{2}}.$$

It follows that $\omega_{RH}(A)=\omega_H(A)$. Now clearly $\omega_U(A)\le\omega_R(A)\le\omega_{RH}(A)$ and so (1) follows from Proposition 5.1.

(2) is very similar and we omit it.

(3) Here we use Lemma 3.4. Suppose A admits an $H^\infty(\Sigma_\sigma)-$calculus and suppose $\sigma'>\nu>\sigma$. We show that the sequence $\{2^k tR(2^k te^{\pm i\nu}):\ k\in\mathbb{Z}\}$ is

Rademacher-bounded with constant independent of t. To do this we note that if $N_1 > N_2$

$$\begin{aligned} &2^{N_1} t R(2^{N_1} t e^{i\nu}, A) - 2^{N_2} t R(2^{N_2} t e^{i\nu}, A) \\ &= - \sum_{j=N_2+1}^{N_1} t 2^{j-1} A R(2^j t e^{i\nu}, A) R(2^{j-1} t e^{i\nu}, A). \end{aligned}$$

Let $k(z) = z(e^{i\nu} - z)^{-1}(e^{i\nu} - 2z)^{-1}$. Let $u(z) = (k(z))^{\frac{1}{2}} \in H^\infty(\Sigma_\sigma)$. We observe that

$$\sup_{N_1 > N_2} \sup_{\epsilon_j = \pm 1} \left\| \sum_{j=N_2+1}^{N_1} \epsilon_j u(2^{-j} t^{-1} A) \right\| \le M < \infty$$

independent of t by Lemma 4.1. Applying Theorem 3.3 yields that

$$\left\{ \sum_{j=N_2+1}^{N_1} k(2^{-j} t^{-1} A) : \; N_1 > N_2 \right\}$$

is Rademacher-bounded with constant independent of t. But this implies that $\{2^{N_1} t R(2^{N_1} t e^{i\nu}, A) - 2^{N_2} t R(2^{N_2} t e^{i\nu}, A) : \; N_1 > N_2\}$ is also Rademacher-bounded with constant independent of t and hence (taking limits) so is $\{2^n t R(2^n t e^{i\nu}, A) : \; n \in \mathbb{Z}\}$. A similar argument for $-\nu$ and an application of Lemma 3.4 shows that $\omega_R(A) \le \nu$. Hence $\omega_R(A) \le \omega_H(A)$. The proof is finished as in (1). □

As a Corollary to the proof of Theorem 5.3 we obtain some additional information on the operator-valued calculus considered in Theorem 4.4:

Corollary 5.4 *Assume that X has property (α) and let $\mathcal{F} \subset \mathcal{L}(X)$ be an R-bounded set. If A is H^∞-sectorial then for any $\sigma > \omega_H(A)$ the set $\{F(A) : \; F \in H^\infty(\Sigma_\sigma, \mathcal{A}), \; F(\zeta) \in \mathcal{F} \; \forall \zeta \in \Sigma_\sigma\}$ is R-bounded.*

Proof. Adapt the proof of Theorem 5.3 (1) using the fact that the set

$$\left\{ \sum_{|k| \le N} T_k h_s^{\pm\nu}(2^k t A) : T_k \in \mathcal{F} \cap \mathcal{A} \right\}$$

is R-bounded, again by Theorem 3.3. □

6. The joint H^∞-calculus and sums of closed operators

First we consider the joint functional calculus.

Theorem 6.1 *Suppose A and B are $H^\infty-$sectorial operators such that B is $WRH^\infty-$sectorial. Then for any $\sigma > \omega_H(A)$ and $\sigma' > \omega_{WRH}(B)$ the pair (A, B) has a joint $H^\infty(\Sigma_\sigma \times \Sigma_{\sigma'})-$calculus.*

Proof. We need only observe that if $f \in H^\infty(\Sigma_\sigma \times \Sigma_{\sigma'})$ then the family $\{f(z, B) : z \in H^\infty(\Sigma_\sigma)\}$ is WR-bounded and then apply Theorem 4.5. □

We can now apply Theorem 5.3 to obtain a result of Lancien, Lancien and Le Merdy [29] (see also [1]). Note that their argument depends on the quite technical discretization developed in [21].

