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On Absolute Bases

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1. Introduction

The theory of bases in general locally convex spaces has proved particularly rich when applied to nuclear spaces. Many interesting and deep results relating the structure of bases and the nuclearity of the space are known. These results show up a strong connexion between absolute bases and nuclear spaces; we have the following theorems if E is a Fréchet space with a basis.

Theorem A (Dynin-Mitiagin [4], Mitiagin [15]). *If E is nuclear, every basis of E is absolute.*

Theorem B (Pietsch [21], see also Bennett-Cooper [1]). *If both E and its strong dual possess absolute bases, then E is nuclear.*

Theorem C (Wojtynski [26]). *If every basis of E is absolute, then E is nuclear.*

In this paper we shall attempt a unified theory of absolute bases in locally convex spaces. The main result of § 2 is to extend the Lindenstrauss-Petczyński theorem, that every unconditional basis of l^1 is absolute, to show that if a barrelled space possesses an absolute basis, then every unconditional basis is absolute. In § 3 we introduce a dual notion, ∞-absolute bases, and obtain a similar result for spaces with ∞-absolute bases. Then in § 4, we study spaces in which there exist both absolute and ∞-absolute bases. In § 5, we study p-absolute bases, a generalization of absolute bases, due to Schock. The main result is that a F-space with a p-absolute basis and a (different) q-absolute basis with $p \neq q$ is nuclear; this extends a result of Schock who establishes this result for spaces with a single basis which is simultaneously p-absolute and q-absolute.

2. Absolute Bases

When working in locally convex spaces which are not necessarily either complete or barrelled it is convenient to distinguish between several types of bases, which are usually lumped together in the literature as "absolute". Suppose (χ_n) is a Schauder basis of the locally convex space (E, τ); we shall make the following definitions.

Definition (1). (χ_n) is *semi-absolute* if whenever $\Sigma a_n \chi_n$ converges then $\Sigma a_n \chi_n$ converges absolutely.

Definition (2). (χ_n) is *pre-absolute* if whenever $\Sigma a_n \chi_n$ is Cauchy then $\Sigma |a_n| \nu(\chi_n) < \infty$ for every continuous semi-norm ν.

Definition (3). (χ_n) is *absolute* if $\Sigma a_n \chi_n$ converges if and only if for every continuous semi-norm $\nu, \Sigma |a_n| \nu(\chi_n) < \infty$.

Definition (4). $(\chi_n$ is *pre-Köthe* if it is pre-absolute and for every continuous semi-norm ν, the semi-norm ν_1 is continuous where

$$\nu_1(\Sigma a_n \chi_n) = \Sigma |a_n| \nu(\chi_n).$$

Definition (5). (χ_n) is *Köthe* if it is pre-Köthe and absolute.

The terminology of (4) and (5) is dictated by the fact that a locally convex space E with a Köthe basis is isomorphic to a Köthe (perfect) sequence space in its normal topology (see [12] and [13] § 30).

We shall denote by τ_1 the topology induced by the semi-norms $\{\nu_1\}$ (see (4)); thus (χ_n) is a pre-Köthe basis if and only if $\tau = \tau_1$ (for clearly $\tau \leqq \tau_1$). The following result is a restatement of a result familiar from the theory of Köthe sequence spaces.

Proposition 1. *If (E, τ) is a locally convex space with a pre-Köthe basis, then:*

(i) *(E, τ) is complete if and only if the basis is a Köthe basis.*

(ii) *τ and the weak topology define the same convergent and Cauchy sequences.*

Proof. See Köthe [13], p. 413 and p. 416.

In this section we shall study the possible types of unconditional bases in spaces with absolute bases.

Let (χ_n) be a basis of a locally convex space, and let (θ_n) be a sequence such that $\sup_n |\theta_n| \leqq 1$ and $\theta_n = 0$ eventually. We define the maps

$$P_\theta\left(\sum_{i=1}^{\infty} a_i \chi_i\right) = \sum_{i=1}^{\infty} \theta_i a_i \chi_i .$$

We shall say that (χ_n) is *u-Schauder* if the maps $\{P_\theta\}$ are equicontinuous.

Proposition 2. *Suppose E is barrelled and (χ_n) is a Schauder basis of E;*

(i) *If (χ_n) is semi-absolute, then (χ_n) is pre-Köthe.*

(ii) *If (χ_n) is unconditional, then (χ_n) is u-Schauder.*

Proof. (i) Let ν be a continuous semi-norm on E; then the set $\{x; \nu_1(x) \leqq 1\}$ is a barrel, and so ν_1 is continuous on E.

(ii) See [9]; the proof is similar to (i).

