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Pelczynski's Property (V) on $C(\Omega, E)$ Spaces

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I. Introduction

Let E and F be Banach spaces and suppose $T: E \to F$ is a bounded linear operator. T is said to be *unconditionally converging* if whenever $\sum_{n=1}^{\infty} x_n$ is a weakly unconditionally Cauchy (w.u.c.) series then $\sum_{n=1}^{\infty} Tx_n$ is an unconditionally convergent series. The Banach space E has Pelczynski's property (V) if every unconditionally converging operator on E is weakly compact. Pelczynski [8] showed that if Ω is a compact Hausdorff space, then $C(\Omega)$ the space of continuous scalar-valued functions on Ω, has property (V). He also introduced property (u). A Banach space E has property (u) if whenever (x_n) is a weakly Cauchy sequence there is a w.u.c. $\sum_{i=1}^{\infty} u_i$ so that $x_n - \sum_{i=1}^{n} u_i \to 0$ weakly. Any order-continuous Banach lattice (in particular, any Banach space with an unconditional basis) has property (u) [7, Vol. II, p. 31]. A Banach space which has property (u) and contains no copy of l_1 has property (V). This follows from [8, Proposition 2, p. 642] and [10, Main Theorem, p. 2411]. It has been asked (Pelczynski [8, Remark 1, p. 645], see also Diestel and Uhl [4, p. 183]) whether if Ω is a compact Hausdorff space, $C(\Omega, E)$ the space of continuous E-valued functions on Ω has (V) whenever E has (V). Our main result is that if E has (u) and contains no copy of l_1 then $C(\Omega, E)$ has property (V). This covers and strengthens practically all known cases.

First, let us fix some notations and terminology. Recall that a series $\sum_{n=1}^{\infty} x_n$ in a Banach space E is said to be weakly unconditionally Cauchy (w.u.c.) if for every $x^* \in E^*$, the series $\sum_{n=1}^{\infty} x^*(x_n)$ is unconditionally convergent. There are many other criteria for w.u.c. that are quite useful, for instance, it can be shown (see [3, Theorem 6, p. 44]) that a series $\sum_{n=1}^{\infty} x_n$ is w.u.c. if and only if

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$\sup_{n\in\mathbb{N}} \sup_{\sigma_i=\pm 1} \left\| \sum_{i=1}^{n} \sigma_i x_i \right\| < \infty$. If E and F are Banach spaces, $L(E, F)$ will stand for the space of bounded linear operators from E to F. Finally, any other notation or terminology used and not defined can be found in [4] and [7].

II. Some Preliminaries Lemmas

Lemma 1. *Let E be a Banach space containing no isomorphic copy of l_1. If E has property (u), then for each $e^* \in E^*$, there exists a w.u.c. series $\sum_{i=1}^{\infty} u_i$ so that:*

(a) $\|u_1 + u_2 + \ldots + u_n\| \leqq 1 + \frac{1}{n}, \quad n \geqq 1$

(b) $\sum_{i=1}^{\infty} e^*(u_i) = \|e^*\|$.

Proof. Let $e^* \in E^*$. Select any sequence (x_n) in E so that $\|x_n\| \leqq 1$ and $e^*(x_n) \to \|e^*\|$. Then by Rosenthal's theorem [10] (x_n) has a weakly Cauchy subsequence (e_n). Since E has property (u), we can find a w.u.c. series $\sum_{i=1}^{\infty} v_i$ so that $e_n - \sum_{i=1}^{n} v_i \to 0$ weakly. Now, by induction we can select an increasing sequence of integers $(p_n : n = 0, 1, 2, \ldots)$ with $p_0 = 0$ and $c_j \geqq 0$ $(j = 1, 2, \ldots)$ so that

$$\sum_{j=p_{n-1}+1}^{p_n} c_j = 1, \qquad n = 1, 2, \ldots$$

and

$$\left\| \sum_{j=p_{n-1}+1}^{p_n} c_j \left(\sum_{i=1}^{j} v_i - e_j \right) \right\| \leqq \frac{1}{n}.$$

This follows from Mazur's theorem since 0 must be in the closed convex hull of $\left\{ e_n - \sum_{i=1}^{n} v_i \right\}_{n=N}^{\infty}$ for any N.

