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# Pathological Linear Spaces and Submeasures

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## 1. Introduction

Let $K$ be a compact Hausdorff space and let $\lambda$ be a probability measure on $K$. We denote by $L_0(K,\lambda)$ the space of all Borel functions $f:K\to\mathbb{R}$ with the topology of convergence in measure. $L_0(K,\lambda)$ is an $F$-space (complete metric topological vector space) if, as usual, we identify functions equal almost everywhere.

Spaces of the type $L_0(K,\lambda)$ are probably the most studied examples of non-locally convex topological vector spaces. In spite of their bad reputation there is some evidence that they are in fact rather well-behaved spaces. Thus for a general topological vector space $X$ it is often very useful to be able to produce non-trivial linear operators $T:X\to L_0(K,\lambda)$, in the same way as linear functionals facilitate the theory of locally convex spaces.

We shall say that a point $x\in X$ is *pathological* it wherever $T:X\to L_0(K,\lambda)$ is a continuous linear operator then $Tx=0$. $X$ is *pathological* if every $x\in X$ is pathological. Note that if $X$ is separable then it suffices in the definition to take $K=(0,1)$ with Lebesgue measure.

The first example of a pathological $F$-space was given in 1973 by Christensen and Herer [3]. A more natural example is the space $L_p/H_p$ where $0<p<1$ (see [1, 5]). However the Christensen-Herer example also showed the connection with *pathological submeasures.*

We recall that if $\mathscr{A}$ is an algebra of subsets of some abstract set $L$ then a *submeasure* $\phi:\mathscr{A}\to\mathbb{R}$ is a map satisfying $\phi(A)\leqq\phi(A\cup B)\leqq\phi(A)+\phi(B)$ and $\phi(\emptyset)=0$. $\phi$ is said to be *pathological* (see [3, 11]) if whenever $\lambda:\mathscr{A}\to\mathbb{R}$ is a (finitely-additive) measure with $0\leqq\lambda(A)\leqq\phi(A)$ $(A\in\mathscr{A})$ then $\lambda\equiv 0$.

Now if $S(\mathscr{A})$ denotes the space of all simple $\mathscr{A}$-measurable functions $f:L\to\mathbb{R}$, then $S(\mathscr{A})$ can be topologized by the topology of convergence in $\phi$-measure (functions differing only on a set of $\phi$-measure zero are identified). The completion of $S(\mathscr{A})$ in this topology may be denoted by $\Lambda(\phi)$. Then $\Lambda(\phi)$ is pathological

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whenever $\phi$ is pathological. In [3] this is essentially established only for countable algebras $\mathscr{A}$; however the argument can be modified for general algebras and in any case we shall give an alternative proof later.

The purpose of this note is first to give a simple geometric criterion for the pathology of a point in a topological vector space. We then use our results to give some further information on pathological submeasures. We show, for example, that if $\mathscr{A}$ is an algebra and $\mu:\mathscr{A}\to L_0(K,\lambda)$ is an additive map which is controlled by a pathological submeasure $\phi:\mathscr{A}\to\mathbb{R}$ then $\mu=0$. This extends a recent result of Talagrand [12] who essentially showed the same result for a bounded map $\mu:\mathscr{A}\to L_0(K,\lambda)$. We also give a criterion for pathology of a submeasure similar to the condition for a point in a vector space.

## 2. A Nikishin-Type Theorem

Let $K$ be a compact Hausdorff space and let $\lambda$ be a probability measure on $K$. We denote by $M^*(K)$ the space of all Borel measurable functions $f:K\to\mathbb{R}\cup\{\infty\}$. We also define for $\varepsilon>0$ and $a>0$

$$V(\varepsilon,a)=\{f\in M^*(K):\lambda(|f|\geqq a)\leqq\varepsilon\}.$$

If $X$ is a real vector space then a linear map $T:X\to M^*(K)$ is *prelinear* if

$$T(\alpha x+\beta y)(s)=\alpha Tx(s)+\beta Ty(s) \quad \text{a.e.}$$

whenever $|Tx(s)|<\infty$, $|Ty(s)|<\infty$.