Theorem 1. *Let E be a locally convex space with a Köthe basis; then any u-Schauder basis of E is a Köthe basis.*

Proof. Let (χ_n) be a Köthe basis of E, with dual sequence (φ_n); suppose (η_n) is a u-Schauder basis of E with dual sequence (ψ_n). Let ν be a continuous semi-norm on E; then ν_1 is also continuous where

$$\nu_1(x) = \sum_{n=1}^{\infty} |\varphi_n(x)|\, \nu(\chi_n).$$

The Hausdorff quotient of the semi-normed space (E, ν_1) is isomorphic either to a dense subspace of l_1 or is finite-dimensional. Suppose Σx_k is unconditionally convergent in E; then Σx_k is unconditionally convergent in (E, ν_1). Hence by Orlicz's Theorem [16] (see also [14], p. 295)

$$\left(\sum_{k=1}^{\infty} [\nu_1(x_k)]^2\right)^{\frac{1}{2}} \leqq \sqrt{3} \sup_n \sup_{|\theta_k| \leqq 1} \nu_1\left(\sum_{k=1}^{n} \theta_k x_k\right).$$

Hence, as (η_n) is unconditional, if

$$\pi(x) = \left(\sum_{n=1}^{\infty} |\psi_n(x)|^2 [\nu_1(\eta_n)]^2\right)^{\frac{1}{2}}$$

then

$$\pi(x) \leqq \sqrt{3} \sup_n \sup_{|\theta_k| \leqq 1} \nu_1\left(\sum_{k=1}^{n} \theta_k \psi_k(x)\, \eta_k\right).$$

As (η_n) is u-Schauder π is continuous; hence the semi-norm π_1 is continuous where

$$\pi_1(x) = \sum_{n=1}^{\infty} |\varphi_n(x)|\, \pi(\chi_n).$$

Then the injection $J:(E, \pi_1) \to (E, \pi)$ is continuous. The Hausdorff quotient of (E, π_1) is, as before, a dense subspace of l_1 or finite-dimensional. The quotient of (E, π) is an inner-product space, and therefore we may use Theorem 4.1 of [14] to deduce that J is absolutely summing. Hence

$$\sum_{n=1}^{\infty} |\psi_n(x)|\, \pi(\eta_n) \leqq K \sup_n \sup_{|\theta_k| \leqq 1} \pi_1\left(\sum_{k=1}^{n} \theta_k \psi_k(x)\, \eta_k\right).$$

As (η_n) is u-Schauder, the semi-norm

$$x \to \sum_{k=1}^{\infty} |\psi_k(x)|\, \nu(\eta_k)$$
$$\leqq \sum_{k=1}^{\infty} |\psi_k(x)|\, \pi(\eta_k)$$

is continuous on E, i.e. (η_k) is pre-Köthe. It follows from Proposition 1 that (η_k) is Köthe.

Clearly the proof yields the following Corollaries.

Corollary 1. *If E possesses a pre-Köthe basis, then any u-Schauder basis of E is pre-Köthe.*

Corollary 2. *If E possesses a pre-Köthe basis and* Σx_k *converges unconditionally in E, then, for any continuous semi-norm* v

$$\sum_{k=1}^{\infty} [v_1(x_k)]^2 < \infty .$$

Theorem 2. *Let E be a barrelled space with a (semi-) absolute Schauder basis; then any unconditional Schauder basis of E is (semi-) absolute.*

Proof. This follows from Theorem 1 and Proposition 2.

We now give an example of a locally convex space with a Köthe basis and an unconditional Schauder basis which is not absolute. Consider the sequence space l_2 with the normal or Köthe topology $|\sigma|(l_2, l_2)$ given by the semi-norms

$$v_f(x) = \sum_{k=1}^{\infty} |f_k x_k|$$

where $f = (f_k) \in l_2$. The usual basis $e^{(n)}$ of l_2 is a Köthe basis of $(l_2, |\sigma|(l_2, l_2))$.

We construct a further unconditional Schauder basis by finding an orthonormal basis for the norm topology. For $n \geqq 1$, let E^n be the space spanned by the vectors $\{e^{(2^n+1)}, \ldots, e^{(2^{n+1})}\}$. In each E^n we construct the so-called Haar system (see [18], p. 7) thus:

$$y^{(n,1)} = \frac{1}{2^{n/2}} \sum_{i=1}^{2^n} e^{2^n+i},$$

$$y^{(n,2^k+s)} = \frac{1}{2^{\frac{n-k}{2}}} \sum_{i=1}^{2^n} \beta_i(k,s)\, e^{2^n+i} \quad (\text{where } 1 \leqq s \leqq 2^k),$$

where

$$\beta_i(k,s) = 1\ (2s-2)\, 2^{n-k-1} + 1 \leqq i \leqq (2s-1)\, 2^{n-k-1}$$

$$\beta_i(k,s) = -1\ (2s-1)\, 2^{n-k-1} + 1 \leqq i \leqq s \cdot 2^{n-k}$$

$$= 0 \text{ elsewhere}.$$

Then $(y^{(n,m)}; m = 1, 2, \ldots, 2^n)$ is an orthonormal basis of E^n; the collection $(y^{(n,m)}; 1 \leqq m \leqq 2^n, 1 \leqq n)$ is, together with $e^{(1)}$ and $e^{(2)}$, an orthonormal basis of l_2. It follows that, in any order, the set $\{y^{(n,m)}\} \cup \{e^{(1)}, e^{(2)}\}$ is an unconditional Schauder basis of $(l_2, |\sigma|(l_2, l_2))$.

