London Mathematical Society Lecture Note Series. 337

Methods in Banach Space Theory

Proceedings of the V Conference on Banach Spaces, Cáceres, Spain, 13–18 September 2004

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 2RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521685689

First published 2006

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library

ISBN-13 978-0-521-68568-9 paperback
ISBN-10 0-521-68568-0 paperback

EXTENSION PROBLEMS FOR $\mathcal{C}(K)$–SPACES AND TWISTED SUMS

N. J. KALTON

1. Introduction

This article can be regarded as an update on the handbook article by Zippin [27]. In this article Zippin drew attention to problems surrounding extensions of linear operators with values in $\mathcal{C}(K)$-spaces. The literature on this subject may be said to start with the work of Nachbin, Goodner and Kelley on the case when K is extremally disconnected around 1950. Thus the subject is over fifty years old, but it still seems that comparatively little is known in the general case. We are particularly interested in extending operators on separable Banach spaces when we can assume the range is $\mathcal{C}(K)$ for K a compact metric space. In this article we will sketch some recent progress on these problems.

2. Linear extension problems

It is, by now, a very classical result that a Banach space X is 1-injective if and only if X is isometric to a space $\mathcal{C}(K)$ where K is extremally disconnected; this is due to Nachbin, Goodner and Kelley [22], [11] and [17]. For a general compact Hausdorff space K the space $\mathcal{C}(K)$ is usually not injective (and, in particular, never if K is metrizable). However it is a rather interesting question to determine conditions when linear operators into arbitrary $\mathcal{C}(K)$–spaces can be extended. This problem was first considered in depth by Lindenstrauss in 1964 [18].

Let us introduce some notation. Suppose X is a Banach space and E is a closed subspace. Then, for $\lambda \geq 1$, we will say that the pair (E, X) has the $(\lambda, \mathcal{C})$-*extension property* if whenever $T_0 : E \to \mathcal{C}(K)$ is a bounded operator then there is an extension $T : X \to \mathcal{C}(K)$ with $\|T\| \leq \lambda \|T_0\|$. We say that X has the $(\lambda, \mathcal{C})$-*extension property* if (E, X) has the $(\lambda, \mathcal{C})$-extension property for every closed subspace E. We will use the term $\mathcal{C}$-*extension property* to

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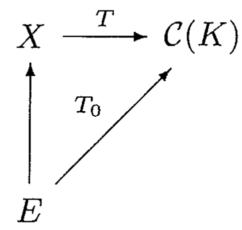
1991 *Mathematics Subject Classification.* Primary: 46B03, 46B20.
The author was supported by NSF grant DMS-0244515.

denote the $(\lambda, \mathcal{C})$-extension property for some $\lambda \geq 1$.



Usually we will want to suppose that X is separable and in this case it obviously suffices to take K metrizable; indeed since every $\mathcal{C}(K)$ for K metrizable is a contractively complemented subspace of $\mathcal{C}[0,1]$ we may even take $K = [0,1]$. Notice that c_0 is separably injective by Sobczyk's theorem [23]. This implies that if one chooses K to be the one-point compactification of $\mathbb{N}$ so that $\mathcal{C}(K) = c \approx c_0$ then one always has extensions when X is separable. A deep result of Zippin [25] shows that c_0 is the unique separably injective separable Banach space.

The spaces $\mathcal{C}(K)$ are $\mathcal{L}_\infty$-spaces, which means that locally they behave like ℓ_∞ and so are injective in a local sense. In 1964, Lindenstrauss [18] showed that if we restrict the operator T_0 to be compact then indeed an extension always exists and we can choose $\lambda = 1 + \epsilon$ for any $\epsilon > 0$. However the extension of bounded operators is more delicate. Indeed consider the Cantor set $\Delta = \{0,1\}^{\mathbb{N}}$ and $\varphi : \Delta \to [0,1]$ be the canonical surjection