The following theorem is similar in spirit to the Nikishin theorems on factorizing operators into $L_0(K,\lambda)$ [9].

**Theorem 2.1.** *Let $X$ be a topological vector space and let $T:X\to M^*(K)$ be a prelinear map. Suppose that for some neighborhood $W$ of $0$ in $X$ we have $T(W)\subset V(\varepsilon,a)$. Then there is a Borel subset $E$ of $K$ with $\lambda(E)\geqq 1-\varepsilon$ so that the map $T_E$ is a continuous linear operator from $X$ into $L_0(K,\lambda)$ where*

$$\begin{aligned} T_E x(s) &= Tx(s) \qquad s\in E\\ &=0 \qquad s\notin E. \end{aligned}$$

*(In particular $|T_E x(s)|<\infty$, $\lambda-$a.e.)*

*Proof.* Let $\|\cdot\|$ be a continuous $F$-norm on $X$ so that $\|x\|\leqq 1$ implies $x\in W$. Let $\Gamma$ be the collection of sequences $\{f_n\}$ in $M^*(K)$ so that $f_n=Tx_n$ where $\sum\|x_n\|<\infty$. For each $\{f_n\}\in\Gamma$ let $C=C\{f_n\}$ be defined by

$$C=\left\{s:\lim_{n\to\infty} f_n(s)=0\right\}.$$

Let $\mathscr{C}$ be the collection of all such subsets of $K$. By interlacing sequences it is clear that if $C_1, C_2\in\mathscr{C}$ then $C_1\cap C_2\in\mathscr{C}$. Hence if we let $E$ be a Borel subset of maximal measure so that $\lambda(E\setminus C)=0$ for $C\in\mathscr{C}$, then

$$\lambda(E)=\inf_{C\in\mathscr{C}}\lambda(C).$$

The map $T_E$ has the property that if $\sum \|x_n\| < \infty$ then $T_E f_n \to 0$ $\lambda$-a.e. and it easily follows that $T_E : X \to L_0(K, \lambda)$ is continuous. It remains therefore to show that $\lambda(E) \geqq 1 - \varepsilon$. For this it suffices to show that if $C = C\{f_n\} \in \mathscr{C}$ then $\lambda(C) \geqq 1 - \varepsilon$.

Let $f_n = Tx_n$ where $\sum \|x_n\| < \infty$. Let $\{\eta_n : n = 1, 2, \ldots\}$ be a sequence of independent identically distributed random variables on some probability space $(\Omega, P)$ each with normal distribution, mean zero and variance one. Let

$$\varrho(\theta) = (\mathscr{E}(|\eta_1|) + 1) \sum_{i=1}^{\infty} \|\theta x_i\| .$$

Then $\varrho(\theta) \to 0$ as $\theta \to 0$.

For $\omega \in \Omega$ let

$$\xi_n(\omega) = \sum_{i=1}^{n} \eta_i(\omega) x_i .$$

Then

$$\mathscr{E}(\|\theta \xi_n\|) \leqq \varrho(\theta), \quad n \in \mathbb{N},$$

and

$$P(\|\theta \xi_n\| > 1) \leqq \varrho(\theta) .$$

Thus for $\theta > 0$,

$$(P \times \lambda)\{(\omega, s) : |T\xi_n(s)| \geqq \theta^{-1}\} \leqq \varepsilon + \varrho(\theta) .$$

If we fix $s \in K$ and $\max_{1 \leqq k \leqq n} |f_k(s)| = \infty$, then the set of $\{\alpha_1, \ldots, \alpha_n\} \in \mathbb{R}^n$ so that $|\alpha_1 f_1(s) + \ldots + \alpha_n f_n(s)| < \infty$ is a proper linear subspace. Hence

$$P(|T\xi_n(s)| = \infty) = 1 .$$

If $\max_{1 \leqq k \leqq n} |f_k(s)| < \infty$, then $T\xi_n(s)$ is normally distributed with mean zero and variance $\sum_{k=1}^{n} |f_k(s)|^2$. Hence if $h_n(s) = \left( \sum_{k=1}^{n} |f_k(s)|^2 \right)^{1/2}$.

