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Minimal and strongly minimal Orlicz sequence spaces

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1. Introduction.

The structure theory of Orlicz sequence spaces was initiated in work of Lindberg [6] and Lindenstrauss and Tzafriri ([7], [8] and [9]; see also [10]) in the early seventies. Recently this study has been continued by Hernandez and Rodriguez-Salinas ([2], [3]). In their work, Lindenstrauss and Tzafriri introduced the class of minimal Orlicz sequence spaces and conjectured that these spaces are prime. The only separable prime spaces known are the ℓ_p and c_0, but Lindenstrauss and Tzafriri gave other examples of minimal Orlicz spaces. The purpose of this note is to show that this conjecture is false in general, but that a smaller non-trivial class of strongly minimal spaces is introduced which still has the potential to contain new prime spaces. Our results are achieved by introducing separate necessary and sufficient conditions for a reflexive Orlicz sequence space ℓ_G to be complemented in another such space ℓ_F.

We now proceed to a more detailed discussion of the basic definitions and our results. We refer to [10] for the basic facts about Orlicz sequence spaces, but review here some key definitions and ideas. It will be convenient to allow Orlicz functions to be possibly non-convex, even though we do not wish to discuss non-locally convex examples. Thus we will for the purposes of this note consider an Orlicz function to be a continuous function $F : [0,\infty) \to [0,\infty)$ satisfying $F(0) = 0$, such that $F(x) > 0$ if $x > 0$ and satisfying, for a suitable constant, C, $F(tx) \leq CtF(x)$ whenever $0 \leq t \leq 1$ and $0 < x < \infty$. We say that F satisfies the Δ_2-condition if $F(2x) \leq KF(x)$ for a suitable constant K and all x. Two Orlicz functions F and G are equivalent if $\log F(x)/G(x)$ is bounded on $(0,\infty)$ and equivalent near zero if $\log F(x)/G(x)$ is bounded on $(0,1)$. Any Orlicz function satisfying our definition and the Δ_2-condition is then equivalent to a convex Orlicz function.

If F is an Orlicz function satisfying the Δ_2-condition then we define the Orlicz function space $L_F = L_F(0,\infty)$ to be the space of all measurable real functions f on $(0,\infty)$ satisfying

$$\int_0^\infty F(|f(t)|)\,dt < \infty$$

and this is isomorphic to a Banach space if we equip it with the quasi-norm whose unit ball B is given by $B = \{f : \int F(|f|)\,dt \leq 1\}$. Similarly the Orlicz sequence space ℓ_F consists of all sequences $(x_n)_{n=1}^\infty$ such that $\sum_{n=1}^\infty F(|x_n|) < \infty$; its unit ball consists of all sequences for which $\sum F(|x_n|) \leq 1$. ℓ_F and ℓ_G coincide as Banach spaces if and only F and G are equivalent near zero. $L_F(0,\infty)$ is reflexive if and only if there exists $\alpha > 0$ and a constant C so that if $0 \le t \le 1$ and $0 < x < \infty$, $F(tx) \le Ct^{1+\alpha}F(x)$. ℓ_F is reflexive if F is equivalent near zero to a function G for which $L_G(0,\infty)$ is reflexive.

If F is convex and L_F is reflexive we define F^* by $F^*(t) = \sup_{0<s<\infty}(st - F(s))$. Then F^* is also a convex Orlicz function satisfying the Δ_2-condition, and L_F^* can be naturally identified with L_{F^*}. Similarly, ℓ_F^* can be naturally identified with ℓ_{F^*}.

Let us now assume that $F(x) = \exp(\phi(\log x))$ for $x > 0$ where ϕ is a uniformly Lipschitz function; this is, in particular the case when F is convex and satisfies the Δ_2-condition. Then we define E_F to be the closure of the set of functions $F_t(x) = F(tx)/F(t)$, for $0 < t \le 1$, in $C[0,1]$; E_F is a compact set in $C[0,1]$. We let C_F be the closed convex hull of E_F. It is shown in [6] (see [10]) that ℓ_G is isomorphic to a closed subspace of ℓ_F if and only if G is equivalent near zero to some $G_1 \in C_F$. Lindenstrauss and Tzafriri [8] showed that if $G \in E_F$ then ℓ_G is isomorphic to a complemented subspace of ℓ_F. They asked if the converse was true, i.e. if ℓ_G is isomorphic to a complemented subspace of ℓ_F, is G equivalent near zero to some $G_1 \in E_F$? The author gave a counter-example to this with $G(x) = x^p$ in [5].