$$\varphi((t_n)_{n=1}^\infty) = \sum_{n=1}^\infty \frac{t_n}{2^n}.$$

Then $\mathcal{C}[0,1]$ can be isometrically embedded into $\mathcal{C}(\Delta)$ via the embedding $f \to f \circ \varphi$. For this embedding, $\mathcal{C}[0,1]$ is uncomplemented in $\mathcal{C}(\Delta)$ (much more is true, cf. [3] p. 21). Thus the identity map on $\mathcal{C}[0,1]$ cannot be extended to $\mathcal{C}(\Delta)$ i.e. $\mathcal{C}(\Delta)$ fails the $\mathcal{C}$-extension property. The existence of this counterexample already implies that ℓ_1 fails the $\mathcal{C}$-extension property. Indeed let $Q : \ell_1 \to \mathcal{C}(\Delta)$ be a quotient map and let $E = Q^{-1}\mathcal{C}[0,1]$. Then the map $Q : E \to \mathcal{C}[0,1]$ cannot be extended to an operator $T : E \to \mathcal{C}[0,1]$. Indeed if such an extension exists then $T = SQ$ where $S : \mathcal{C}(\Delta) \to \mathcal{C}(\Delta)$ is a bounded operator, which is a projection of $\mathcal{C}(\Delta)$ onto $\mathcal{C}[0,1]$. Thus any space that contains ℓ_1 fails the $\mathcal{C}$-extension property. However in 1971, Lindenstrauss and Pełczyński [19] gave a positive result:

Theorem 2.1. *The space c_0 has the $(1+\epsilon, \mathcal{C})$-extension property for every $\epsilon > 0$.*

For a discussion of which spaces can replace $\mathcal{C}(K)$-spaces in this theorem see [8]. Twenty years later Zippin [26] gave a stronger result for ℓ_p when $p > 1$.

Theorem 2.2. *For $p > 1$ the spaces ℓ_p have the $(1, \mathcal{C})$-extension property.*

The characterization of spaces with the $\mathcal{C}$-extension property remains mysterious. It is for example not known if L_p for $1 < p < \infty$ has the $\mathcal{C}$-extension property but it is known that if $p \neq 2$ then L_p fails the $(1, \mathcal{C})$–extension property [12]. Recently the author [15] has characterized separable Banach spaces with the $(1+\epsilon, \mathcal{C})$-extension property in terms of properties of types. We will not discuss this in detail, but we note the following application:

Theorem 2.3. *Let X be a separable Orlicz sequence space not containing ℓ_1. Then X has the $\mathcal{C}$-extension property.*

Note that we do not claim the $(1, \mathcal{C})$ or $(1+\epsilon, \mathcal{C})$-extension property; this is a renorming theorem, so that X can be renormed to have the $(1+\epsilon, \mathcal{C})$-extension property.

3. Extensions by $C(K)$-spaces

An *extension* of a Banach space X by a space Y is a short exact sequence:

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0.$$

More informally we refer to Z as an extension of X by Y if Z is a Banach space with a subspace isometric to Y so that Z/Y is isometric to X. Such an extension *splits* if it reduces to a direct sum, i.e. Y is complemented in Z. We write Ext $(X, \mathcal{C}) = \{0\}$ if *every* extension of X by a $C(K)$–space splits.

Now suppose $T_0 : Y \to C(K)$ is a bounded linear operator. Then we can construct an extension of X by $C(K)$ by the pushout construction. Then T_0 has a bounded extension $T : Z \to C(K)$ if and only if this extension splits:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T_0} & \overset{T}{\swarrow} & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C(K) & \longrightarrow & Z' & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

This means that Ext $(X, \mathcal{C}) = \{0\}$ if and only if whenever Y is a Banach space and E is a subspace of Y with $Y/E \approx X$ then (E, Y) has the $\mathcal{C}$-extension property. For the special case when $Y = \ell_1$ one obtains a complete classification of subspaces of ℓ_1 with the $\mathcal{C}$-extension property (first noted by Johnson and Zippin [12]):

Theorem 3.1. *Let E be a subspace of ℓ_1; then (E, ℓ_1) has the $\mathcal{C}$-extension property if and only if Ext $(\ell_1/E, \mathcal{C}) = \{0\}$.*

Johnson and Zippin [12] went on to prove:

Theorem 3.2. *Let E be a subspace of ℓ_1 which is weak*-closed as a subspace of c_0^*; then (E, ℓ_1) has the $\mathcal{C}$-extension property.*

Recently the author [15] refined their arguments to show that in fact under these hypotheses (E, ℓ_1) has the $(1+\epsilon, \mathcal{C})$-extension property for every $\epsilon > 0$. (The original argument yielded only $3+\epsilon$ in general.)