$$\int_K P(h_n(s)|\eta_1| \geqq a\theta^{-1}) d\lambda(s) \leqq \varepsilon + \varrho(\theta) .$$

Here the integrand is taken to be one if $h_n(s) = \infty$. Now let $n \to \infty$ and put $B = \{s : h_n(s) \to \infty\}$; then

$$\lambda(B) \leqq \varepsilon + \varrho(\theta) .$$

Letting $\theta \to 0$, $\lambda(B) \leqq \varepsilon$. In particular $\lambda(C) \geqq 1 - \varepsilon$, where $C = \left\{ s : \lim_{n \to \infty} f_n(s) = 0 \right\}$.

## 3. Finitely Additive Measures

In this section let $D$ be a discrete set, and let $\nu$ be a finitely-additive measure on the collection $\mathscr{P}D$ of all subsets of $D$. Suppose also $\nu(D) = 1$. We let $M(D)$ be the space of all functions $f : D \to \mathbb{R}$, and set $V(\varepsilon, a) \subset M(D)$ to be the set of $f$ such that $\nu(|f| \geqq a) \leqq \varepsilon$.

**Theorem 3.1.** *Suppose $X$ is a topological vector space and $T:X\to M(D)$ is a linear map. Suppose $Tx_0=1_D$ and that $W$ is a neighborhood of zero so that $T(W)\subset V(\varepsilon,a)$ where $0<\varepsilon<1$. Then there is a compact Hausdorff space $K$, a probability measure $\lambda$ on $K$ and a continuous linear operator $S:X\to L_0(K,\lambda)$ so that $Sx_0=1$ and if $\sigma>0$ $S(W)\subset V(\varepsilon(1-\varepsilon)^{-1},a+\sigma)$.*

*Proof.* We let $K=\beta D$, the Stone-Cech compactification of $D$. For each $f\in M(D)$ there is a unique continuous extension $f^*:K\to\mathbb{R}\cup\{\infty\}$ (the one-point compactification of $\mathbb{R}$). We consider the prelinear map $T^*:X\to M^*(K)$ defined by $T^*x=(Tx)^*$. Let $v^*$ be the regular Borel measure on $K$ induced by $v$. Then if $w\in W$ and $\sigma>0$,

$$\begin{aligned} v^*(|T^*w|\geqq a+\sigma)&\leqq v(|Tw|\geqq a)\\ &\leqq\varepsilon. \end{aligned}$$

Thus by Theorem 2.1, there is a Borel subset $E$ of $K$ with $v^*(E)\leqq\varepsilon$ so that the map $x\mapsto(T^*x)\cdot 1_{K\setminus E}$ is continuous.

Define $\lambda(B)=v^*(K\setminus E)^{-1}v^*(B\cap(K\setminus E))$ for a Borel set $B$. Then the map $S=T^*$ is continuous from $X$ into $L_0(K,\lambda)$ and $Sx_0=1_K$. If $w\in W$, then for $\sigma>0$,

$$\begin{aligned} \lambda(|T^*w|\geqq a+\sigma)&\leqq(1-v^*(E))^{-1}\varepsilon\\ &=\varepsilon(1-\varepsilon)^{-1}. \end{aligned}$$

We shall also need a finitely-additive version of a lemma of Musial et al. [8]. The proof is virtually identical – the key uses of Fubini's theorem only need products with *finite* probability spaces.

**Lemma 3.2.** *Suppose $f_1,\dots,f_n\in M(D)$ and $\alpha_1f_1+\dots+\alpha_nf_n\in V(\varepsilon,a)$ for every $\alpha_i=+1$. Then $\beta_1f_1+\dots+\beta_nf_n\in V(8\varepsilon,8a)$ for every $\beta_i$ with $-1\leqq\beta_i\leqq1$.*

We note in this section a classical criterion of Kelley [7] on the existence of finitely-additive measures on $D$.