However, these considerations lead Lindenstrauss and Tzafriri to introduce the class of minimal Orlicz functions. We define an Orlicz function satisfying the above conditions to be minimal if $F \in E_G$ whenever $G \in E_F$. It is easy to see that if F is minimal then it is equivalent to a convex and minimal Orlicz function. In fact if $\tilde{F}$ is equivalent to F and is convex then there exists $G \in E_{\tilde{F}}$ which is minimal; further some $F_1 \in E_G$ is equivalent to F by the minimality of F so that F_1 is both minimal and convex and equivalent to F. We shall refer to any Orlicz sequence space ℓ_F where F is equivalent near zero to a minimal Orlicz function as a minimal Orlicz sequence space. It is shown in [8] that if ℓ_F is a minimal Orlicz sequence space and $G \in E_F$ then $\ell_F \approx \ell_G$ and this suggests the conjecture that each such space is prime. In [8] non-trivial examples of minimal Orlicz sequence spaces were constructed. More recently, Hernandez and Rodriguez-Salinas ([3]) gave an explicit example, which actually was introduced for different purposes in [4]. This example, which is reflexive, is given by

$$(*) \qquad F(t) = t^p \exp\Big(\sum_{n=0}^{\infty}(1-\cos(2\pi(\log t)/2^n))\Big)$$

where $1 < p < \infty$.

Let us define a minimal Orlicz sequence space ℓ_F to be *strongly minimal* if, whenever ℓ_G is an Orlicz sequence space which is isomorphic to a complemented subspace of ℓ_F then G is equivalent to function in E_F. We shall show that there is a non-trivial minimal reflexive Orlicz sequence space which is not strongly minimal and in fact contains a complemented copy of ℓ_p for some p; this space cannot be prime so that the Lindenstrauss-Tzafriri conjecture is false. However, we show that the space ℓ_F with F given by (*) is strongly minimal. This does not show that the space is prime; however it does suggest the possibility in view of the following:

THEOREM 1.1. *Let ℓ_F be a strongly minimal reflexive Orlicz sequence space. Let X be a complemented subspace of ℓ_F with an unconditional basis. Then X contains a complemented subspace isomorphic to ℓ_F. In particular if $X \approx X \times X$ then $X \approx \ell_F$.*

PROOF: It is easy to show that some subsequence of the unconditional basis is equivalent to the canonical basis of an Orlicz sequence space ℓ_G. But then G is equivalent near zero to a function in E_F and so (see [8]) $\ell_G \approx \ell_F$. The last assertion is a well-known form of the Pelczynski decomposition technique.∎

In view of Theorem 1.1 we can relate our example to three open problems (this observation is due to Peter Casazza). If ℓ_F is not a new prime space then it either has a complemented subspace X which fails to have an unconditional basis, or ℓ_F fails to have the Schroeder-Bernstein Property [1].

This research was initiated during a visit to Spain in May 1989. The author would like to thank Francisco Hernandez for some interesting discussions on this topic and the Universities of Madrid and Zaragoza for their hospitality.

2. Complemented subspaces of Orlicz sequence spaces.

We recall that a basic sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X dominates a basic sequence $(y_n)_{n=1}^{\infty}$ in a Banach space Y provided there is a constant M so that for all $a_1, \ldots, a_n$ and $n \in \mathbb{N}$,

$$\|\sum_{i=1}^{n} a_i y_i\| \le M \|\sum_{i=1}^{n} a_i x_i\|.$$

If (x_n) is a basis of X then (x_n^*) denotes the biorthogonal functionals in X^*. The following lemma is very well-known and we only sketch the proof.

LEMMA 2.1. *Let X and Y be reflexive Banach spaces with symmetric bases $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$, respectively. In order that X be isomorphic to a complemented subspace of Y it is necessary and sufficient that there is an increasing sequence of positive integers $(p_n)_{n=0}^{\infty}$ with $p_0 = 0$ and block basic sequences $u_n = \sum_{i=p_{n-1}+1}^{p_n} \alpha_i y_i$, $\phi_n = \sum_{i=p_{n-1}+1}^{p_n} \beta_i y_i^*$ so that (x_n) dominates (u_n), (x_n^*) dominates (ϕ_n) and $\inf |\phi_n(u_n)| > 0$.*

PROOF: One direction follows easily from standard "gliding hump" techniques. For the other assume that $(u_n), (\phi_n)$ satisfy the given conditions. We may assume without loss of generality that $\phi_n(u_n) = 1$ for all n. Define $A : Y \to X$ by $Ay = \sum \phi_n(y)x_n$ and $B : X \to Y$ by $Bx = \sum x_n^*(x)u_n$. It suffices to show that both A and B are well-defined and bounded. For $y \in Y$, $x^* \in X^*$, and $m \in \mathbb{N}$,

$$\begin{aligned} |\sum_{k=1}^{m} x^*(x_k)\phi_k(y)| &\le \|y\| \|\sum_{k=1}^{m} x^*(x_k)\phi_k\| \\ &\le M\|y\|\|x^*\| \end{aligned}$$

for a suitable constant M independent of m, y, x^*. This shows that A is indeed bounded. A very similar argument shows that B is bounded. ∎

LEMMA 2.2. *Let F be a convex Orlicz function such that $L_F(0,\infty)$ is reflexive (so that F and F^* are both continuous.) Then for $0 \le x < \infty$ we have $x \le F^{-1}(x)(F^*)^{-1}(x) \le 2x$.*