Let

$$s_n = \sum_{j=p_{n-1}+1}^{p_n} c_j \left(\sum_{i=1}^{j} v_i \right).$$

Then $\|s_n\| \leqq 1 + \frac{1}{n}$ and $e^*(s_n) \to \|e^*\|$. Let $s_0 = 0$ and put $u_n = s_n - s_{n-1}$, $n = 1, 2, \ldots$. Then if $\sigma_i = \pm 1$ for $i = 1, 2, \ldots, n$

$$\sum_{i=1}^{n} \sigma_i u_i = \sum_{j=1}^{p_n} t_j v_j,$$

where $|t_j| \leqq 1, j = 1, \ldots, p_n$. Thus $\sum_{i=1}^{\infty} u_i$ is a w.u.c. series as required.

Lemma 2. *Let E be a separable Banach space containing no isomorphic copy of l_1 and with property (u). Then there is a sequence of maps $\theta_n : E^* \to E$ so that each θ_n is* *universally measurable (for the weak*-topology on E^*) and for each $e^* \in E^*$*

(i) $\sum_{i=1}^{\infty} \theta_i(e^*)$ is a w.u.c. series

(ii) $\|\theta_1(e^*)+\ldots+\theta_n(e^*)\| \leqq 1+\frac{1}{n}, \quad n \geqq 1$

(iii) $\sum_{i=1}^{\infty} \langle \theta_i(e^*), e^* \rangle = \|e^*\|$.

Proof. Let V be the closed unit ball of E^*. Then V is a compact metric space in the weak*-topology. Consider the Polish space $E^{\mathbb{N}} \times V$. In $E^{\mathbb{N}} \times V$ let B be the set of $\{(e_n), e^*\}$ so that

$$\sup_{n \in \mathbb{N}} \sup_{\sigma_i = \pm 1} \left\| \sum_{i=1}^{n} \sigma_i e_i \right\| < \infty ,$$

$$\|e_1 + e_2 + \ldots + e_n\| \leqq 1 + \frac{1}{n}, \quad n \geqq 1 ,$$

and

$$\sum_{i=1}^{\infty} e^*(e_i) = \|e^*\| .$$

Noting, in particular, that the norm is weak*-Borel on V we see that B is a Borel subset of $E^{\mathbb{N}} \times V$ and hence is an analytic set [2, Proposition 8.2.3, p. 262]. Define $\psi: B \to V$ by $\psi\{(e_n), e^*\} = e^*$. Then ψ is surjective by Lemma 1. Now by [2, Theorem 8.5.3, p. 286] there exists a universally measurable map $\zeta: V \to B$ so that $\zeta\psi(e^*) = e^*$, for $e^* \in V$. Now for each $n \geqq 1$ define $\theta_n: E^* \to E$ so that if $\|e^*\| \leqq 1$, $\zeta(e^*) = \{(\theta_n(e^*)), e^*\}$ while if $\|e^*\| > 1$,

$$\theta_n(e^*) = \theta_n\left(\frac{e^*}{\|e^*\|}\right).$$

III. The Main Theorem

Theorem 3. *Let E be a Banach space containing no isomorphic copy of l_1 and with property (u). Then if Ω is a compact Hausdorff space, $C(\Omega, E)$ has property (V).*

Proof. First note that since sequences of continuous functions in $C(\Omega, E)$ take their values into a separable subspace of E, it suffices to prove the theorem for a separable Banach space E.