Now if

$$\sum_{m,n} |a_{n,m}|^2 < \infty$$

we have that $\sum_{m,n} a_{n,m} y^{(n,m)}$ converges. If $y^{(n,m)}$ is an absolute basis of $(l_2, |\sigma|(l_2, l_2))$ then for any $f \in l_2$

$$\Sigma |a_{n,m}| \, v_f(y^{(n,m)}) < \infty .$$

Therefore

$$\sum_{n,m} [v_f(y^{(n,m)})]^2 < \infty$$

for any $f \in l_2$. However, we set $f_1 = f_2 = 0$ and

$$f_k = \frac{1}{n \cdot 2^{n/2}} \; 2^n + 1 \leqq k \leqq 2^{n+1} .$$

Then

$$\Sigma |f_k|^2 = \Sigma \frac{1}{n^2} < \infty , \quad \text{and so} \quad f \in l_2 .$$

We also have

$$v_f(y^{(n,1)}) = \frac{1}{n}$$

$$v_f(y^{(n,2)}) = \frac{1}{n}$$

and for

$$2^k < m \leqq 2^{k+1} \; 1 \leqq k \leqq n-1$$

$$v_f(y^{(n,m)}) = \frac{1}{2^{\frac{(n-k)}{2}}} \cdot 2^{n-k} \frac{1}{n \cdot 2^{n/2}}$$

$$= \frac{1}{n \cdot 2^{k/2}} .$$

Therefore

$$\sum_{m=1}^{2^n} [v_f(y^{(n,m)})]^2 = \frac{2}{n^2} + \sum_{k=1}^{n-1} \frac{1}{n^2}$$

$$= \frac{n+1}{n^2} .$$

Hence

$$\sum_{m,n} [v_f(y^{(n,m)})]^2 = \infty$$

and $\{y^{(n,m)}\} \cup \{e^{(1)}\} \cup \{e^{(2)}\}$ is not an absolute basis of $\{l_2, |\sigma|(l_2, l_2)\}$.

We give one further result in this section related to Corollary 2 of Theorem 1. A series Σx_k is subseries convergent if for every subseries x_{k_n}, Σx_{k_n} converges.

Proposition 3. *Let E be a locally convex space with a semi-absolute Schauder basis. If Σx_k is subseries convergent, then for each continuous semi-norm v*

$$\Sigma [v(x_k)]^2 < \infty .$$

Proof. We first observe that τ_1 is an $\langle E, E'\rangle$-polar topology (since each v_1 is lower-semi-continuous). Secondly (E, τ_1) is separable and the given basis is a pre-Köthe basis of (E, τ_1); hence by Theorem 8 of [11] Σx_k is τ_1-subseries convergent. By Corollary 2 to Theorem 1, for continuous v

$$\Sigma [v(x_k)]^2 \leqq \Sigma [v_1(x_k)]^2 < \infty .$$

3. Dual Absolute Bases

Definition (6). A Schauder basis (χ_n) of a locally convex space is ∞-*absolute* if whenever $(a_n \chi_n)$ is bounded and $c_n \to 0$ then $\Sigma c_n a_n \chi_n$ converges.

It is clear that an ∞-absolute basis is weakly unconditional, for if $\Sigma a_n \chi_n$ converges then $(a_n \chi_n)$ is bounded and hence for $f \in E'$ and $c_n \to 0$ $\Sigma c_n a_n f(\chi_n)$ converges. Therefore $\Sigma |a_n| \, |f(\chi_n)| < \infty$, and the convergence of $\Sigma a_n \chi_n$ is weakly unconditional. If E is also sequentially complete, it follows that $\Sigma a_n \chi_n$ is subseries convergent weakly and therefore in the original topology (and, in particular, (χ_n) is unconditional).

In this section we shall show that ∞-absolute bases are "dual" to absolute bases and obtain some results similar to Theorems 1 and 2.

Proposition 4. *Let E be a sequentially complete locally convex space with a Schauder basis (χ_n); let (φ_n) be the dual sequence to (χ_n). The following are equivalent:*

(i) (χ_n) *is ∞-absolute,*

(ii) (φ_n) *is a semi-absolute basis of $(E', \beta(E', E))$,*

(iii) (φ_n) *is a pre-Köthe basis of $(E', \beta(E', E))$.*

Proof. (i)$\Rightarrow$(iii): Let B be a bounded subset of E, and let v_B be the associated $\beta(E', E)$-continuous semi-norm on E', i.e.

$$v_B(f) = \sup_{x \in B} |f(x)| .$$

Let

$$v_B(\varphi_n) = v_n \qquad n = 1, 2 \ldots .$$

Then (χ_n) is a simple basis of E (see [8], p. 379) and so the set $\left\{\sum_{k=1} \varphi_k(x) \chi_k ; x \in B, n = 1, 2, \ldots\right\}$ is bounded in E. It follows that the set $\{v_n \chi_n ; n = 1, 2, \ldots\}$ is also bounded and for $c_n \to 0$, $\Sigma c_n v_n \chi_n$ converges. Hence for $f \in E'$

$$\sum_{n=1}^{\infty} v_n |f(\chi_n)| < \infty ,$$

and so

$$f = \sum_{n=1}^{\infty} f(\chi_n) \varphi_n \quad \text{absolutely in} \quad \beta(E', E) .$$