PROOF: The right-hand side inequality is obvious from $uv \le F(u) + F^*(v)$ upon substituting $F(u) = F^*(v) = x$. For the left-hand inequality, suppose $x = F(u)$ and note that $F(u)/u \le F'(u)$ where F' denotes the left-hand derivative of F. Then $F^*(F'(u)) = uF'(u) - F(u)$ and $(F^*)'(F'(u)) \le u$. Thus $(F^*)'(x/u) \le u$ and hence $F^*(x/u) \le x$. Thus $x \le u(F^*)^{-1}(x) = F^{-1}(x)(F^*)^{-1}(x)$.∎

THEOREM 2.3. *Let F, G be convex Orlicz functions such that $L_F(0,\infty)$ and $L_G(0,\infty)$ are reflexive. Then for ℓ_G to be isomorphic to a complemented subspace of ℓ_F it is necessary and sufficient that there is a constant C and a sequence μ_n of probability measures each with compact support in $(0,1]$ such that:*

(1)
$$\int F(F^{-1}(t)x)\frac{d\mu_n(t)}{t} \le CG(x) \qquad 2^{-n} \le x \le 1$$

(2)
$$\int F^*((F^*)^{-1}(t)x)\frac{d\mu_n(t)}{t} \le CG^*(x) \qquad 2^{-n} \le x \le 1.$$

PROOF: Suppose (1) and (2) hold. Since $\int t^{-1}d\mu_n < \infty$ for each n, we may find an increasing sequence of positive integers $(p_n)_{n=0}^\infty$ with $p_0 = 0$ and a sequence of nonnegative measurable functions $(f_n)_{n=0}^\infty$ with support f_n contained in $[p_{n-1}, p_n]$, so that f_n is nonincreasing on $[p_{n-1}, p_n]$ and such that $\lambda(f_n > t) = \int_{(t,1]} s^{-1}d\mu_n(s)$.

Let $u_n(t) = F^{-1}(f_n(t))$ and let $v_n(t) = (F^*)^{-1}(f_n(t))$. Then clearly

$$\int_0^\infty F(u_n(t))dt = \int_0^\infty F^*(v_n(t))dt = \int_0^\infty f_n(t)dt = 1.$$

Thus $\|u_n\|_{L_F} = \|v_n\|_{L_{F^*}} = 1$.

Suppose $a_1, \ldots, a_n \in \mathbb{R}$ with $\sum_{i=1}^n G(|a_i|) \le C^{-1}$. Let $J = \{i : 1 \le i \le n,\ |a_i| \ge 2^{-i}\}$. Then

$$\begin{aligned}\int_0^\infty F(\sum_{i\in J} a_iu_i(t))dt &= \sum_{i\in J}\int_{p_{i-1}}^{p_i} F(|a_i|F^{-1}(f_i(t))dt \\ &= \sum_{i\in J}\int F(|a_i|F^{-1}(t))\frac{d\mu_i(t)}{t} \\ &\le C\sum_{i\in J} G(|a_i|) \le 1.\end{aligned}$$

Thus $\|\sum_{i\in J} a_iu_i\|_{L_F} \le 1$ while $\|\sum_{i\notin J} a_iu_i\|_{L_F} \le \sum_{i=1}^n 2^{-i} \le 1$. It follows that the unit vector basis of ℓ_G dominates (u_n) and a similar argument shows that the unit vector basis of ℓ_{G^*} dominates (v_n).

Now let P be the natural averaging projection of L_F onto ℓ_F, (which we identify as the subspace of L_F of functions constant on each interval $(n-1, n]$ for $n \in \mathbb{N}$) i.e.

$$Pf = (\int_{n-1}^n f(t)dt)_{n=1}^\infty.$$

We also use P for the same projection on L_{F^*}. Let $\tilde{u}_n = Pu_n$ and $\tilde{v}_n = Pv_n$. Since P is bounded on both L_F and L_{F^*}, (u_n) dominates the block basic sequence $(\tilde{u}_n)$ and (v_n) dominates the sequence $(\tilde{v}_n)$. Now since u_n and v_n are each nonincreasing on $[p_{n-1}, p_n]$ we have

$$\begin{aligned}\sum_{i=p_{n-1}}^{p_n} \tilde{u}_{n,i}\tilde{v}_{n,i} &= \sum_{i=p_{n-1}+1}^{p_n}\int_{i-1}^i u_n(t)dt\int_{i-1}^i v_n(t)dt \\ &\ge \sum_{i=p_{n-1}+1}^{p_n} u_n(i)v_n(i) \\ &\ge \int_{p_{n-1}+1}^{p_n} u_n(t)v_n(t)dt \\ &= \int_{p_{n-1}+1}^{p_n} F^{-1}(f_n(t))(F^*)^{-1}(f_n(t))dt \\ &\ge \int_{p_{n-1}+1}^{p_n} f_n(t)dt \\ &= 1 - \int_{p_{n-1}}^{p_{n-1}+1} f_n(t)dt.\end{aligned}$$