In terms of extensions this means:

Corollary 3.3. *If X is the dual of a subspace of c_0 then Ext $(X, \mathcal{C}) = \{0\}$.*

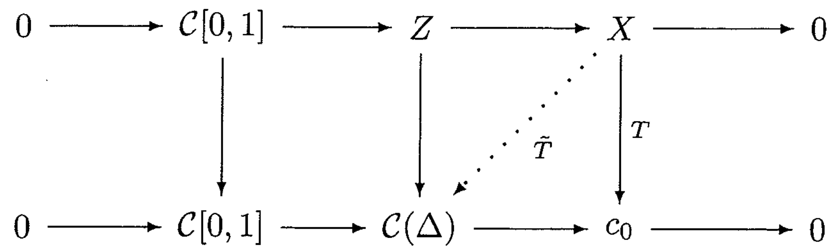
This suggests a natural problem:

Problem 1. *Let X be a separable Banach space. Is it true that Ext $(X, \mathcal{C}) = \{0\}$ if and only if X is isomorphic to the dual of a subspace of c_0?*

This is equivalent (via the automorphism results of Lindenstrauss and Rosenthal [20]) to asking if whenever (E, ℓ_1) has the $\mathcal{C}$-extension property if and only if there is an automorphism $U : \ell_1 \to \ell_1$ such that $U(E)$ is weak*-closed. There is some evidence for a positive answer to Problem 1. The author proved the following results in [14]:

Theorem 3.4. *Let X be a separable Banach space such that Ext $(X, \mathcal{C}) = \{0\}$. Then*
(i) X has the Schur property.
(ii) If X has a (UFDD) then X is isomorphic to the dual of a subspace of c_0.

The method of proof revolved around taking one non-trivial extension of $\mathcal{C}(K)$ namely the example created in §2, and performing a pullback construction for an arbitrary operator $T : X \to c_0$:



The existence of $\tilde{T} : X \to \mathcal{C}(\Delta)$ which lifts T is equivalent to the splitting of the pullback sequence. Thus if Ext $(X, \mathcal{C}) = \{0\}$ one can always lift T and this allows us to make deductions about the structure of X.

For a characterization of spaces X such that Ext $(X, \mathcal{C}) = \{0\}$ see [7]. We also remark that if K is a fixed countable compact metric space one may expect that Ext $(X, \mathcal{C}(K)) = \{0\}$ more often. The first non-trivial case is $K = \omega^\omega$ which is discussed in [5].

4. Universal extensions and automorphisms

In a recent paper Castillo and Moreno [6] related extension properties with the Lindenstrauss-Rosenthal automorphism theorems [20]. In their paper, Lindenstrauss and Rosenthal showed that if E and F are two isomorphic subspaces of c_0 of infinite codimension then there is an automorphism $U : c_0 \to c_0$ so that $U(E) = F$. In somewhat less exact language, one can say that there is at most one embedding, up to automorphism, of a separable Banach space into c_0. This is related to Sobczyk's theorem. They also investigated embeddings of separable spaces into ℓ_∞ and proved dual results for ℓ_1 (which we have already mentioned).

Now by Miljutin's theorem [21] all $\mathcal{C}(K)$-spaces are isomorphic for K uncountable and compact metric. The classical Banach-Mazur theorem states that every separable Banach space embeds into $\mathcal{C}[0,1]$ isometrically. The problem of obtaining automorphism results in $\mathcal{C}(K)$-spaces is clearly related to the extension problem; we will now make this relationship precise.