Corollary 6.2 *(Lancien, Lancien and Le Merdy [29]) If X has property (A) then if A and B are $H^\infty-$sectorial operators, for any $\sigma > \omega_H(A)$ and $\sigma' > \omega_H(B)$ the pair (A, B) has a joint $H^\infty(\Sigma_\sigma \times \Sigma_{\sigma'})-$calculus.*

If A and B are sectorial operators on a Banach space X with $\omega(A)+\omega(B) < \pi$ then one can show that the closure $\overline{A + B}$ of $A + B$ with $\mathcal{D}(A + B) = \mathcal{D}(A) \cap \mathcal{D}(B)$ is a sectorial operator with $\omega(A + B) \le \max(\omega(A), \omega(B))$ (see [15]). However, in many applications one needs to show that $A + B$ is already closed on $\mathcal{D}(A) \cap \mathcal{D}(B)$. We now give a criterion for this.

Theorem 6.3 *Suppose A and B are sectorial operators such that A is $H^\infty-$ -sectorial and B is R-sectorial and $\omega_H(A) + \omega_R(B) < \pi$. Then $A + B$ is closed on the domain $\mathcal{D}(A) \cap \mathcal{D}(B)$, there is a constant C such that*

$$\|Ax\| + \|Bx\| \le C\|Ax + Bx\| \qquad x \in \mathcal{D}(A) \cap \mathcal{D}(B) \tag{6.1}$$

and $(A+B)$ is invertible if either A or B is invertible. Furthermore if X has property (α) then $A + B$ is again R-sectorial with $\omega_R(A + B) \le \max(\omega(A), \omega(B))$.

Remarks. (1) Let us compare this theorem with the well-known Dore-Venni Theorem. It is shown in [18] that (6.1) holds if X is a UMD-space and A, B both have bounded imaginary powers (BIP), with

$$\|A^{is}\| \le Ce^{\theta_A|s|}, \quad \|B^{is}\| \le Ce^{\theta_B|s|}$$

where $\theta_A + \theta_B < \pi$. In a UMD-space, R-sectoriality is weaker than (BIP) (see [11]) and $H^\infty-$sectoriality is stronger than (BIP). But in many applications A is an operator known to have an H^∞-calculus, e.g. $A = -\Delta$ or $A = d/dt$ on $L_p(X)$ where X is UMD and $1 < p < \infty$. Thus the weaker assumption on B does lead to more general results, see e.g. Theorem 6.5 below.

(2) Some special cases of Theorem 6.3 where shown in [42] Theorem 5.2, [43] and more recently in [11] (e.g. if A is the generator of a strongly continuous group in a UMD-space).

(3) An extension to non-sectorial sums will be given in a forthcoming paper [40].

Proof. Choose σ, σ' with $\omega_H(A) < \sigma$, $\omega_R(B) < \sigma'$ and $\sigma + \sigma' < \pi$. The function $f(w,z) = w(w+z)^{-1}$ is in $H^\infty(\Sigma_\sigma \times \Sigma_{\sigma'})$ and the set $f(w,B) = -wR(-w,B)$ for $w \in \Sigma_\sigma$ is an R-bounded family. Applying Theorem 4.5 we have $f \in \mathcal{H}(A,B)$. We can see this implies (6.1) either by applying Proposition 2.7 in [29] or by the following simple direct argument based on our construction of the functional calculus. Defining φ_n as in (2.1) we note that $A\varphi_n(A)^2$ and $B\varphi_n(B)^2$ are bounded operators since $z\varphi_n(z)^2 \in H_0^\infty(\Sigma_\tau)$ for any $\tau < \pi$. Now if $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$ we have

$$f(A,B)(A+B)\varphi_n(A)^2\varphi_n(B)^2x = \varphi_n(A)^2\varphi_n(B)^2Ax.$$

Thus

$$\|\varphi_n(A)^2\varphi_n(B)^2Ax\| \le C\|\varphi_n(A)^2\varphi_n(B)^2(A+B)x\|$$

where $C = \|f(A,B)\|$. Letting $n \to \infty$ yields the result.