Now we split into two cases. If $\liminf \int_{p_{n-1}}^{p_{n-1}+1} f_n dt < \frac{1}{2}$ we can pass to a subsequence and apply Lemma 1 to obtain the conclusion. In the other case we can suppose $\int_{p_{n-1}}^{p_{n-1}+1} f_n dt \ge \frac{1}{2}$ for all n. In this case $f_n(p_{n-1} + \frac{1}{4}) \ge \frac{1}{4}$, and hence $\tilde{u}_{n,p_{n-1}+1} \ge \frac{1}{4}F^{-1}(\frac{1}{4})$. Thus $(\tilde{u}_n)$ dominates the unit vector basis of ℓ_F; similarly $(\tilde{v}_n)$ dominates the unit vector basis of ℓ_{F^*}. As $(\tilde{u}_n)$ and $(\tilde{v}_n)$ are dominated respectively by the unit vector bases of ℓ_G and ℓ_{G^*} we conclude that G is equivalent to F in this case.

Conversely, let us suppose that ℓ_G is equivalent to a complemented subspace of ℓ_F. We denote by (e_n) and (e_n^*) the canonical bases in ℓ_F and ℓ_F^*. We may suppose that there exist normalized block basic sequences (u_n) in ℓ_F and (v_n) in ℓ_{F^*}, equivalent respectively to the unit vector bases of ℓ_G and ℓ_{G^*}, of the form $u_n = \sum_{p_{n-1}+1}^{p_n} \alpha_ie_i$ and $v_n = \sum_{p_{n-1}+1}^{p_n} \beta_ie_i^*$ where $0 = p_0 < p_1 < \ldots < p_n < \ldots$ and so that for some $\delta > 0$,

$$|\sum_{i=p_{n-1}+1}^{p_n} \alpha_i\beta_i| \ge \delta.$$

Note also that

$$\sum_{i=p_{n-1}+1}^{p_n} |\alpha_i\beta_i| \le \sum_{i=p_{n-1}}^{p_n} (F(|\alpha_i|) + F^*(|\beta_i|)) \le 2.$$

Let $K = 4/\delta$ and $A_n = \{i : p_{n-1}+1 \le i \le p_n,\ \max(F(|\alpha_i|), F^*(|\beta_i|)) \le K|\alpha_i\beta_i|\}$. Let B_n be the complement of A_n relative to $\{i : p_{n-1}+1 \le i \le p_n\}$. Then

$$\begin{aligned}\sum_{i\in B_n} |\alpha_i\beta_i| &\le \frac{1}{K}\sum_{i\in B_n} F(|\alpha_i|) + F^*(|\beta_i|)\\ &\le \frac{2}{K} = \frac{\delta}{2}.\end{aligned}$$

Thus if $\sigma_n = \sum_{i\in A_n} |\alpha_i\beta_i|$ then $\sigma_n \ge \frac{\delta}{2}$.

Now for $i \in A_n$ we have

$$|\alpha_i| \le F^{-1}(K|\alpha_i||\beta_i|) \le \frac{2K|\alpha_i\beta_i|}{(F^*)^{-1}(K|\alpha_i\beta_i|)}$$

so that $|\beta_i| \ge (2K)^{-1}(F^*)^{-1}(K|\alpha_i\beta_i|)$. This and a similar inequality for α_i imply the existence of a constant $c_1 > 0$ depending only on F, F^* and δ so that, for $i \in A_n$,

$$\begin{aligned}|\alpha_i| &\ge c_1 F^{-1}(|\alpha_i\beta_i|)\\ |\beta_i| &\ge c_1 (F^*)^{-1}(|\alpha_i\beta_i|).\end{aligned}$$

Now let $\tilde{u}_n = \sum_{i\in A_n} \alpha_i e_i$. Plainly the canonical basis of ℓ_G dominates $(\tilde{u}_n)$. Thus there exists a constant C_1 so that for any $\gamma > 0$ there exists $n_0 = n_0(\gamma)$ so that if $n \ge n_0$ and $\gamma \le x \le 1$,

$$\sum_{i\in A_n} F(x|\alpha_i|) \le C_1 G(x).$$

Thus for $n \ge n_0$ and $\gamma \le x \le 1$,

$$\sum_{i\in A_n} F(xc_1F^{-1}(|\alpha_i\beta_i|)) \le C_1G(x),$$

and by utilizing the Δ_2-condition, we obtain for a suitable constant C_2 depending only F, F^* and δ,

$$\sum_{i\in A_n} F(xF^{-1}(|\alpha_i\beta_i|) \le C_2G(x).$$

A similar argument can be applied to $\tilde{v}_n = \sum_{i\in A_n} \beta_i e_i^*$ in ℓ_{F^*}. By passing to a subsequence we can for a suitable constant C_3 require that for $2^{-n} \le x \le 1$,

$$\begin{aligned}\sum_{i\in A_n} F(xF^{-1}(|\alpha_i\beta_i|) &\le C_3G(x)\\ \sum_{i\in A_n} F^*(x(F^*)^{-1}(|\alpha_i\beta_i|)) &\le C_3G^*(x).\end{aligned}$$