Let us say that a separable Banach space has the *separable universal $\mathcal{C}$-extension property* if (X,Y) has the $\mathcal{C}$-extension property whenever Y is a separable Banach space containing X. In effect one may always suppose that $Y=\mathcal{C}[0,1]$. The following result is a more precise version of a theorem of Castillo-Moreno [6] (see [16]):

Theorem 4.1. *Let X be a separable Banach space. The following conditions on X are equivalent:*
(i) X has the separable universal $\mathcal{C}$-extension property.
(ii) If X_1 and X_2 are two subspaces of $\mathcal{C}[0,1]$ with $X \approx X_1 \approx X_2$ then there is an automorphism $U : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ with $U(X_1) = X_2$.

Given this it becomes rather interesting to determine which spaces have the separable universal $\mathcal{C}$-extension property. It is a trivial consequence of Sobczyk's theorem [23] that c_0 has this property; in fact it has the separable universal $(2,\mathcal{C})$-extension property with the obvious meaning. It is then a consequence of Theorem 2.1 that every subspace of c_0 has the separable universal $(2+\epsilon,\mathcal{C})$-extension property. To see this we observe that if X is a subspace of c_0 which is also a subspace of a separable Banach space Y then we can form a separable superspace Z so that $X \subset c_0 \subset Z$ and $X \subset Y \subset Z$; just let Z be the quotient of $c_0 \oplus_1 Y$ by the subspace $\{(x,-x): x \in X\}$ and identify X with the subspace of Z spanned by the cosets of $\{(x,0): x \in X\}$.

The obvious place to start looking for more spaces is to consider space with the separable universal $(1,\mathcal{C})$-extension property. However, in 1964, Lindenstrauss [18] showed that these spaces are exactly the finite-dimensional polyhedral spaces. There are no infinite-dimensional examples. The next obvious try is to consider the separable universal $(1+\epsilon,\mathcal{C})$-extension property for every $\epsilon > 0$. This was first done by Speegle [24], whose main result is that such a space cannot have a uniformly smooth norm. Speegle also asked whether ℓ_1 has this property.

In fact we have recently shown that the answer to Speegle's question is positive:

Theorem 4.2. *Let X be almost isometric to the dual of a subspace of c_0. Then X has the separable universal $(1+\epsilon,\mathcal{C})$-extension property for every $\epsilon > 0$.*

This extends Theorem 3.2 because a weak*-closed subspace of ℓ_1 is the dual of a quotient of c_0, and Alspach [1] showed that a quotient of c_0 is almost isometric to a subspace of c_0. Let us notice here a connection with Problem 1. It is a result of Bourgain [4] that ℓ_1 contains an uncomplemented copy X of ℓ_1. Now (X,ℓ_1) has the $\mathcal{C}$-extension property and so Ext $(\ell_1/X,\mathcal{C}) = \{0\}$.

Problem 2. *Suppose X is a subspace of ℓ_1 which is isomorphic to ℓ_1; is ℓ_1/X the dual of a subspace of c_0?*

Unfortunately, Bourgain's construction in local in nature and so if one creates the example in a natural way the space ℓ_1/X is simply an ℓ_1−sum of finite-dimensional spaces. This Problem asks for a global construction.

Theorem 4.2 is not the complete answer to the characterization of spaces with the $(1+\epsilon, \mathcal{C})$-extension property for every $\epsilon > 0$. We also have:

Theorem 4.3. *Let X be a subspace of $L_1(0,1)$ whose unit ball is compact for the topology of convergence in measure. Then X has the separable universal $(1+\epsilon, \mathcal{C})$-extension property for every $\epsilon > 0$.*

In [10] an example is given of a subspace of L_1 where the unit ball is compact for convergence in measure and yet X is not almost isometric to the dual of a subspace of c_0. Another example constructed in [15] is a Nakano space ℓ_{p_n} where $\lim_{n\to\infty} p_n = 1$.

We now have a fairly rich class of spaces for which the equivalent conditions of Theorem 4.1 hold; this class includes all weak*-closed subspaces of ℓ_1 and all subspaces of c_0. It is not hard to see we can expand the class by taking direct sums (e.g. $c_0 \oplus \ell_1$) and with slightly more work, extensions. Thus any extension of c_0 by ℓ_1 satisfies Theorem 4.1. The fact that there are non-trivial extensions of c_0 by ℓ_1 is proved in [5].