**Proposition 3.3.** *Let $\mathscr{C}$ be a collection of subsets of $D$. Suppose that whenever $C_1,\dots,C_n\in\mathscr{C}$*

$$\inf_{d\in D}\frac{1}{n}\sum_{k=1}^{n}1_{C_k}(d)\leqq\varepsilon.$$

*Then there is a finitely-additive positive measure $v$ on $D$ so that $v(D)=1$ and $v(C)\leqq\varepsilon$ for every $C\in\mathscr{C}$.*

## 4. Pathological Vector Spaces

**Lemma 4.1.** *Let $X$ be a real vector space and suppose $W\subset X$ is a subset. Suppose $x_0\in X$ and $\varepsilon>0$. Then the following conditions are equivalent:*

(i) *There is a discrete set $D$ and, a finitely additive measure $v$ on $D$ with $v(D)=1$, and a linear map $T:X\to M(D)$ with $Tx_0=l_D$ and $T(W)\subset V(\varepsilon,1)$.*

(ii) *Whenever $\{w_1,\dots,w_n\}$ is a finite sequence in $W$ there is a subset $F$ of $\{1,2,\dots,n\}$ with $|F|\geqq n(1-\varepsilon)$ and $x\notin\mathrm{co}\{\pm w_i:i\in F\}$.*

*Proof.* (i)⇒(ii). Suppose $\{w_1, \dots, w_n\}$ is a finite sequence in $W$. Let $A_i = \{d : |Tw_i(d)| \geqq 1\}$. Then $\lambda(A_i) \leqq \varepsilon$ and so

$$\int \sum_{i=1}^{n} 1_{A_i} d\lambda \leqq n\varepsilon .$$

Now there is a set $B$ which is an atom in the algebra generated by $\{A_1, \dots, A_n\}$ so that

$$\sum_{i=1}^{n} 1_{A_i}(d) \leqq n\varepsilon , \quad d \in B .$$

Let $F = \{i : B \cap A_i = \emptyset\}$. Then $|F| \geqq n(1-\varepsilon)$ and $|Tw_i(d)| < 1$ for $i \in F$ and $d \in B$. If $g \in \mathrm{co}\{\pm Tw_i : i \in F\}$ then $|g(d)| < 1$ for $d \in B$ and hence $l_K \notin \mathrm{co}\{\pm Tw_i : i \in F\}$.

(ii)⇒(i). Let $D$ be the subset of $X'$ (the algebraic dual of $X$) consisting of all $d$ so that $d(x_0) = 1$. Define $T: X \to M(D)$ by $Tx(d) = d(x)$.

For each $w \in W$, let $A_w = \{d \in D : |d(w)| \geqq 1\}$. For every finite sequence $\{w_1, \dots, w_n\}$ in $W$, we can find $F \subset \{1, 2, \dots, n\}$ with $|F| \geqq n(1-\varepsilon)$ so that $x_0 \notin \mathrm{co}\{\pm w_i : i \in F\}$. Thus there exists $d \in D$ with $|d(w_i)| < 1$ for $i \in F$. Hence

$$\sum_{i=1}^{n} 1_{A_{w_i}}(d) \leqq n\varepsilon .$$

Now by Proposition 3.3 there is a finitely-additive measure $v$ on $D$ with $v(D) = 1$ and $v(A_w) \leqq \varepsilon$ for every $w \in W$. This proves (i).

We can now state our main theorem characterizing pathological points in a topological vector space. Contrast the conditions below with the standard conditions below with the standard conditions for the existence of a continuous linear functional $f$ so that $f(x_0) \neq 0$.