Case 1. Assume Ω is metrizable and let $T: C(\Omega, E) \to X$ be an unconditionally converging operator and let $G: \sum \to L(E, X^{**})$ be its representing measure [4, p. 181–182], where $\sum$ denotes the σ-field of Borel subsets of Ω. Recall that for each $x^* \in X^*$, the measure $G_{x^*}: \sum \to E^*$ defined by $\langle x, G_{x^*}(B) \rangle = \langle x^*, G(B)x \rangle$ is the representing measure of T^*x^*. Since T is unconditionally converging G takes its values in $L(E, X)$ and if (B_n) is a decreasing sequence of Borel subsets of Ω with empty intersection then $\lim_n \|G\|(B_n)=0$, where $\|G\|(B)=\sup\{|G_{x^*}|(B): \|x^*\|\leqq 1\}$ (see [4, p. 182]). Therefore, there is a regular probability measure λ on Ω such that

$$\lim_{\lambda(B)\to 0} \|G\|(B)=0.$$

(This follows for example from [4, I.2.4 and I.2.5]).

Let U be the unit ball of $C(\Omega, E)$ and let $W=\overline{T(U)}$ in X. We will show that W is weakly compact by invoking a theorem of James [6] (cf. also [9]). We need to show that each $x^*\in X^*$ attains a maximum on W.

Let $x^*\in X^*$ with $\|x^*\|\leqq 1$, then

$$\sup_{w\in W} x^*(w)=\|T^*x^*\|.$$

Since G_{x^*} is λ-continuous by [5, Sect. 13, Theorem 5], the measure G_{x^*} has a weak* λ-derivative $h:\Omega\to E^*$ so that for $f\in C(\Omega,E)$

$$\langle f, T^*x^*\rangle=\int_\Omega \langle f(\omega), h(\omega)\rangle d\lambda(\omega)$$

and

$$\|T^*x^*\|=\int_\Omega \|h(\omega)\|\, d\lambda.$$

The fact that E is separable ensures that h is Lusin λ-measurable from Ω to (E^*, weak*), and hence h is weak*-Borel λ-measurable [12, Theorem 5, p. 26].

Now, for each $\omega\in\Omega$, let

$$\psi_n(\omega)=\theta_n(h(\omega)),$$

where θ_n are the maps obtained in Lemma 2.

Each $\theta_n:\Omega\to E$ is universally measurable, and we recall that

$$\sup_n \sup_{\sigma_i=\pm 1}\left\|\sum_{i=1}^n \sigma_i\psi_i(\omega)\right\|=M(\omega)<\infty, \quad \omega\in\Omega,$$

$$\|\psi_1(\omega)+\psi_2(\omega)+\ldots+\psi_n(\omega)\|\leqq 1+\frac{1}{n}, \quad \omega\in\Omega,$$

$$\sum_{i=1}^\infty \langle\psi_i(\omega), h(\omega)\rangle=\|h(\omega)\|, \quad \omega\in\Omega.$$

Note that $\omega\to M(\omega)$ is a λ-measurable function. Since E is separable, by [12, Theorem 5, p. 26] the functions ψ_n and M are Lusin λ-measurable. Let $\varepsilon_n>0$ be any decreasing sequence so that $\sum_{h=1}^\infty \varepsilon_n<\infty$. Choose $\delta_n>0$ a decreasing sequence so that if $\lambda(B)<\delta_n$ then $\|G\|(B)<\frac{1}{4}\varepsilon_n$. For each $n\geqq 1$ let Ω_n be a closed subset of Ω so that $\lambda(\Omega_n)\geqq 1-\delta_n$, each ψ_k is continuous on Ω_n and

$$\sup_{\omega\in\Omega_n} M(\omega)=M_n<\infty.$$

Note that we can assume that (Ω_n) is an increasing sequence.