Now let

$$C = \left\{ \sum_{n=1}^{\infty} c_n v_n \chi_n ;\, c_n \to 0\ |c_n| \leqq 1 \right\} ;$$

for $f \in E'$

$$|f(x)| \leqq \sum_{n=1}^{\infty} v_n |f(\chi_n)|$$

so that C is bounded. Furthermore

$$\begin{aligned} v_C(f) &= \sup_{x \in C} |f(x)| \\ &= \sum_{n=1}^{\infty} v_n |f(\chi_n)| \\ &= v_{B,1}(f) \quad \text{(see Definition 4)} \end{aligned}$$

so that (φ_n) is indeed pre-Köthe.

(iii)$\Rightarrow$(ii): Immediate.

(ii)$\Rightarrow$(i): Suppose $(a_n \chi_n)$ is a bounded subset of E; and let

$$v(f) = \sup_n |a_n f(\chi_n)| .$$

For $f \in E'$

$$f = \sum_{n=1}^{\infty} f(\chi_n)\, \varphi_n \quad \text{absolutely in} \quad \beta(E', E)$$

and so

$$\sum_{n=1}^{\infty} |f(\chi_n)|\, v(\varphi_n) < \infty ,$$

i.e.

$$\sum_{n=1}^{\infty} |a_n|\, |f(\chi_n)| < \infty .$$

Hence the set

$$B = \left\{ \sum_{k=1}^{n} \theta_k a_k \chi_k ;\ |\theta_k| \leqq 1,\ n = 1, 2 \ldots \right\}$$

is bounded and, if $c_n \to 0$

$$\sum_{n+1}^{n+p} c_k a_k \chi_k \in \left(\sup_{n+1 \leqq k \leqq n+p} |c_k| \right) B .$$

As E is sequentially complete $\Sigma\, c_k a_k \chi_k$ converges.

Proposition 5. *If E is a barrelled space with a semi-absolute Schauder basis (χ_n), then (φ_n) is an ∞-absolute basis of $(E', \sigma(E', E))$ and an ∞-absolute basic sequence in $(E', \beta(E', E))$.*

Proof. The assertion concerning $(E', \sigma(E', E))$ follows automatically from Proposition 4.

Since (χ_n) is simple, it follows (see [8], Theorem 2.6) that (φ_n) is a $\beta(E', E)$-basic sequence. Suppose now that $(a_n \varphi_n)$ is bounded in

$(E', \beta(E', E))$; then $(a_n \varphi_n)$ is equicontinuous, and as (χ_n) is a pre-Köthe basis of E (Proposition 2)

$$v_1(x) = \Sigma\ |a_n|\, |\varphi_n(x)|$$

is a continuous semi-norm on E. Letting $U = \{x; v_1(x) \leqq 1\}$ we see that $\Sigma\, c_n a_n \varphi_n$ converges uniformly on U and therefore strongly.

Proposition 6. *Let E be a sequentially complete locally convex space with an ∞-absolute Schauder basis (χ_n); the following are equivalent:*

(i) *If $(a_n \chi_n)$ is bounded then $\Sigma\, a_n \chi_n$ converges.*

(ii) *E is semi-reflexive.*

(iii) *E is a semi-Montel space (i.e. every bounded subset of E is relatively compact).*

Proof. (i) $\Rightarrow$ (iii): We use Theorem 5.6 of [10]; suppose

$$\eta_n = \sum_{m_{n-1}+1}^{m_n} a_k \chi_k$$

is a bounded block basic sequence; then the set $\{a_k \chi_k; k = 1, 2 \ldots\}$ is bounded (using the fact that (χ_n) is simple). Therefore $\Sigma\, a_k \chi_k$ converges and

$$\lim_{n\to\infty} \eta_n = 0 .$$

(iii) $\Rightarrow$ (ii) Immediate.

(ii) $\Rightarrow$ (i) By the result of Cook [2], (χ_n) is boundedly-complete.

If $(a_k \chi_k)$ is bounded, and $c_k \to 0$, then $\Sigma\, c_k a_k \chi_k$ converges; therefore for $f \in E'$, $\Sigma\, c_k a_k f(\chi_k)$ converges and so

$$\Sigma\ |a_k|\, |f(\chi_k)| < \infty .$$

Hence $\Sigma\, a_k \chi_k$ converges.