5. Homogeneous Zippin selectors

Suppose E is a subspace of a Banach space X. Then Zippin [26, 27] introduced a criterion for (E, X) to have the $(\lambda, \mathcal{C})$-extension property. We say that a map $\Phi : B_{E^*} \to X^*$ is a *Zippin selector* of Φ is weak*-continuous and $\Phi(e^*)|_E = e^*$ for every $e^* \in B_{E^*}$. Then [26] (E, X) has the (λ, C) extension property if and only if there is a Zippin selector $\Phi : B_{E^*} \to \lambda B_{X^*}$.

In certain special cases one can find a homogeneous selector, i.e. one can choose Φ so that $\Phi(\alpha e^*) = \alpha\Phi(e^*)$ for every $e^* \in E^*$. Indeed suppose $X = \ell_p$ where $1 < p < \infty$ and E is any closed subspace. Define $\Phi(e^*)$ to be the unique norm-preserving extension of e^* to ℓ_p. Then Φ is homogeneous and weak*-continuous. To see this suppose (e_n^*) is a sequence in B_{E^*} so that e_n^* converges weak* to e^*. To show that $(\Phi(e_n^*))_{n=1}^\infty$ converges weak* to $\Phi(e^*)$ it suffices to show this for some subsequence. We therefore select $e_n \in B_E$ so that $e_n^*(e_n) = \|e_n^*\|$ and suppose, by passing to a subsequence that (e_n) weakly converges to some $e \in E$. Then using the special properties of ℓ_p it is quite clear that $\Phi(e_n^*)$ converges weak* to some x^* so that $\|x^*\| = \|e\|^{p-1}$ and $x^*(e) = \|e\|^p$. Now $x^*|_E = e^*$ and, if $e^* = 0$ we have $e = 0$ and $x^* = 0$; if ! not $\|e^*\| \geq e^*(e)/\|e\| = \|e\|^{p-1} = \|x^*\|$ so that $\Phi(e^*) = x^*$.

If we have a homogeneous Zippin selector for (E, X) we can extend Φ to be defined on homogeneous on E^* and continuous for the bounded weak*-topology (equivalently weak*-continuous on bounded sets). We define

$$\|\Phi\| = \sup\{\|\Phi(e^*)\| : \|e^*\| \leq 1\}.$$

Now consider the embedding of ℓ_1 into $\mathcal{C}(B_{\ell_\infty})$. It is shown in [16] that there is a homogeneous Zippin selector $\|\Phi\|$ selector for $(\ell_1, \mathcal{C}(B_{\ell_\infty}))$ with $\|\Phi\| = 1$. It follows that:

Theorem 5.1. *Suppose X is a separable Banach space containing ℓ_1. Then for any $\epsilon > 0$ there is a homogeneous Zippin selector Φ so that $\|\Phi\| < 1 + \epsilon$.*

To see this, use Theorem 4.2. There is a linear operator $T : X \to \mathcal{C}(B_{\ell_\infty})$ with $\|T\| < 1 + \epsilon$ and $Tx = x$ for $x \in \ell_1$. Define $\Psi : \ell_1^* \to X^*$ by $\Psi = T^* \circ \Phi$.

In the general the $(\lambda, \mathcal{C})$-extension property on a pair (E, X) does not imply the existence of a homogeneous Zippin selector Φ with $\|\Phi\| = \lambda$. In fact if E is non-separable there are examples where no homogeneous Zippin selector exists [9]. However Castillo and Suarez [9] recently applied an old result of Benyamini [2] to obtain:

Theorem 5.2. *If E is a separable subspace of a Banach space X so that (E, X) has the $\mathcal{C}$-extension property then there is a homogeneous Zippin's selector $\Phi : E^* \to X^*$.*

It turns out that the existence of homogeneous Zippin selectors is important for c_0–products. In fact we can now prove [16]:

Theorem 5.3. *If X has the separable universal $\mathcal{C}$-extension property then $c_0(X)$ also has the separable universal $\mathcal{C}$-extension property. The space $c_0(\ell_1)$ has the separable universal $(2 + \epsilon, \mathcal{C})$-extension property.*

The space $c_0(\ell_1)$ is the space with the most complicated structure that we know satisfies Theorem 4.1. We now turn to the question raised by Castillo and Moreno [6]: does a separable Hilbert space satisfy this Theorem? Indeed Speegle's theorem [24] shows that ℓ_2 fails the separable universal $(1 + \epsilon, \mathcal{C})$-extension property but, as the example of c_0 shows, this cannot resolve the question in general. Let us start by considering the canonical inclusion $\ell_2 \subset \mathcal{C}(B_{\ell_2^*})$. Theorem 5.2 implies:

Theorem 5.4. *Suppose $1 < p < \infty$. Then there is a homogeneous Zippin selector for $(\ell_p, \mathcal{C}(B_{\ell_p^*}))$.*

This does not seem to immediately help us decide whether ℓ_p has the separable universal $\mathcal{C}$-extension property. However if ℓ_p is embedded in some X so that (ℓ_p, X) has the $\mathcal{C}$-extension property then the argument of Theorem 5.1 shows that (ℓ_p, X) has a homogeneous Zippin selector Φ. This allows to make a renorming of X by setting, for example:

$$|x| = \sup\{|\langle x, \Phi(e^*)\rangle| : \ e^* \in B_{\ell_p^*}\}$$

and then

$$\|x\|_1 = (\tfrac{1}{2}\|x\|^p + \tfrac{1}{2}|x|^p)^{1/p}.$$

Thus $\|\cdot\|_1$ is an equivalent norm on X which agrees with the original norm on ℓ_p. However it has an additional property. There exists a constant $c > 0$ so that

$$\lim_{n\to\infty} \|x + u_n\|_1^p \geq \|x\|_1^p + c^p \lim_{n\to\infty} \|u_n\|_1^p$$

whenever $x \in X$, $(u_n)_{n=1}^\infty$ is a weakly null sequence in ℓ_p and all the limits exist. This condition as it turns out is also sufficient for the $\mathcal{C}$-extension property:

Theorem 5.5. *Suppose $1 < p < \infty$. Suppose $\ell_p \subset X$ where X is a separable Banach space. In order that (ℓ_p, X) has the $\mathcal{C}$-extension property it is necessary and sufficient that there is an equivalent norm $\|\cdot\|_1$ on X ℓ_p so that for some $c > 0$,*

$$\lim_{n\to\infty} \|x + u_n\|_1^p \geq \|x\|_1^p + c^p \lim_{n\to\infty} \|u_n\|_1^p$$

whenever $x \in X$, $(u_n)_{n=1}^\infty$ is a weakly null sequence in ℓ_p and all the limits exist.

Thus our problem is reduced to a renorming question. Let us note here that we do not require that the new norm $\|\cdot\|_1$ coincides with the original norm on ℓ_p. To see what this means let us suppose we have $1 < p < \infty$ and H is an Hilbertian subspace of L_p. Then if $p \geq 2$, H is complemented by a result of Kadets and Pełczyński [13] and so (H, L_p) has the $\mathcal{C}$-extension property. On the other hand, if $1 < p < 2$ then the hypothesis of Theorem 5.5 holds for the original norm on L_p. To see this, observe first that for a suitable constant $a > 0$ we have an inequality

$$|1 + t|^p \geq 1 + pt + a\min(|t|^p, |t|^2) \qquad -\infty < t < \infty.$$

Then suppose $\|f\|_p = 1$ and $(g_n)_{n=1}^\infty$ is a weakly null sequence with $\|g_n\|_p \leq 1$. Let sgn $t = t/|t|$ if $t \neq 0$ and let sgn $0 = 0$.