**Theorem 4.2.** *Let $X$ be a topological vector space and suppose $x_0 \in X$. Then the following conditions are equivalent:*

(i) *$x_0$ is pathological.*

(ii) *For any $\varepsilon$, $0 < \varepsilon < 1$, and any neighborhood $W$ of $0$, there is a finite set $\{w_1, \dots, w_n\}$ in $W$ so that whenever $F \subset \{1, 2, \dots, n\}$ with $|F| \geqq n(1-\varepsilon)$ then $x_0 \in \mathrm{co}\{\pm w_i : i \in F\}$.*

(iii) *For some fixed $\varepsilon$, $0 < \varepsilon < 1$, and any neighborhood $W$ of $0$ there is a finite sequence $\{w_1, \dots, w_n\}$ in $W$ so that if $F \subset \{1, 2, \dots, n\}$ with $|F| \geqq n(1-\varepsilon)$ then $x_0 \in \mathrm{co}\{\pm w_i : i \in F\}$.*

*Proof.* (i)⇒(ii). If (ii) fails for some $0 < \varepsilon < 1$ and $W$ there is, by Lemma 4 a discrete set $D$ with a finitely additive normalized measure $v$ and a linear map $T: X \to M(D)$ so that $Tx_0 = l_D$ and $T(W) \subset V(\varepsilon, 1)$. By Theorem 3.1, $x_0$ is not pathological.

(ii)⇒(iii). Automatic.

(iii)⇒(i). If $x_0$ is not pathological there exists a linear operator $T: X \to L_0(K, \lambda)$ with $Tx_0 = 1_K$. Choose $W$ a zero-neighborhood so that $T(W) \subset V(\varepsilon, 1)$. The existence of $T$ implies that (iii) must fail [repeat the proof of Lemma 4.1(i)⇒(ii)].

## 5. Pathological Submeasures

We now apply our results to the study of pathological submeasures (see [2, 3, 11]). Before proceeding we shall state two criteria for the pathology of a submeasure. Let $\mathscr{A}$ be an algebra of subsets of a set $L$.

**Theorem 5.1.** *Let $\phi:\mathscr{A}\to\mathbb{R}$ be a pathological submeasure.*

(i) [2, Theorem 5]. *For every $\varepsilon>0$ there exist $A_1,\ldots,A_n\in\mathscr{A}$ so that $\phi(A_i)\leqq\varepsilon$ and*

$$\frac{1}{n}\sum_{i=1}^{n} 1_{A_i}\geqq 1-\varepsilon\,.$$

(ii) [2, Theorem 2]. *If $\varrho:\mathscr{A}\to\mathbb{R}$ is a normalized (finitely-additive) measure and $\varepsilon>0$ then there is a set $A\in\mathscr{A}$ with $\phi(A)\leqq\varepsilon$ and $\varrho(A)\geqq 1-\varepsilon$.*

Let $X$ be a metrizable topological vector space, $F$-normed by $x\mapsto\|x\|$. Let $\mu:\mathscr{A}\to X$ be any additive set-function. Then a submeasure $\phi:\mathscr{A}\to X$ is said to *control* $\mu$ if given $\varepsilon>0$ there exists $\delta>0$ so that $\phi(A)\leqq\delta$ implies $\|\mu(A)\|\leqq\varepsilon$. Every additive set function has a control submeasure $\|\mu\|$ defined by

$$\|\mu\|(A)=\sup(\|\mu(B)\|:B\subset A, B\in\mathscr{A})\,.$$

$\mu$ is said to be *exhaustive* if $\|\mu(A_n)\|\to 0$ whenever $(A_n)$ is a disjoint sequence. If $\mu$ has a control measure then $\mu$ is exhaustive; the converse is the (unsolved) Maharam problem.

However in the special case $X=L_0(K,\lambda)$ the converse is known from recent results [6, 12]. A classical result of Orlicz [10] implies that if $\mu$ is *bounded* then $\mu$ is exhaustive. The converse is proved for countably additive $\mu$ in [6, 12]. This also give the result for exhaustive $\mu$, either by a standard extension procedure or by using the Drewnowski reduction of exhaustive measures to countably additive measures [4]. Now if $\mu:\mathscr{A}\to L_0(K,\lambda)$ is bounded then as shown in [12] we can actually find a function $f\in L_0(K,\lambda)$ with $f>0$ a.e. so that the map $A\mapsto f\cdot\mu(A)$ is bounded into $L_2(K,\lambda)$. Hence there is a (finitely-additive) control measure for $\mu$.