By the Borsuk-Dugundji theorem [13] there is an extension operator $S_n: C(\Omega_n, E)\to C(\Omega, E)$ so that $\|S_n\|=1$ and $S_nf(\omega)=f(\omega)$ for $f\in C(\Omega_n, E)$ and

$\omega \in \Omega_n$. Let $g_{n,k} = S_n(\psi_k|\Omega_n)$, and let

$$x_{n,k} = T\left(\sum_{j=1}^{k} g_{n,j}\right).$$

Note that for every $n \in \mathbb{N}$ and $k \in \mathbb{N}$

$$\left\|\sum_{j=1}^{k} g_{n,j}\right\| \leqq 1 + \frac{1}{k}$$

so that $\frac{k}{k+1} x_{n,k} \in W$.

Also

$$\left\|\sum_{j=1}^{k} \sigma_j g_{n,j}\right\| \leqq M_n$$

for $k \in \mathbb{N}$ and all $\sigma_j = \pm 1$. Hence as T is unconditionally converging $\lim_{k\to\infty} x_{n,k}$ exists for all $n \geqq 1$. Let $w_n = \lim_{k\to\infty} x_{n,k}$. Then $w_n \in W$ for all $n \geqq 1$.

For $n \in \mathbb{N}$ and $k \in \mathbb{N}$

$$\sum_{j=1}^{k} g_{n,j}(\omega) = \sum_{j=1}^{k} g_{n+1,j}(\omega), \qquad \omega \in \Omega_n,$$

while for $\omega \notin \Omega_n$

$$\left\|\sum_{j=1}^{k} g_{n,j}(\omega) - \sum_{j=1}^{k} g_{n+1,j}(\omega)\right\| \leqq 2\left(1 + \frac{1}{k}\right) \leqq 4.$$

Note that

$$\left\|\int_\Omega \sum_{j=1}^{k} g_{n,j}(\omega) - \sum_{j=1}^{k} g_{n+1,j}(\omega)\, d\lambda(\omega)\right\|$$

$$\leqq \left(\sup_{\omega \in \Omega\setminus\Omega_n} \left\|\sum_{j=1}^{k} g_{n,j}(\omega) - \sum_{j=1}^{k} g_{n+1,j}(\omega)\right\|\right) \|G\|\,(\Omega\setminus\Omega_n) < \varepsilon_n .$$

Hence

$$\|x_{n,k} - x_{n+1,k}\| \leqq \varepsilon_n \quad \text{for all} \quad k \in \mathbb{N}.$$

We conclude then that

$$\|w_n - w_{n+1}\| \leqq \varepsilon_n$$

and hence (w_n) is a convergent sequence. Let $w = \lim_{n\to\infty} w_n$, then $w \in W$. Now for each $n \in \mathbb{N}$

$$\begin{aligned} x^*(w_n) &= \lim_{k\to\infty} x^*(x_{n,k}) \\ &= \sum_{j=1}^{\infty} \langle g_{n,j}, T^*x^* \rangle \\ &= \sum_{j=1}^{\infty} \int_\Omega \langle g_{n,j}(\omega), h(\omega) \rangle\, d\lambda(\omega). \end{aligned}$$

Note that

$$\sum_{j=1}^{\infty} |\langle g_{n,j}(\omega), h(\omega)\rangle| \leqq M_n \|h(\omega)\|, \qquad \omega \in \Omega,$$

so that by the Dominated convergence theorem of Lebesgue,

$$x^*(w_n) = \int_{\Omega} \sum_{j=1}^{\infty} \langle g_{n,j}(\omega), h(\omega)\rangle \, d\lambda(\omega).$$

If $\omega \in \Omega_n$

$$\sum_{j=1}^{\infty} \langle g_{n,j}(\omega), h(\omega)\rangle = \sum_{j=1}^{\infty} \langle \psi_j(\omega), h(\omega)\rangle = \|h(\omega)\|.$$

If $\omega \notin \Omega_n$ and $k \in \mathbb{N}$

$$\left|\sum_{j=1}^{k} \langle g_{n,j}(\omega), h(\omega)\rangle\right| \leqq \left(1 + \frac{1}{k}\right) \|h(\omega)\|.$$

Hence

$$x^*(w_n) \geqq \int_{\Omega_n} \|h(\omega)\| \, d\lambda(\omega) - \int_{\Omega \setminus \Omega_n} \|h(\omega)\| \, d\lambda(\omega).$$

Letting $n \to \infty$ we get

$$x^*(w) \geqq \int_{\Omega} \|h(\omega)\| \, d\lambda(\omega) = \|T^* x^*\| = \sup_{v \in W} x^*(v).$$

We conclude that W is weakly compact as required.