Now assume X has property (α). For $\max(\sigma,\sigma') < \rho < \pi$ and $|\arg\mu| \ge \rho$ consider the functions $f_\mu(w,z) = \mu(\mu - w - z)^{-1} \in H^\infty(\Sigma_\sigma \times \Sigma_{\sigma'})$. Note that

$$f_\mu(w,B) = \frac{\mu}{\mu - w}((\mu - w)R(\mu - w, B)).$$

Since $\mu(\mu - w)^{-1}$ is bounded uniformly for $|\arg\mu| \ge \rho$ and $w \in \Sigma_\sigma$ and also $|\arg(\mu - w)| \ge \sigma'$ this collection of operators in R-bounded. Now, Corollary 5.4 yields that the set $f_\mu(A,B) = \mu R(\mu, A+B)$ is R-bounded for $|\arg\mu| \ge \rho$. □

Applying Theorem 5.3 gives:

Corollary 6.4 *Suppose X has property (Δ) (e.g. if X has analytic (UMD)). Suppose A and B are H^∞−sectorial operators such that $\omega_H(A) + \omega_H(B) < \pi$. Then $A + B$ is closed on the domain $\mathcal{D}(A) \cap \mathcal{D}(B)$ and (6.1) holds.*

Example. We now show by example that both Corollaries 6.2 and 6.4 are nearly optimal. To do this we let (ϵ_j) and (η_k) be as before two sequences of mutually independent Rademachers on some probability space $(\Omega, \mathbb{P})$. We define $\mathrm{Rad}_2(X)$ to be the subspace of $L_2(\Omega; X)$ spanned by functions of the form $\epsilon_j\eta_k x$ for $j,k \in \mathbb{N}$ and $x \in X$. Let A be defined by

$$A\left(\sum_{j,k}\epsilon_j\eta_k x_{jk}\right) = \sum_{j,k}(2j+1)!\epsilon_j\eta_k x_{jk}$$

with the natural domain and let

$$B\left(\sum_{j,k}\epsilon_j\eta_k x_{jk}\right)=\sum_{j,k}(2k)!\epsilon_j\eta_k x_{jk}$$

with its natural domain.

Both A and B are H^∞−sectorial with $\omega_H(A)=\omega_H(B)=0$. Clearly $f\in\mathcal{H}(A,B)$ if and only if

$$\sum_{j,k}\epsilon_j\eta_k x_{jk}\to\sum_{j,k}f((2j+1)!,(2k)!)\epsilon_j\eta_k x_{jk}$$

is a bounded operator.

If (A,B) has a joint $H^\infty(\Sigma_\sigma\times\Sigma_{\sigma'})$−calculus then since the map $f\to(f((2j+1)!,(2k)!))_{j,k}$ maps $H^\infty(\Sigma_\sigma\times\Sigma_{\sigma'})$ onto $\ell_\infty(\mathbb{N}^2)$ we must have that X has property (α) (and hence so does $\mathrm{Rad}_2(X)$.)

If $f(w,z)=w(w+z)^{-1}$ defines a bounded operator then by a limiting argument the map

$$\sum_{j,k}\epsilon_j\eta_k x_{jk}\to\sum_{k\le j}\epsilon_j\eta_k x_{jk}$$

is also bounded, i.e. X (and $\mathrm{Rad}_2(X)$) has property (Δ). □

Let us note that we can recapture the main result of [42] on maximal regularity. Suppose $-A$ is the generator of an analytic semigroup i.e. $\omega(A)<\frac{\pi}{2}$. Then A has maximal L_p− regularity for $1<p<\infty$ if the Cauchy problem

$$y'(t)+Ay(t)=f(t),\qquad t>0,\quad y(0)=0 \tag{6.2}$$

has for every $f\in L_p(\mathbb{R}_+,X)$ a solution $y:\mathbb{R}_+\to X$ which satisfies the estimate

$$\|y'\|_{L_p(\mathbb{R}_+,X)}+\|Ay\|_{L_p(\mathbb{R}_+,X)}\le C\|f\|_{L_p(\mathbb{R}_+,X)}. \tag{6.3}$$