$$\begin{aligned}
&\int |f + g_n|^p dt \\
&\geq 1 + p\int |f|^{p-1}(\operatorname{sgn} f)g_n\,dt + a\int_{|g_n|<|f|} |f|^{p-2}|g_n|^2 dt + a\int_{|g_n|\geq|f|} |g_n|^p dt \\
&\geq 1 + p\int |f|^{p-1}(\operatorname{sgn} f)g_n\,dt + a\left(\int_{|g_n|<|f|} |g_n|^p dt\right)^{2/p} + a\left(\int_{|g_n|\geq|f|} |g_n|^p dt\right)^{2/p} \\
&\geq 1 + p\int |f|^{p-1}(\operatorname{sgn} f)g_n\,dt + \frac{a}{2}\|g_n\|_p^2.
\end{aligned}$$

Note that

$$\lim_{n\to\infty} \int |f|^{p-1}(\operatorname{sgn} f)g_n\,dt = 0.$$

From this it follows easily that the norm on L_p has the property that for some $c > 0$ we have

$$\lim_{n\to\infty} \|f + g_n\|_p^2 \geq \|f\|_p^2 + c^p \lim_{n\to\infty} \|g_n\|_p^2$$

whenever $f \in L_p$, $(g_n)_{n=1}^\infty$ is a weakly null sequence in L_p and all the limits exist. In fact it easy to see via renorming that any Banach space with a 2-concave unconditional basis satisfies a similar condition. These considerations are, however, a form of overkill. We do not require a condition on every weakly null sequence; instead we need the conditions for weakly null sequences in the given Hilbertian subspace.

Surprisingly when one attempts a more delicate analysis one finds that the (UMD)-property of Burkholder begins to play a role. Recall that a Banach space X has the (UMD)-property if for some (respectively, every) $1 < p < \infty$

there is a constant $C = C(p)$ so that for any finite X-valued martingale $(M_n)_{n=0}^N$ one has an estimate

$$(\mathbb{E}\|\sum_{j=1}^{N}\epsilon_j dM_j\|^p)^{1/p} \leq C\mathbb{E}\|M_N\|^p)^{1/p} \qquad \epsilon_j = \pm 1,\ j = 1, 2, \ldots, N$$

where $dM_j = M_j - M_{j-1}$.

The connection is expressed in the following theorem:

Theorem 5.6. *Suppose $1 < p < \infty$ and $\ell_p \subset X$ where X is a Banach space with (UMD). Then X can be given an equivalent norm so that*

$$\lim_{n\to\infty}(\tfrac{1}{2}\|x+u_n\|^p + \tfrac{1}{2}\|x-u_n\|^p) \geq \|x\|^p + c^p \lim_{n\to\infty}\|u_n\|^p$$

whenever $x \in X$, $(u_n)_{n=1}^\infty$ is a weakly null sequence in ℓ_p and all the limits exist.

This is not quite what we need but one quickly gets:

Theorem 5.7. *Suppose $1 < p < \infty$ and $\ell_p \subset X$ where X is a Banach space with (UMD). If X has an unconditional basis (or even a (UFDD)) then (ℓ_p, X) has the $\mathcal{C}$-extension property.*

This result applies when $X = L_r$ for some $1 < r < \infty$ or is a reflexive Schatten ideal. We do not know whether the result remains true if one removes the (UFDD) hypothesis: however the (UMD) hypothesis is necessary:

Theorem 5.8. *If $1 < p < \infty$ there is a super-reflexive Banach space X with unconditional basis containing ℓ_p so that (ℓ_p, X) fails to have the $\mathcal{C}$-extension property.*

This answers the question of Castillo and Moreno negatively. There must be at least two non-automorphic embeddings of a Hilbert space into $\mathcal{C}[0,1]$. However the methods are very specific to ℓ_p-spaces and it is natural to ask:

Problem 3. *Is there any super-reflexive example of a separable Banach space with the separable universal $\mathcal{C}$-extension property?*

Of course we can eliminate any space which contains a complemented copy of ℓ_p for $1 < p < \infty$ (such as L_p). The example in Theorem 5.8 proves that there are super-reflexive spaces failing the $\mathcal{C}$-extension property (thus answering a question of Zippin [27]). However we may still ask:

Problem 4. *Does every separable Banach space with (UMD) have the $\mathcal{C}$-extension property?*

We do not even know the answer for L_p when $1 < p < \infty$. See also [27].

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