General Case. Let Ω be an arbitrary compact Hausdorff space, and let $T: C(\Omega, E) \to X$ be an unconditionally converging operator, and let $\{\psi_n\}_{n \geqq 1}$ be a sequence contained in the unit ball of $C(\Omega, E)$. Similarly as in [1, Theorem 8] we can construct a compact metric space $\bar{\Omega}$ a continuous mapping from Ω onto $\bar{\Omega}$, an operator $\bar{T}: C(\bar{\Omega}, E) \to X$ and a sequence $\{\bar{\psi}_n\}_{n \geqq 1}$ in the unit ball of $C(\bar{\Omega}, E)$ such that $\bar{T}(\psi_n) = T(\psi_n)$ for all $n \in \mathbb{N}$. Moreover since T is unconditionally converging, it is immediate that $\bar{T}$ is unconditionally converging too. By Case 1, the operator $\bar{T}$ is weakly compact and therefore $\{T\psi_n\}_{n \geqq 1}$ has a weakly convergent subsequence. Hence we conclude that $C(\Omega, E)$ has property (V).

We conclude by applying our result to the case when E is isomorphic to a closed subspace of an order-continuous Banach lattice [7, Vol. II, p. 7]. The next Lemma can be deduced from [14, Theorem 16]. We shall include a proof for the sake of completeness.

Lemma 5. *Let E be a closed subspace of an order-continuous Banach lattice F. If E has property (V) then E contains no subspace isomorphic to l_1.*

Proof. Suppose G is a closed subspace of E isomorphic to l_1. Since G is a separable subspace of F, there is a band F_0 with weak order unit in F so that $G \subset F_0$ [7, Vol.

II, Proposition 1.a.9]. We show that the adjoint j_0^* of the inclusion map $j_0: G \to F_0$ cannot be unconditionally converging. Let ψ be a strictly positive linear functional on F_0, this of course is guaranteed by [7, Vol. II, Proposition 1.b.15]. If j_0^* is unconditionally converging, then $j_0^*[-\psi, \psi]$ is weakly compact in G^*, this follows from [8, Theorem 1] and the fact that the principal ideal in F_0^* generated by ψ is an AM-space (see [11, p. 102]). Now if $f^* \in F_0^*$ and $f^* \geqq 0$, then $f^* \wedge n\psi \uparrow f^*$ weak* and hence if j_0^* is unconditionally converging $j_0^*(f^* \wedge n\psi)$ will converge in norm to $j_0^* f^*$, thus $G^* = j_0^*(F^*)$ will be weakly compactly generated. As $G^* \simeq l_\infty$ this is a contradiction.

Since F_0 is complemented in F [7, Vol. II, Proposition 1.a.11], the adjoint of the inclusion map $j: G \to F$, j^* fails to be unconditionally converging. However, j^* factors through E^*. Hence E^* contains a copy of c_0 and thus is not weakly sequentially complete; this contradicts property (V) (see [8]).

Theorem 6. *Let E be a Banach space isomorphic to a closed subspace of an order-continuous Banach lattice. Then E has property (V) if and only if $C(\Omega, E)$ has property (V).*

Proof. If E has property (V), then E contains no copy of l_1 by Lemma 5; E has property (u) automatically.

Conversely, if $C(\Omega, E)$ has property (V), then E will have property (V) since it is trivially complemented in $C(\Omega, E)$.

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