If we denote by $\tilde{B}$ the derivative d/dt on $\tilde{X}=L_p(\mathbb{R}_+,X)$ and by $\tilde{A}$ the extended operator $(\tilde{A}f)(t)=Af(t)$ then (6.3) is equivalent to:

$$\|\tilde{A}y\|_{\tilde{X}}+\|\tilde{B}y\|_{\tilde{X}}\le C\|(\tilde{A}+\tilde{B})y\|_{\tilde{X}}.$$

Thus we can apply Theorem 6.3:

Theorem 6.5 *Suppose X is a Banach space with (UMD) and suppose A is an R-sectorial operator on X with $\omega_R(A)<\frac{\pi}{2}$. Then A has maximal L_p-regularity for* $1<p<\infty$.

Proof. Since X has (UMD), we have that $\tilde{B}$ is H^∞−sectorial and $\omega_H(\tilde{B})=\frac{\pi}{2}$ (see e.g. [25]). It is easy to see that $\omega_R(\tilde{A})=\omega_R(A)$ so the result is a consequence of Theorem 6.3. □

Remarks. (1) If X has (UMD) and A is $H^\infty-$sectorial then $\omega_H(A) < \frac{\pi}{2}$ implies maximal L_p-regularity by Corollary 6.4.

(2) It is shown in [26] that any non-Hilbertian Banach space with an unconditional basis admits a sectorial operator A with $\omega(A) < \frac{\pi}{2}$ but failing maximal L_p-regularity.

(3) It is shown in [42] that the condition $\omega_R(A) < \frac{\pi}{2}$ actually characterizes L_p-maximal regularity. The operator approach to maximal regularity has a long history, see e.g. [15],[16], [22] and [43].

7. L_1-spaces and $C(K)$-spaces

We recall that a GT-space is a Banach space X so that the Grothendieck theorem is valid, i.e. if $T : X \to \ell_2$ is bounded then T is absolutely summing and for some C independent of T, $\pi_1(T) \le C\|T\|$; see [39] for a full discussion. Examples of such spaces are L_1-spaces, their quotients by reflexive spaces and L_1/H_1 ([7], [9] and [39]).

Proposition 7.1 *Suppose X is a GT-space of cotype 2 and that A is a $H^\infty-$sectorial operator on X. Then if $\omega_H(A) < \nu$ and $0 < s < 1$ there is a constant C so that if $x \in X$,*

$$C^{-1}\|x\| \le \int_{\Gamma_\nu} \left\|A^s R(\zeta, A)x\right\| \frac{|d\zeta|}{|\zeta|^s} \le C\,\|x\| \qquad x \in X. \tag{7.1}$$

Proof. This is just a version of a classical result of Lindenstrauss and Pełczyński [35] on uniqueness of unconditional bases in ℓ_1. Let $g(z) = (h_s^\nu(z))^{\frac{1}{2}}$. Then for some constant C_0 independent of x, t we have if $|\alpha_k| \le 1$ for $|k| \le N$,

$$\left\|\sum_{|k|\le N} \alpha_k g(2^k tA)x\right\| \le C_0\,\|x\| \qquad x \in X, t > 0$$

by Lemma 4.1. Now since X has cotype 2, there exists a constant C_1 independent of x, t, so that

$$\left(\sum_{k\in\mathbb{Z}} \|g(2^k tA)x\|^2\right)^{\frac{1}{2}} \le C_1\|x\| \qquad x \in X, t > 0. \tag{7.2}$$

Now suppose $x \in X$ and $t > 0$ and choose by the Hahn-Banach theorem $x_k^* \in X^*$ with $\|x_k^*\| = 1$ and $\langle h_s^\nu(2^k tA)x, x_k^*\rangle = \|h_s^\nu(2^k tA)x\|$. Consider the map $S_x : X \to \ell_2(\mathbb{Z})$ defined by $S_x y = (\langle g(2^k tA)y, x_k^*\rangle)_{k\in\mathbb{Z}}$. By (7.2) S is bounded and $\|S_x\| \le C_1$. Hence for some constant C_2 we have $\pi_1(S_x) \le C_2$.