Now let μ_n be the probability measure supported on a finite subset of $(0,1]$ given by

$$\mu_n = \frac{1}{\sigma_n}\sum_{i\in A_n} |\alpha_i\beta_i|\epsilon_{|\alpha_i\beta_i|}$$

where ϵ_a denotes the Dirac measure at a. Then

$$\begin{aligned}\int F(xF^{-1}(t))\frac{d\mu_n(t)}{t} &= \frac{1}{\sigma_n}\sum_{i\in A_n} F(xF^{-1}(|\alpha_i\beta_i|))\\ &\le 2C_3\delta^{-1}G(x)\end{aligned}$$

as long as $2^{-n} \le x \le 1$. We also obtain the similar inequality for F^* and G^* and hence the theorem is proved.∎

Let us now introduce some notation. If ℓ_F is a reflexive Orlicz sequence space and $0 < \lambda < \infty$ we shall say that G is λ-represented in F if there is a constant C and a sequence ν_n of probability measures with compact support in $(0,1]$ such that for $2^{-n} \le x \le 1$,

$$\left(\int \max\left(\frac{F(tx)}{G(x)F(t)}, \frac{G(x)F(t)}{F(tx)}\right)^{\lambda} d\nu_n(t)\right)^{\frac{1}{\lambda}} \le C.$$

THEOREM 2.4. *Let ℓ_F be a reflexive Orlicz sequence space. Then there exist constants $0 < \lambda_0 < 1 < \lambda_1 < \infty$ with the property that for any Orlicz sequence space ℓ_G to be isomorphic to a complemented subspace of ℓ_F it is necessary that G is λ_0-represented in F and sufficient that G is λ_1-represented in F.*

PROOF: Clearly it suffices to consider the case when F is convex. Since ℓ_F is reflexive, we may assume that L_F is reflexive so that there is a constant $0 < \alpha < 1$ and a constant $c > 0$ so that whenever $\xi \ge 1$ and $0 \le x < \infty$, we have $F(\xi x) \ge c\xi^{1+\alpha}F(x)$ and $F^*(\xi x) \ge c\xi^{1+\alpha}F^*(x)$.

For ξ, x we define $A(\xi, x) = \{t : F(t)G(x) \ge \xi F(tx)\}$. Also let $B(\xi, x) = \{t : F^*(tx) \ge \xi F^*(t)G^*(x)\}$. We make two claims:

CLAIM 1: If $0 < y \le 1$, and $\xi \ge 2$ then

$$F(A(\xi, G^{-1}(y))) \subset F^*(B(c\xi^{\alpha}2^{-(1+\alpha)}, (G^*)^{-1}(y))).$$

CLAIM 2: If $0 < y \le 1$, and $\xi \ge 2$, then

$$F^*(B(\xi, (G^*)^{-1}(y))) \subset F(A(c\xi^{\alpha}2^{-(1+\alpha)}, G^{-1}(y))).$$

PROOF OF CLAIM 1: Suppose $F^{-1}(s) \in A(\xi, G^{-1}(y))$. Then $sy \ge \xi F(F^{-1}(s)G^{-1}(y))$ so that $F^{-1}(s)G^{-1}(y) \le F^{-1}(sy/\xi)$. By Lemma 2.2, this implies

$$\frac{1}{2}\xi(F^*)^{-1}(sy/\xi) \le (F^*)^{-1}(s)(G^*)^{-1}(y).$$

Thus applying F^* to both sides we obtain

$$\frac{c}{2^{1+\alpha}}\xi^{\alpha}sy \le F^*((F^*)^{-1}(s)(G^*)^{-1}(y)).$$

Now if we substitute $\tau = (F^*)^{-1}(s)$ and $z = (G^*)^{-1}(y)$,

$$\frac{c}{2^{1+\alpha}}\xi^{\alpha}F^*(\tau)G^*(z) \le F^*(\tau z).$$

This implies that $\tau = (F^*)^{-1}(s) \in B(c\xi^\alpha 2^{-(1+\alpha)}, (G^*)^{-1}(y))$ as required.

PROOF OF CLAIM 2: Suppose $(F^*)^{-1}(s) \in B(\xi, (G^*)^{-1}y)$. Then

$$\xi sy \leq F^*((F^*)^{-1}(s)(G^*)^{-1}(y))$$

so that $(F^*)^{-1}(\xi sy) \leq (F^*)^{-1}(s)(G^*)^{-1}(y)$. Hence $\frac{1}{2}\xi F^{-1}(s)G^{-1}(y) \leq F^{-1}(\xi sy)$ and so $F(\frac{1}{2}\xi F^{-1}(s)G^{-1}(y)) \leq \xi sy$. Now, by the assumptions on F,

$$\frac{c}{2^{1+\alpha}}\xi^\alpha F(F^{-1}(s)G^{-1}(y)) \leq sy$$

and $F^{-1}(s) \in A(c\xi^\alpha/2^{1+\alpha}, G^{-1}(y))$ as required.