Theorem 3. *Let E be a sequentially complete locally convex space with an ∞-absolute Schauder basis; then any unconditional Schauder basis of E is ∞-absolute.*

Proof. By Proposition 4, $(E', \beta(E', E))$ possesses a pre-Köthe basis. Suppose Σf_k converges unconditionally in $(E', \sigma(E', E))$; then the set

$$C = \left\{ \sum_{k=1}^{n} \theta_k f_k;\ |\theta_k| \leqq 1,\ n = 1, 2 \ldots \right\}$$

is $\sigma(E', E)$ bounded and therefore $\beta(E', E)$-bounded. If $\gamma \in E'' = (E', \beta(E', E))'$ then

$$\sup_{f \in C} |\gamma(f)| < \infty ,$$

i.e.

$$\sum_{k=1}^{\infty} |\gamma(f_k)| < \infty .$$

By Proposition 1 (ii) Σf_k is $\beta(E', E)$-Cauchy. In particular any unconditional Schauder basis (χ_n) of E is shrinking; we must show that (φ_n) is u-Schauder in $(E', \beta(E', E))$.

Let

$$P_\theta f = \sum_{i=1}^{\infty} \theta_i f(\chi_i)\, \varphi_i \quad \text{where} \quad |\theta_i| \leqq 1\,,$$

and $\theta_n = 0$ eventually. Each P_θ is $\sigma(E', E)$-continuous and for $x \in E$

$$|P_\theta f(x)| \leqq \sum_{i=1}^{\infty} |f(\chi_i)|\, |\varphi_i(x)|$$

so that the maps P_θ are weakly (and therefore strongly) pointwise bounded. By Lemma 2.5 of [8], they form an equicontinuous collection on $(E', \beta(E', E))$, and (φ_n) is u-Schauder.

Hence we may apply Theorem 2, Corollary 1, to deduce that (φ_n) is pre-Köthe, and Proposition 4 to deduce that (χ_n) is ∞-absolute.

4. Absolute and ∞-absolute Bases in Nuclear Spaces

Definition (7). An ∞-absolute Schauder basis (χ_n) is *∞-Köthe* if the topology on E may be given by the semi-norms v_∞ where

$$v_\infty(\Sigma\, a_n x_n) = \sup_n |a_n|\, v(x_n)$$

for each continuous semi-norm v.

Proposition 7. *If (E, τ) is a Fréchet space then an ∞-absolute Schauder basis is ∞-Köthe.*

Proof. Let $(v^{(1)}, v^{(2)}, \ldots)$ be a fundamental sequence of semi-norms on E; then $\{v_\infty^{(1)}, v_\infty^{(2)}, \ldots\}$ defines a topology τ^* on E. For any $n, v_\infty^{(n)}$ is continuous on (E, τ), using the fact that (E, τ) is barrelled; therefore $\tau^* \leqq \tau$. We now show that (E, τ^*) is complete; let x_n be a τ^*-Cauchy sequence. Then for each m

$$\lim_{n\to\infty} \varphi_m(x_n) = a_m \quad \text{exists}$$

and

$$|a_m|\, v^{(k)}(\chi_m) = \lim_{n\to\infty} |\varphi_m(x_n)|\, v^{(k)}(\chi_m) \quad \text{each} \quad k$$

uniformly in m. Now

$$\lim_{m\to\infty} \varphi_m(x_n)\, v^{(k)}(\chi_m) = 0 \quad \text{each} \quad n$$

so that we deduce

$$\lim_{m\to\infty} |a_m|\, v^{(k)}(\chi_m) = 0\,, \qquad k = 1, 2, \ldots.$$

Therefore

$$\lim_{m\to\infty} a_m \chi_m = 0\ (\tau)\,.$$

As τ is metrizable there is a sequence $c_n \to 0$ such that $\{c_n^{-1} a_n \chi_n\}$ is bounded; hence $\Sigma a_n \chi_n$ converges in (E, τ) to x, say. It follows quickly that $x_n \to x$ (E, τ^*) and so (E, τ^*) is complete. Hence by the Closed Graph Theorem $\tau = \tau^*$.

We now give a criterion for a locally convex space with a basis to be nuclear. This result is essentially known, being contained in results of Dynin and Mitiagin [4], Grothendieck [5] (Chapitre 2, p. 59), and Pietsch [20] (p. 88) or [19].

Theorem 4. *Let E be a locally convex space with a Schauder basis (χ_n), and further suppose that the maps $x \to \varphi_n(x)\chi_n$ are equicontinuous. Then the following statements are equivalent:*

(i) *E is nuclear.*

(ii) *(χ_n) is Köthe and ∞-Köthe.*

(iii) *For any continuous semi-norm v there exists a continuous semi-norm ω with $\sum_{k=1}^{\infty} \frac{v(\chi_k)}{\omega(\chi_k)} < \infty$ (where $0/0 = 0$).*

Proof. (i)$\Rightarrow$(ii): This is contained in Theorems 8 and 9 (pp. 89–91) of [15].