Summarizing we state:

**Theorem 5.2.** *Let $\mu:\mathscr{A}\to L_0(K,\lambda)$ be an additive set function. The following conditions are equivalent:*

(i) *$\mu$ is exhaustive.*

(ii) *$\mu(\mathscr{A})$ is bounded.*

(iii) *There is a control measure for $\mu$.*

**Lemma 5.3.** *Let $\phi:\mathscr{A}\to\mathbb{R}$ be a pathological submeasure, and suppose $\mu:\mathscr{A}\to L_0(K,\lambda)$ is an additive set function. Suppose for some $\delta>0$, $\varepsilon>0$, and $a>0$ we have $\mu(A)\in V(\varepsilon,a)$ whenever $\phi(A)\leqq\delta$. Then for any $\sigma>0$, $\mu(\mathscr{A})\subset V(\varepsilon,a+\sigma)$.*

*Proof.* Let $S(\mathscr{A})$ be the space of simple $\mathscr{A}$-measurable functions and let $\|\cdot\|_\infty$ be the usual sup-norm on $S(\mathscr{A})$. Let $J:S(\mathscr{A})\to L_0(K,\lambda)$ be the natural integration operator. Fix any $\theta>0$.

For each $A\in\mathscr{A}$ we may determine a Borel set $A^*$ in $K$ of minimal $\lambda$-measure so that the linear map $f\mapsto 1_{K\setminus A^*}J(f1_A)$ is continuous on $S(\mathscr{A})$. $A^*$ is unique up to sets of measure zero and $(A\cup B)^*=A^*\cup B^*$.

By Theorem 5.1(i) we can find $A_1, \dots, A_n \in \mathscr{A}$ with $\phi(A_i) \leqq \delta$ so that

$$\frac{1}{n}\sum_{i=1}^{n} 1_{A_i} \geqq \frac{16}{16+\theta}.$$

Now

$$\frac{1}{n}\sum_{i=1}^{n} 1_{A_i^*} \geqq \frac{16}{16+\theta} l_{L^*} \quad \text{a.e.}$$

and hence

$$\lambda(L^*) \leqq \frac{16+\theta}{16} \max_i \lambda(A_i^*).$$

Fix $i \leqq n$. Suppose $B_1, \dots, B_m \subset A_i$ are disjoint. The $\sum \alpha_i 1_{B_i} \in V(2\varepsilon, 2a)$ wherever $\alpha_i = \pm 1$. Now by the lemma of Musial et al. [8] (compare Lemma 3.2), $Jf \in V(16\varepsilon, 16a)$ whenever $\|f\|_\infty \leqq 1$, $\operatorname{supp} f \subset A_i$. We conclude from Theorem 2.1 that $\lambda(A_i^*) \leqq 16\varepsilon$ and hence $\lambda(L^*) \leqq (16+\theta)\varepsilon$.

The measure $A \mapsto \mu(A) 1_{K \setminus L^*}$ is bounded and so has a control measure. By 5.1(ii) there exists $A_0 \in \mathscr{A}$ with $\phi(A_0) < \delta$ so that whenever $B \cap A_0 = \emptyset$ then $\mu(B) 1_{K \setminus L^*} \in V(\theta\varepsilon, \theta a)$. Thus $\mu(B) \in V((16+2\theta)\varepsilon, \theta a)$. If $A \subset A_0$ then $\mu(A) \in V(\varepsilon, a)$. Combining for $A \in \mathscr{A}$, $\mu(A) \in V((17+2\theta)\varepsilon, (1+\theta)a)$; as $\theta > 0$ is arbitrary we obtain the lemma.

We shall later have need of a modification of Lemma 5.3.