Now

$$\sum_{|k|\le N}\|h_s^\nu(2^k tA)x\| = \sum_{|k|\le N}\langle h_s^\nu(2^k tA)x, x_k^*\rangle$$
$$\le \sum_{|k|\le N}\|S_x g(2^k tA)x\|$$
$$\le C_2 \sup_{|\alpha_k|\le 1}\left\|\sum_{|k|\le N}\alpha_k g(2^k tA)x\right\|$$
$$\le C_2 C_0\|x\|.$$

If we rewrite the integral in (7.1) in a similar way as in (4.3) and (4.4) and integrate the last inequality for $1 \le t \le 2$, we obtain the upper estimate in (7.1). The lower estimate follows from the equation:

$$x = \lim_{n\to\infty}\frac{-1}{2\pi i}\int_{\Gamma_\nu}\varphi_n(\zeta)\zeta^{-s}A^s R(\zeta, A)x\, d\zeta. \ \square$$

The same argument yields:

Proposition 7.2 *Suppose X^* is a GT-space of cotype 2 and that A is a H^∞−sectorial operator on X. Then if $\omega_H(A) < \nu$ and $0 < s < 1$ there is a constant C so that if $x \in X$,*

$$\frac{1}{C}\|x^*\| \le \int_{\Gamma_\nu}\|(A^s R(\zeta, A))^* x^*\|\frac{|d\zeta|}{|\zeta|^s} \le C\|x^*\| \qquad x^* \in X^*.$$

Remarks. Let us point out that Proposition 7.1 implies that very few operators on L_1 can have an H^∞−calculus. This statement can be made much more precise but since the techniques required are rather specialized we will defer this to a later paper and instead note the following simple application, which effectively shows that no reasonable differential operator on L_1 can have an H^∞−calculus.

Proposition 7.3 *Suppose X is a GT-space of cotype 2 and A is an H^∞−sectorial operator on X. If Y is an infinite-dimensional closed reflexive subspace of $\mathcal{D}(A)$ (with the graph norm) then A is bounded on Y (and so Y is closed in X).*

Proof. We use the notation of Proposition 7.1. In particular notice that (7.1) implies the existence of an isomorphic embedding $T : X \to L_1(\Gamma_\nu, |d\zeta|;\ X)$ defined by

$$Tx(\zeta) = |\zeta|^{-\frac{1}{2}}A^{\frac{1}{2}}R(\zeta, A)x.$$

Fix $0 \ne \lambda \in \Gamma_\nu$. Then $R(\lambda, A)$ maps X isomorphically onto $\mathcal{D}(A)$ (with the graph norm). Let $Y_0 = R(\lambda, A)^{-1}Y$; then Y_0 is an infinite-dimensional reflexive subspace of X. Then (since Y_0 contains no copy of ℓ_1) the set $\{\|Ty(\zeta)\|;\ \|y\| \le$

$1, \; y \in Y_0\}$ is equi-integrable. We show that this implies that the (bounded) operator $A^{\frac{1}{2}}R(\lambda, A)$ satisfies a lower bound on Y_0. Indeed if not there exists a sequence (y_n) in Y_0 so that $\|y_n\| = 1$ and $\|A^{\frac{1}{2}}R(\lambda, A)y_n\| \to 0$. But, by the resolvent equation, $\|A^{\frac{1}{2}}R(\zeta, A)y_n\| \to 0$ for all $\zeta \in \Gamma_\nu \setminus \{0\}$. Now by (7.1) and equi-integrability, we have $\|y_n\| \to 0$ which gives a contradiction. Now applying the same argument to $Y_1 = A^{\frac{1}{2}}R(\lambda, A)Y_0$ gives a lower bound on $AR(\lambda, A)^2$ on Y_0. Thus $R(\lambda, A)$ has a lower bound on Y_0. Since $Y = R(\lambda, A)Y_0$ this implies the result. □

Let us note, with respect to differential operators, that embeddings of Sobolev spaces into $L_1(\Omega)$ are Dunford-Pettis operators.