(ii)$\Rightarrow$(iii): Let v be a continuous semi-norm; then as (χ_n) is both Köthe and ∞-Köthe there is a continuous semi-norm ω with $v_1(x) \leqq \omega_\infty(x)$ for all $x \in E$. Thus

$$\sum_{n=1}^{\infty} |\varphi_n(x)|\, v(\chi_n) \leqq \sup_n |\varphi_n(x)|\, \omega(\chi_n)$$

and it follows that

$$\sum_{n=1}^{\infty} \frac{v(\chi_n)}{\omega(\chi_n)} < \infty .$$

(iii)$\Rightarrow$(i): Given v, we determine ω such that

$$\sum_{k=1}^{\infty} \frac{v(\chi_k)}{\omega(\chi_k)} < \infty .$$

As the maps $x \to \varphi_n(x)\chi_n$ are equicontinuous the semi-norm ω_∞ is also continuous. Consider the identity map $I : (E, \omega_\infty) \to (E, v)$; then

$$Ix = \sum_{n=1}^{\infty} \varphi_n(x)\, \chi_n$$

where $\|\varphi_n\|_{(\omega_\infty)} = \frac{1}{\omega(\chi_n)}$ and $\|\chi_n\|_{(v)} = v(\chi_n)$ so that

$$\Sigma\, \|\varphi_n\|\, \|\chi_n\| < \infty ,$$

i.e. I is nuclear. Thus E is nuclear.

We remark that by Theorem 1 it is enough to assume in (ii) that E possesses a Köthe basis and a (different) ∞-Köthe basis. We now consider the weaker assumption that E possesses both an absolute basis and an ∞-absolute basis.

Theorem 5. *Let E be sequentially complete and possess both an absolute Schauder basis and a (different) ∞-absolute Schauder basis. Suppose Σx_k converges unconditionally in E; then*

$$\sum_{k=1}^{\infty} \nu(x_k) < \infty$$

for a continuous semi-norm ν on E.

Proof. By Theorem 3 E possesses a Schauder basis (χ_n) which is both absolute and ∞-absolute. Let τ_1 be, as usual, the topology given by the semi-norms

$$\nu_1(x) = \sum_{n=1}^{\infty} |\varphi_n(x)| \, \nu(\chi_n)$$

for continuous ν.

As (E, τ_1) is separable and τ_1 is an $\langle E, E' \rangle$-polar topology we deduce, using Theorem 8 of [11], that if Σx_k is unconditionally convergent in E, then Σx_k converges unconditionally in τ_1. Then for any continuous semi-norm ν

$$\sup_n \sup_{|\theta_k| \leq 1} \nu_1 \left(\sum_{k=1}^{n} \theta_k x_k \right) < \infty ,$$

i.e.

$$\sup_n \sup_{|\theta_k| \leq 1} \sum_{j=1}^{\infty} \left| \sum_{k=1}^{n} \theta_k \varphi_j(x_k) \right| \nu(\chi_j) < \infty$$

and in particular

$$\sup_n \sup_{|\theta_k| \leq 1} \sup_j \left| \sum_{k=1}^{n} \theta_k \varphi_j(x_k) \right| \nu(\chi_j) < \infty ,$$

i.e.

$$\sup_j \sum_{k=1}^{\infty} |\varphi_j(x_k)| \, \nu(\chi_j) < \infty .$$

Thus the set $\left\{ \left(\sum_{k=1}^{\infty} |\varphi_j(x_k)| \right) \chi_j \; j = 1, 2 \ldots \right\}$ is bounded in (E, τ) and for $c_j \to 0$ we have that $\sum_{j=1}^{\infty} c_j \left(\sum_{k=1}^{\infty} |\varphi_j(x_k)| \right) \chi_j$ converges in (E, τ). As the convergence is absolute we obtain

$$\sum_{j=1}^{\infty} |c_j| \sum_{k=1}^{\infty} |\varphi_j(x_k)| \, \nu(\chi_j) < \infty$$

whenever $c_j \to 0$. Therefore

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\varphi_j(x_k)| \, \nu(\chi_j) < \infty$$

or

$$\sum_{k=1}^{\infty} \nu_1(x_k) < \infty .$$

Hence

$$\sum_{k=1}^{\infty} \nu(x_k) < \infty .$$

Theorem 6. *Let E be a sequentially complete locally convex space such that both E and $(E', \beta(E', E))$ possess semi-absolute Schauder bases; then, if E is* (a) *semi-reflexive or* (b) *barrelled, an unconditionally convergent series in E is absolutely convergent.*

Proof. We first reduce case (b) to case (a). In case (b), E is complete (see [7]) and furthermore possesses a boundedly-complete basis. We may then use results of [9] (Theorems 2.2 and 2.3) or [25] to show that if E is not reflexive it contains a complemented subspace isomorphic to l_1; but this implies that $(E', \beta(E', E))$ is inseparable and could not have a basis. Hence E is reflexive and (b) reduces to (a).

Suppose that (φ_k) is a semi-absolute Schauder basis of $(E', \beta(E', E))$; then (χ_k), the dual sequence of (φ_k) is an unconditional Schauder basis of $(E', \beta(E', E))' = E$ in the weak topology $\sigma(E, E')$. By a theorem of Dubinsky and Retherford [3], (χ_n) is a Schauder basis of E; then by Proposition 4 (χ_n) is ∞-absolute. The result then follows by Theorem 5.