We proceed to the proof of the theorem. We pick λ_0 so that $0 < \lambda_0 < \alpha$. Suppose first that ℓ_G is isomorphic to a complemented subspace of ℓ_F. Then we may choose measures μ_n as in Theorem 2.3. Let $\nu_n = \mu_n \circ F$. Then we have

$$\int \frac{F(tx)}{F(t)G(x)} d\nu_n(t) \leq C \tag{3}$$

for $2^{-n} \leq x \leq 1$.

Next suppose $2^{-n} \leq (G^*)^{-1}(G(x)) = y \leq 1$. Then for $\xi \geq 2$,

$$\begin{aligned}\nu_n(A(\xi,x)) &= \mu_n(F(A(\xi,x))) \\ &\leq \mu_n(F^*B(\frac{c}{2^{1+\alpha}}\xi^\alpha, y)) \\ &\leq \frac{2^{1+\alpha}}{c}\xi^{-\alpha}\int \frac{F^*((F^*)^{-1}(t)y)}{G^*(y)}\frac{d\mu_n(t)}{t} \\ &\leq Cc^{-1}2^{1+\alpha}\xi^{-\alpha}.\end{aligned}$$

Now if $\lambda_0 < \alpha$ this leads to an estimate

$$\left(\int \left(\frac{F(t)G(x)}{F(tx)}\right)^{\lambda_0} d\nu_n(t)\right)^{1/\lambda_0} \leq C_1$$

for $2^{-n} \leq y \leq 1$ where C_1 depends only on C, α and λ_0. Passing to a suitable subsequence and combining with (3) gives the result in one direction.

For the converse direction, pick λ_1 so that $\lambda_1\alpha > 1$. This time we suppose ν_n are given as in the definition of λ-representability. Let $\mu_n = \nu_n \circ F^{-1}$. Then since $\lambda_1 > 1$ it is easy to see that (1) of Theorem 2.3 is satisfied.

Now suppose x is such that $2^{-n} \leq G^{-1}(G^*(x)) = y \leq 1$. Suppose $\xi \geq 2$; then for a suitable constant C independent of n,

$$\begin{aligned}\mu_n(F^*B(\xi,x)) &\leq \mu_n(F(A(\frac{c}{2^{1+\alpha}}\xi^\alpha, y)) \\ &= \nu_n(A(\frac{c}{2^{1+\alpha}}\xi^\alpha, y)) \\ &\leq C\xi^{-\lambda_1\alpha}.\end{aligned}$$

This leads to an estimate, for $2^{-n} \leq y \leq 1$,

$$\int \frac{F^*(tx)}{F^*(t)G^*(x)} d\mu_n \circ F^*(t) \leq C_1$$

for a suitable C_1. Changing variables and passing to a suitable subsequence gives (2) of Theorem 2.3.∎

3. Examples of minimal Orlicz sequence spaces.

We will now describe a method of construction of minimal Orlicz functions suggested by the work of Hernandez and Rodriguez-Salinas [3]. First we fix $p > 1$. Identify the unit circle $\mathbf{T}$ with $\mathbf{R}/2\pi\mathbf{Z}$. We shall suppose that $(f_n)_{n=0}^\infty$ is a sequence of C^1-functions on $\mathbf{T}$ (i.e. 2π-periodic functions on $\mathbf{R}$) satisfying $f_n(0) = 0$ and such that the series $\sum_{n=0}^\infty 2^{-n}L_n$ converges where $L_n = \|f_n'\|_\infty$. For convenience we let $R_n = \sum_{k=n+1}^\infty 2^{-k}L_k$ so that $\lim_{n\to\infty} R_n = 0$. We then define the C^1-function ϕ on $\mathbf{R}$ by

$$\phi(u) = \sum_{n=0}^\infty f_n(\frac{2\pi u}{2^n}).$$

We define for $t > 0$, $F(t) = t^p \exp(\phi(-\log t))$.

We also introduce the functions g_n on $\mathbf{T}$ defined by $g_n(\theta) = \sum_{k=0}^n f_k(2^{n-k}\theta)$. Let us then note that for any u, v we have

$$|\phi(u+v) - \phi(v) - g_n(\frac{2\pi(u+v)}{2^n}) + g_n(\frac{2\pi v}{2^n})| \leq \sum_{k=n+1}^\infty |f_k(\frac{2\pi(u+v)}{2^k}) - f_k(\frac{2\pi v}{2^k})|$$

and hence

$$|\phi(u+v) - \phi(v) - g_n(\frac{2\pi(u+v)}{2^n}) + g_n(\frac{2\pi v}{2^n})| \leq 2\pi|u|R_n. \tag{4}$$

PROPOSITION 3.1. *ℓ_F is a minimal reflexive Orlicz sequence space.*

PROOF: First, for any $\epsilon > 0$ there exists $N = N(\epsilon)$ so that

$$2\pi R_N = 2\pi\sum_{n=N+1}^\infty 2^{-n}L_n < \epsilon$$

and hence ϕ satisfies an estimate $|\phi(u+v) - \phi(v)| < M(\epsilon) + \epsilon|u|$ where $M(\epsilon) = 2\|g_N\|_\infty$. From this it follows without difficulty that F is equivalent an Orlicz function G with the property that $x^{-\alpha}G(x)$ is increasing for some $\alpha > 1$. Thus ℓ_F is a reflexive Orlicz space.