In view of this comparative rarity of $H^\infty-$sectorial operators in this setting, it is not surprising that we may substantially improve the results of this paper for these special spaces. Our first deduction is that for these special spaces, Theorem 4.5 can be improved by removing the U-boundedness assumption.

Theorem 7.4 *Suppose X is a Banach space such that either X or X^* is a GT-space of cotype 2. Suppose A, B are sectorial operators such that A admits an $H^\infty-$calculus. Suppose $f \in H^\infty(\Sigma_\sigma \times \Sigma_{\sigma'})$ where $\sigma > \omega_H(A)$ and $\sigma' > \omega(B)$. Suppose that for each $z \in \Sigma_\sigma$, $f_z(w) = f(z, w) \in \mathcal{H}(B)$ and*

$$\sup_{z \in \Sigma_\sigma} \|f(z, B)\| < \infty.$$

Then $f \in \mathcal{H}(A, B)$ (i.e. $f(A, B)$ is bounded).

Proof. Let us assume that X is a GT-space of cotype 2, the other case is similar. By Proposition 4.2 if $f_n(w, z) = \varphi_n(w)\varphi_n(z)f(w, z)$ we can write

$$\begin{aligned} f_n(A, B)x &= \frac{-1}{2\pi i}\int_{\Gamma_\nu} \zeta^{-s} f_n(\zeta, B)A^s R(\zeta, A)x \, d\zeta \\ &= \frac{-1}{2\pi i}\int_{\Gamma_\nu} \zeta^{-s}\varphi_n(\zeta)f(\zeta, B)A^s R(\zeta, A)\varphi_n(B)x \, d\zeta \end{aligned}$$

and this leads immediately to $\sup_n \|f_n(A, B)\| < \infty$ which implies the boundedness of $f(A, B)$. □

It is clear now if A is $H^\infty-$sectorial on L_1 with $\omega_H(A) < \pi/2$ then one can apply the above result to conclude that A has L_1-maximal regularity. More generally we have the following result (suggested by a question of Gilles Lancien):

Theorem 7.5 *Suppose either X or X^* is a GT-space of cotype 2 (e.g. if $X = L_1,\; C(K)$ or the disk algebra $A(\mathbb{D})$). If A is an $H^\infty-$sectorial operator on X with $\omega_H(A) < \pi/2$ then A has L_p-maximal regularity for $1 < p < \infty$.*

Proof. Let us prove this for X a GT-space of cotype 2 as the other case is dual. Suppose $\omega_H(A) < \nu < \pi/2$ and $0 < s < 1$. If $u > 0$ then for $x \in X$ by Proposition 4.2

$$Ae^{-uA}x = \frac{-1}{2\pi i}\int_{\Gamma_\nu} \zeta^{1-s}e^{-u\zeta}A^sR(\zeta, A)x\,d\zeta.$$

Now suppose $f \in L_p(\mathbb{R}, X)$. We will estimate the norm of S_δ where

$$S_\delta f(v) = \int_\delta^\infty Ae^{-uA}f(v-u)du.$$

If we let $G(t) = A^sR(|t|e^{i(\operatorname{sgn} t)\nu}, A)$ then we have an estimate

$$\|Ae^{-uA}x\| \le C_0\int_{-\infty}^\infty |t|^{1-s}e^{-cu|t|}\|G(t)x\|dt$$

where $c = \cos\nu > 0$ and so

$$\|S_\delta f(v)\| \le C_0\int_{-\infty}^\infty\int_{-\infty}^{v-\delta} |t|^{1-s}e^{c(u-v)|t|}\|G(t)f(u)\|du\,dt.$$

Now if $g \in L_q(\mathbb{R})$ with $g \ge 0$, where $\frac{1}{p}+\frac{1}{q} = 1$, we have

$$\begin{aligned}\int_{u+\delta}^\infty e^{c(u-v)|t|}g(v)\,dv &\le c|t|\int_u^\infty\int_v^\infty e^{c(u-w)|t|}g(v)\,dw\,dv\\ &= c|t|\int_u^\infty (w-u)e^{c(u-w)|t|}\frac{1}{w-u}\int_u^w g(v)\,dv\,dw\\ &\le \frac{1}{c|t|}\sup_{w>u}\frac{1}{w-u}\int_u^w g(v)\,dv\end{aligned}$$

Hence the left-hand side is estimated by $C_1|t|^{-1}(\mathcal{M}g)(u)$ where $\mathcal{M}$ is the Hardy-Littlewood maximal function.