Corollary (Pietsch [4]). *If E is an F-space then the hypotheses of Theorem 5 imply that E is nuclear.*

Proof. We use a result of Grothendieck [5] (Chapitre II, Théoreme 8, Corollaire 1), that an F-space in which every unconditionally convergent series converges absolutely is nuclear.

5. *p*-absolute Bases

Schock [24] introduces a useful generalization of absolute bases.

Definition (8). A Schauder basis (χ_n) is *p-absolute* $1 \leqq p < \infty$ if $\Sigma\, a_n \chi_n$ converges if and only if for every continuous semi-norm ν,

$$\Sigma\, |a_n|^p \, [\nu(\chi_n)]^p < \infty .$$

Definition (9). (χ_n) is *p-Köthe* if it is p-absolute and the semi-norms

$$\nu_p(\Sigma\, a_n \chi_n) = (\Sigma\, |a_n|^p \, [\nu(\chi_n)]^p)^{1/p}$$

are a fundamental collection for the topology on E.

This is not the definition given by Schock, who calls p-Köthe "p-absolute". We remark that Rosenberger and Schock [23] have defined p-Köthe bases for $0<p<1$, and have shown that any F-space with a p-Köthe basis, $p<1$, is nuclear. It may be verified without difficulty that a p-absolute basis is unconditional, and a p-Köthe basis is u-Schauder.

Proposition 8. *A p-absolute basis of an F-space is p-Köthe.*

Proof. The proof is similar to Proposition 7.

It is easy to show that a basis which is Köthe and ∞-Köthe is automatically p-Köthe for $1<p<\infty$, so that Theorem 4 implies that "most" bases of nuclear spaces are p-Köthe for any p. There is a converse to this statement, due to Schock [24] for F-spaces (see also Gubitz [6], p. 81, where it is extended to LF-spaces).

Theorem 7. *Let E be a locally convex space with a Schauder basis (χ_n) which is both p-Köthe and q-Köthe with $1\leqq p<q\leqq\infty$; then E is nuclear.*

Proof. We assume $q<\infty$; the case $q=\infty$ is similar. Given a continuous semi-norm v, then v_p is also continuous and so there exists a continuous semi-norm ω with

$$v_p(x)\leqq\omega_q(x) \qquad x\in E,$$

i.e.

$$\sum_{n=1}^{\infty}|\varphi_n(x)|^p\,[v(\chi_n)]^p\leqq\left(\sum_{n=1}^{\infty}|\varphi_n(x)|^q\,[\omega(\chi_n)]^q\right)^{p/q}.$$

Thus for any sequence $a_n\geqq 0$, $a_n=0$ eventually,

$$\sum_{n=1}^{\infty}a_n[v(\chi_n)]^p\leqq\left(\sum_{n=1}^{\infty}a_n^{q/p}[\omega(\chi_n)]^q\right)^{p/q}.$$

Let s be such that

$$\frac{1}{s}+\frac{p}{q}=1$$

then we may deduce

$$\Sigma\left[\frac{v(\chi_n)}{\omega(\chi_n)}\right]^{ps}<\infty,$$

i.e.

$$\Sigma\left[\frac{v(\chi_n)}{\omega(\chi_n)}\right]^{r}<\infty$$

where $\dfrac{1}{r}=\dfrac{1}{p}-\dfrac{1}{q}$.

Let m be an integer such that $m\geqq r$; starting with $v=v^{(0)}$ we may determine continuous semi-norms $v^{(1)},\dots,v^{(m)}$ such that

$$\Sigma\left[\frac{v^{(k-1)}(\chi_n)}{v^{(k)}(\chi_n)}\right]^r<\infty \qquad k=1,2\dots m$$

and a repeated application of Hölder's inequality

$$\Sigma \frac{v(\chi_n)}{v^{(m)}(\chi_n)} < \infty$$

so that, by Theorem 4, E is nuclear.

Proposition 9. *A sequentially complete locally convex space with a p-absolute Schauder basis, where* $1 < p < \infty$, *is semi-reflexive.*

Proof. Let τ be the original topology on E, and let τ_p be the topology of the semi-norms v_p where

$$v_p(x) = \left(\sum_{n=1}^{\infty} |\varphi_n(x)|^p [v(\chi_n)]^p\right)^{1/p}.$$

Clearly each v_p is lower-semi-continuous in τ, and therefore as τ is sequentially complete a τ-bounded subset of E is τ_p-bounded.

We show that the p-absolute basis (χ_n) is boundedly-complete; for if $\left(\sum_{i=1}^{n} a_i \chi_i\right)$ is a bounded sequence in τ, then for each τ-continuous semi-norm v,

$$\sup_n v_p\left(\sum_{i=1}^{n} a_i \chi_i\right) < \infty,$$

$$\sum_{i=1}^{\infty} |a_i|^p [v(\chi_i)]^p < \infty$$

and $\sum_{i=1}^{\infty} a_i \chi_i$ converges.