To demonstrate minimality, suppose $G \in E_F$. Then there is a sequence $\sigma_k \geq 0$ such that if $G(x) = x^p\exp(\psi(-\log x))$ then $\lim_{k\to\infty}(\phi(u+\sigma_k) - \phi(\sigma_k)) = \psi(u)$, uniformly on compact subsets of $[0,\infty)$. By passing to a subsequence we may suppose that $2\pi\sigma_k/2^n$ converges in $\mathbf{T}=\mathbf{R}/2\pi\mathbf{Z}$ for each fixed n to some a_n where $0\le a_n<2\pi$. Thus

$$|\psi(u)-g_n(\frac{2\pi u}{2^n}+a_n)+g_n(a_n)|\le 2\pi R_n|u|.$$

If we let $\tau_n=2^n(1-(2\pi)^{-1}a_n)$ then

$$|\psi(u+\tau_n)-\psi(\tau_n)-g_n(\frac{2\pi u}{2^n})|\le 4\pi R_n|u|$$

and hence

$$|\psi(u+\tau_n)-\psi(\tau_n)-\phi(u)|\le 6\pi R_n|u|.$$

This implies that $F\in E_G$ and so F is minimal.∎

We now impose an additional constraint.

PROPOSITION 3.2. *Suppose for some increasing sequence of integers N_n we have $\sup 2^{N_n}R_{N_n}=A<\infty$. If $F(x)$ is equivalent to x^r for some r then $\|g_{N_n}\|_\infty$ is bounded. Furthermore $G(x)=x^p\exp(\psi(-\log x))$ is $\lambda-$represented in F if and only if there is a constant C so that for every n there is a probability measure μ_n on $\mathbf{T}$ such that*

$$\int \exp(\lambda|g_{N_n}(\theta+\theta_0)-g_{N_n}(\theta)-h_n(\theta_0)|)d\mu_n(\theta)\le C,\qquad 0\le\theta_0<2\pi \tag{5}$$

where $h_n(\theta)=\psi(2^{N_n}\theta/2\pi)$.

PROOF: Notice first that $|\phi(u)-g_{N_n}(2\pi u/2^{N_n})|\le 2\pi A$ for $|u|\le 2^{N_n}$. In particular $|\phi(2^{N_n})|\le 2\pi A$. Thus if F is equivalent to x^r then $r=p$. But then ϕ is bounded and so there is also a uniform bound on g_{N_n}.

For the second part, we observe that if G is $\lambda-$represented in F then there is a constant C_0 and a sequence of compactly supported probability measures ν_n on $[0,\infty)$ such that

$$\int \exp(\lambda|\phi(u+v)-\phi(u)-\psi(v)|)d\nu_n(u)\le C_0$$

for $0\le v\le 2^{N_n}$. If $0\le v\le 2^{N_n}$, (4) gives us the estimate

$$|\phi(u+v)-\phi(u)-g_{N_n}(\frac{2\pi(u+v)}{2^{N_n}})+g_{N_n}(\frac{2\pi u}{2^{N_n}})|\le 2\pi A.$$

If we define μ_n on $\mathbf{T}$ by $\int f(\theta)d\mu_n(\theta)=\int f(2\pi t/2^{N_n})d\nu_n(t)$ (5) will follow with $C=C_0e^{2\pi\lambda A}$.

Conversely we assume (5) we quickly get that for $0\le\theta_0<2\pi$,

$$\int \exp(\lambda|\phi(\frac{2^{N_n}(\theta+\theta_0)}{2\pi})-\phi(\frac{2^{N_n}\theta}{2\pi})-\psi(\frac{2^{N_n}\theta_0}{2\pi})|)d\mu_n\le Ce^{2\pi\lambda A}.$$

It then easily follows by a change of variables that G is $\lambda-$represented in F.∎

THEOREM 3.3. *Suppose $1<p<\infty$. There exists a minimal reflexive Orlicz sequence space ℓ_F which is not isomorphic to ℓ_p but which contains ℓ_p as a complemented subspace. In particular, ℓ_F is not prime.*

PROOF: We pick a sequence of C^1-functions $(h_n)_{n=0}^\infty$ on $\mathbf{T}$ with $h_n(0)=h_n(2\pi)=0$ and such that if $M_n=\|h_n\|_\infty$ then we have both $M_n>2^n+\sum_{k=1}^{n-1}M_k$ and

$$\int_0^{2\pi}\exp(4^n|h_n(\theta)|)\frac{d\theta}{2\pi}\le 2$$

for all n.