Hence

$$\int_{-\infty}^\infty g(v)\|S_\delta f(v)\|dv \le C_0C_1\int_{-\infty}^\infty\int_{-\infty}^\infty |t|^{-s}\mathcal{M}g(u)\|G(t)f(u)\|du\,dt.$$

However Proposition 7.1 implies that we have an estimate

$$\int_{-\infty}^\infty |t|^{-s}\|G(t)f(u)\|dt \le C_2\|f(u)\|.$$

Substituting in we have

$$\int_{-\infty}^\infty g(v)\|S_\delta f(v)\|dv \le C_0C_1C_2\int_{-\infty}^\infty \mathcal{M}g(u)\|f(u)\|du$$

and since M is bounded on L_q this establishes a uniform bound on the operators S_δ. Letting $\delta \to 0$ yields the result. □

References

1. D. Albrecht, E. Franks, A. McIntosh, Holomorphic functional calculi and sums of operators. Bull. Austral. Math. Soc. **58** (1998), 291–305
2. H. Amann, M. Hieber, G. Simonett, Bounded H_∞-calculus for elliptic operators. Differential Integral Equations **7** (1994), 613–653
3. P. Auscher, A. McIntosh, A. Nahmod, The square root problem of Kato in one dimension, and first order elliptic systems. Indiana Univ. Math. J. **46** (1997), 659–695
4. E. Berkson, T.A. Gillespie, Spectral decompositions and harmonic analysis on UMD spaces, Studia Math. **112** (1994) 13–49
5. G. Blower, A multiplier characterization of analytic UMD spaces. Studia Math. **96** (1990), 117–124
6. J. Bourgain, Some remarks on Banach spaces in which martingale differences are unconditional, Arkiv Math. **21** (1983) 163–168
7. J. Bourgain, New Banach space properties of the disc algebra and H^∞. Acta Math. **152** (1984), 1–48
8. J. Bourgain, Vector-valued singular integrals and the H^1-BMO duality. Probability theory and harmonic analysis (Cleveland, Ohio, 1983), 1–19, Monographs Textbooks Pure Appl. Math., 98, Dekker, New York, 1986
9. J. Bourgain, W.J. Davis, Martingale transforms and complex uniform convexity. Trans. Amer. Math. Soc. **294** (1986), 501–515
10. P. Clément, B. de Pagter, F. Sukochev, H. Witvliet, Schauder decomposition and multiplier theorems, to appear
11. P. Clément, J. Prüss, An operator-valued transference principle and maximal regularity on vector-valued L_p-spaces, In: Lumer, G., Weis, L. (eds.) Proc. of the 6th. International Confernce on Evolution equations. Marcel Dekker, 2000 (to appear)
12. T. Coulhon, X.T. Duong, Maximal regularity and kernel bounds: observations on a theorem by Hieber and Prüss. Adv. Differential Equations **5** (2000) 343–368
13. T. Coulhon, D. Lamberton, Régularité L^p pour les équations d'évolution, Séminaire d'Analyse Fonctionnelle Paris **VI–VII** (1984–85), 155–165
14. M. Cowling, I. Doust, A. McIntosh, A. Yagi, Banach space operators with a bounded H^∞-calculus, J. Austral. Math. Soc. **60** (1996) 51–89
15. G. Da Prato, P. Grisvard, Sommes d'opérateurs linéaires et équations différentielles opérationnelles, J. Math. Pures Appl. **54** (1975), 305–387
16. G. Dore, L^p regularity for abstract differential equations (In "Functional Analysis and related topics", editor: H. Komatsu), Lect. Notes in Math. 1540, Springer Verlag (1993)
17. G. Dore, Maximal regularity in L^p spaces for an abstract Cauchy problem. Adv. Differential Equations **5**(1–3) (2000), 1-3, 293–322
18. G. Dore, A. Venni, On the closedness of the sum of two closed operators, Math. Z. **196** (1987), 189–201
19. X.T. Duong, H^∞ functional calculus of second order elliptic partial differential operators on L^p spaces, Proc. Centre for Math. Analysis ANU, Canberra **24** (1989), 91–102
20. X.T. Duong, A. McIntosh, Functional calculi of second-order elliptic partial differential operators with bounded measurable coefficients. J. Geom. Anal. **6** (1996), 181–205
21. E. Franks, A. McIntosh, Discrete quadratic estimates and holomorphic functional calculi in Banach spaces. Bull. Austral. Math. Soc. **58** (1998), 271–290
22. P. Grisvard, Equations différentielles abstraites, Ann. Sci. Ecole Norm. Sup. (4) **2** (1969), 311–395
23. U. Haagerup, G. Pisier, Factorization of analytic functions with values in noncommutative L_1-spaces and applications. Canad. J. Math. **41** (1989), 882–906