We next show that (χ_n) is shrinking; for, if not, there is a bounded block basic sequence (η_n) (see [10], Theorem 5.4) and $f \in E'$ with $f(\eta_n) = 1$ for all n. Suppose $\Sigma |a_k|^p < \infty$ but Σa_k does not converge. For any semi-norm v we have

$$\sup_n v_p(\eta_n) < \infty$$

and therefore

$$\Sigma |a_n|^p v_p^p(\eta_n) < \infty.$$

It follows that $\Sigma a_n \eta_n$ converges and therefore $\Sigma a_n f(\eta_n)$ converges, i.e. Σa_n converges contrary to assumption. Hence (χ_n) is boundedly-complete and shrinking, and by Cook's theorem ([2]) E is semi-reflexive.

One would like to obtain a duality theorem for p-absolute or p-Köthe bases; however this only seems possible for Fréchet spaces. The following result is due to Schock [24], in view of Proposition 9.

Theorem 8. *Let E be a Fréchet space with a p-absolute basis (χ_n) for* $1 < p < \infty$. *Then (φ_n) is q-Köthe basis of $(E', \beta(E', E))$ where* $\frac{1}{p} + \frac{1}{q} = 1$.

We now turn to unconditional convergence in spaces with p-Köthe bases. Theorem 1, Corollary 2 has an obvious extension.

Proposition 10. *Let E be a locally convex space with a p-Köthe basis; suppose Σx_k converges unconditionally in E. Then, for every continuous semi-norm, if $p \geqq 2$*

$$\Sigma [\nu(x_k)]^p < \infty$$

or if $p \leqq 2$

$$\Sigma [\nu(x_k)]^2 < \infty .$$

Proof. The proof is the same as that of Theorem 1, Corollary 2; we use Orlicz's Theorem [16] on the space l_p, and the fact that E has a basic collection of semi-norms ν such that the Hausdorff quotient of (E, ν) is isomorphic to a dense subspace of l_p or a finite-dimensional space.

Theorem 9. *Let E be a barrelled space with a p-Köthe basis and a (different) q-Köthe basis where $\infty \geqq p > q$ and $p > 2$. Then E is nuclear.*

Proof. Let $r = \max(q, 2)$; then if (χ_n) is the p-Köthe basis of E, we have, by Proposition 10, that for $x \in E$

$$\sum_{n=1}^{\infty} |\varphi_n(x)|^r [\nu(\chi_n)]^r < \infty$$

for continuous semi-norm ν. If we define

$$\nu_r(x) = \left(\sum_{n=1}^{\infty} |\varphi_n(x)|^r [\nu(\chi_n)]^r \right)^{1/r}$$

then ν_r is lower-semi-continuous, and therefore continuous on E, as E is barrelled. Furthermore

$$\nu_p(x) \leqq \nu_r(x) \qquad x \in E$$

so that the semi-norms ν_r determine the topology on E. Hence (χ_n) is r-Köthe with $r < p$. By Theorem 7, E is nuclear.

Theorem 10. *Let E be a Fréchet space with a p-absolute basis and a (different) q-absolute basis with $1 \leqq q < p \leqq \infty$. Then E is nuclear.*

Proof. By Theorem 9, we may restrict attention to the case $1 \leqq q < p \leqq 2$. If $q = 1$, then Theorem 1 and Theorem 7 together show that E is nuclear. Hence we may suppose $1 < q < p \leqq 2$.

Then Theorem 8 shows that $(E', \beta(E', E))$ has an r-Köthe basis and an s-Köthe basis where

$$\frac{1}{r} + \frac{1}{p} = 1 ,$$

$$\frac{1}{s} + \frac{1}{q} = 1$$

so that $\infty > s > r \geqq 2$.

Furthermore $(E', \beta(E', E))$ is barrelled (Proposition 9) and so by Theorem 9 is nuclear. It follows that E is also nuclear (see Pietsch [20], p. 70).

Pietsch [22] Satz 8, shows that there is a non-nuclear Fréchet space E with a fundamental sequence of semi-norms $(v_1, v_2, \ldots)$ and a fundamental sequence $(\omega_1, \omega_2, \ldots)$ such that (E, v_n) is isomorphic to a subspace of l_p each n and (E, ω_n) is isomorphic to a subspace of l_q, each n, where $p \neq q$. Theorem 10 shows that this cannot occur if the (v_n) and (ω_n) are determined respectively by a p-absolute and a q-absolute basis.

Finally let us point out two results due to Wojtyński [26].

Theorem 11. *Let E be a Fréchet space with a p-absolute basis $(1 \leqq p < \infty)$ in which every basis is unconditional; then E is nuclear.*

Theorem 12. *Let E be a Fréchet space with 2-absolute basis (or more generally a projective limit of Hilbert spaces). Then every unconditional basis of E is 2-absolute.*

We remark that we have seen that Theorem 12 remains true with 2 replaced by 1 or ∞ (Theorems 1 and 3). However, Pełczyński [17] has shown that for $p \neq 1, 2, \infty$, the spaces l^p possess unconditional bases which are not p-absolute.

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