**Lemma 5.4.** *Let $\phi: \mathscr{A} \to \mathbb{R}$ be a pathological submeasure, and let $v$ be a normalized finitely-additive measure on a discrete set $D$. Suppose $\mu: \mathscr{A} \to M(D)$ is an additive set function and that for some $\delta > 0$, $\varepsilon > 0$, and $a > 0$, $\mu(A) \in V(\varepsilon, a)$ whenever $\phi(A) \leqq \delta$. Then if $\sigma > 0$, $\mu(\mathscr{A}) \subset V(17\varepsilon, a+\sigma)$.*

The proof, which we omit, is essentially the same. One takes $K = \beta D$ and $\lambda$ the natural probability measure on $K$ induced by $v$. The integration operator $J$ induces a prelinear map $J^*: S(\mathscr{A}) \to M^*(K)$; Lemma 3.2 is used in place of the lemma of Musial et al. to show that $J^*$ essentially satisfies the same conditions as $J$ in the proof of Lemma 5.3.

**Theorem 5.5.** *Let $\phi: \mathscr{A} \to \mathbb{R}$ be a pathological submeasure and let $\mu: \mathscr{A} \to L_0(K, \lambda)$ be any additive map controlled by $\phi$. Then $\mu = 0$.*

*Proof.* This immediate from Lemma 5.3.

**Theorem 5.6.** *Let $\tau$ be any vector topology on $S(\mathscr{A})$ such that $1_{A_n} \to 0(\tau)$ whenever $\phi(A_n) \to 0$. Then $(S(\mathscr{A}), \tau)$ is pathological. In particular $\Lambda(\phi)$ (see the introduction) is pathological.*

*Proof.* It suffices to show that $1_A (A \in \mathscr{A})$ is pathological in $(S(\mathscr{A}), \tau)$. If $T: S(\mathscr{A}) \to L_0(K, \lambda)$ is continuous then $A \mapsto T(1_A)$ is a finitely additive set function controlled by $\phi$. Hence $T(1_A) = 0$ for each $A \in \mathscr{A}$.

*Remark.* Of course the pathology of $\Lambda(\phi)$ can be proved more easily by the methods of [3]. However as our example where these methods do not work consider the "$L_1$-space" of $\phi$ i.e. topologize $S(\mathscr{A})$ by the (pseudo-) quasi-norm

$$\|f\| = \int_0^\infty \phi(|f| \geqq t)dt.$$

Theorem 4.2 can be applied to $\Lambda(\phi)$ to give a criterion for $\phi$ to be pathological. However we conclude by obtaining a much better criterion using only characteristic functions.

**Theorem 5.7.** *Let $\phi:\mathscr{A}\to\mathbb{R}$ be a pathological submeasure. Suppose $\varepsilon>0$ and if $0<\delta<1/17$. Then there exist $A_1, \ldots, A_n \in \mathscr{A}$ with $\phi(A_i)<\varepsilon$ and so that if $F\subset\{1,2,\ldots,n\}$ with $|F|\geqq(1-\delta)n$ then $(1-\varepsilon)1_L\in \mathrm{co}\{\pm 1_{A_i}: i\in F\}$.*

*Proof.* If the theorem is false we can by Lemma 4.1 produce a discrete set $D$, a normalized finitely-additive measure $\nu$ on $D$ and linear map $T:S(\mathscr{A})\to M(D)$ with $T1_L=(1-\varepsilon)^{-1}1_D$ and $T(1_A)\in V(\delta,1)$ wherever $\phi(A)\leqq\varepsilon$. By Lemma 5.4 $T(1_L)\in V(17\delta, 1+\sigma)$ for any $\sigma>0$. Since $17\delta<1$ this is a contradiction.

*Remarks.* The condition $\delta<\frac{1}{17}$ seems artificial but we do not know how to remove it. It is not difficult to obtain the standard criterion Theorem 5.1(i) from Theorem 5.7, but we do know any direct proof of Theorem 5.7 from Theorem 5.1.

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