24. M. Hieber, J. Prüss, Heat kernels and maximal L^p-L^q estimates for parabolic evolution equations. Comm. Partial Differential Equations **22** (1997), 1647–1669
25. M. Hieber, J. Prüss, Functional calculi for linear operators in vector-valued L^p-spaces via the transference principle. Adv. Differential Equations **3** (1998), 847–872
26. N.J. Kalton, G. Lancien, A solution to the problem of L^p−maximal regularity, Math. Z. **235** (2000), 559–568
27. N.J. Kalton, L. Weis, in preparation
28. P. Kunstmann, Maximal L_p−regularity for second order elliptic operators with uniformly continuous coefficients on domains, preprint
29. F. Lancien, G. Lancien, C. Le Merdy, A joint functional calculus for sectorial operators with resolvent, Proc. London Math. Soc. **77** (1998) 387–414
30. G. Lancien, Counterexamples concerning sectorial operators, Archiv Math. **71** (1998) 388–398
31. G. Lancien, C. Le Merdy, A generalized H^∞-calculus for operators on subspaces of L_p and applications to maximal regularity, Illinois J. Math. **42** (1998) 470–480
32. C. Le Merdy, Counterexamples on L^p-maximal regularity, Math. Zeit. **230** (1999), 47–62
33. C. Le Merdy, H^∞-functional calculus and applications to maximal regularity, Publications Mathématiques de Besançon, Fasicule 16 (1999)
34. J. Lindenstauss, L. Tzafriri, Classical Banach spaces I, Springer-Berlin (1977)
35. J. Lindenstrauss, A. Pełczyński, Absolutely summing operators in L_p-spaces and their applications. Studia Math. **29** (1968) 275–326
36. A. McIntosh, Operators which have an H_∞ functional calculus. Miniconference on operator theory and partial differential equations (North Ryde, 1986), 210–231, Proc. Centre Math. Anal. Austral. Nat. Univ., 14, Austral. Nat. Univ., Canberra, 1986
37. G. Pisier, Some results on Banach spaces without local unconditional structure, Comp. Math. **37** (1978) 3–19
38. G. Pisier, Holomorphic semigroups and the geometry of Banach spaces, Annals of Math. (2) **115** (1982), 375–392
39. G. Pisier, Factorization of linear operators and geometry of Banach spaces. CBMS Regional Conference Series in Mathematics, 60e American Mathematical Society, Providence, R.I., 1986
40. Z. Strkalj, Sums of non-sectorial operators, in preparation
41. H. Tanabe, Equations of evolution, Pitman, London-San Francisco-Melbourne, 1979
42. L. Weis, Operator-valued Fourier multiplier theorems and maximal regularity, to appear
43. L. Weis, A new approach to maximal L_p-regularity. In: Lumer, G., Weis, L. (eds.) Proc. of the 6th. International Confernce on Evolution equations, Marcel Dekker, 2000 (to appear)