Let $B_n=\|h_n'\|_\infty$. We pick a strictly increasing sequence of integers N_k such that $B_k2^{N_{k-1}-N_k}\le 1$ for all k, where $N_0=0$. Then define f_n by $f_n=h_{N_k}$ if $n=N_k$ and $f_n=0$ if $n\notin(N_k)_{k=0}^\infty$. Thus

$$\begin{aligned}2^{N_k}R_{N_k}&=\sum_{j=k+1}^\infty B_j2^{N_k-N_j}\\&\le\sum_{j=k+1}^\infty 2^{N_k-N_{j-1}}\\&\le 2\end{aligned}$$

so that we can apply Proposition 3.2. Observe first that F cannot be equivalent to any x^r since $\|g_{N_k}\|_\infty\ge 2^k$.

To complete the proof we show that x^p is $\lambda-$represented in F for every λ. Thus Theorem 2.4 will give the result. To show this we estimate by convexity of the exponential function,

$$\int_0^{2\pi}\exp(\lambda|g_{N_k}(\theta+\theta_0)-g_{N_k}(\theta)|)\frac{d\theta}{2\pi}\le\int_0^{2\pi}\exp(2\lambda|g_{N_k}(\theta)|)\frac{d\theta}{2\pi}.$$

However writing $g_{N_k}=2^{-(k+1)}(0)+\sum_{j=0}^k 2^{-(j+1)}(2^{j+1}f_{N_j})$ and again using convexity the integral is estimated by

$$2^{-(k+1)}+\sum_{j=0}^k 2^{-(j+1)}\int\exp(\lambda 2^{j+2}|f_{N_j}(\theta)|)\frac{d\theta}{2\pi}=$$

$$=2^{-(k+1)}+\sum_{j=0}^k 2^{-(j+1)}\int\exp(\lambda 2^{j+2}|h_j(\theta)|)\frac{d\theta}{2\pi}$$

which is bounded, independent of k, for every λ. We can now apply Proposition 3.2. ∎

We now turn to the construction of strongly minimal Orlicz sequence spaces. We recall that ℓ_F is strongly minimal if whenever ℓ_G is isomorphic to a complemented subspace of ℓ_F then G is equivalent to F. Our example is of the above form with $f_n(x)=\alpha(1-\cos x)$ for every n, so that $\phi(u)=\alpha(\sum_{n=0}^\infty(1-\cos(2\pi u/2^n)))$. This example was first discussed in a function space context by Johnson, Maurey, Schechtman and Tzafriri [4] and later its minimality was proved by Hernandez and Rodriguez-Salinas

[3], who also observe that for small enough α, F is actually convex, (it is, of course, always equivalent to a convex function).

THEOREM 3.4. *Suppose $1 < p < \infty$ and $|\alpha| > 0$. Then if*

$$F(x) = x^p \exp \alpha\Big(\sum_{n=0}^{\infty}\Big(1 - \cos\Big(\frac{2\pi \log x}{2^n}\Big)\Big)\Big)$$

then F is not equivalent to any x^r and ℓ_F is a strongly minimal Orlicz sequence space.

PROOF: In this case we can apply Proposition 3.2 (take $N_n = n$.) Thus our proof reduces to analyzing the functions $g_n(\theta) = \alpha \sum_{k=0}^{n}(1 - \cos(2^k\theta))$ on $\mathbf{T}$. The fact that F is non-equivalent to any x^r is proved in [4], or may be proved from Proposition 3.2 by estimating $\|g_n\|_2$. We therefore have only to establish that if $G(x) = x^p \exp(\psi(-\log x))$ is λ-represented for some $\lambda > 0$ in F then G is equivalent to some $G_1 \in E_F$. This in turn will be achieved by establishing the following result of possibly independent interest.

THEOREM 3.5. *For any K there is a constant $C = C(K)$ (independent of n) so that if $g_n(\theta) = \sum_{k=0}^{n}(1 - \cos 2^k\theta)$, μ is a probability measure on $\mathbf{T}$ and h is a function on $\mathbf{T}$ such that for every $0 \le \theta_0 < 2\pi$ we have*

$$\int |g_n(\theta + \theta_0) - g_n(\theta) - h(\theta_0)|^2 d\mu(\theta) \le K^2$$

then there exists σ, $0 \le \sigma < 2\pi$ such that

$$|h(\theta) - (g_n(\theta + \sigma) - g_n(\sigma))| \le C$$

for $0 \le \theta < 2\pi$.

PROOF: We first introduce the angular distance on $\mathbf{T}$, $\delta(\theta_1, \theta_2) = d(\theta_1 - \theta_2, 2\pi\mathbf{Z})$. We will be interested in ways of measuring the spread of μ. For $0 \le k \le n$, we define:

$$\alpha_k = \int\int \sin^2(2^{k-1}(\theta_1 - \theta_2))d\mu(\theta_1)d\mu(\theta_2)$$
$$\beta_k = \int\int \delta(2^k\theta_1, 2^k\theta_2)d\mu(\theta_1)d\mu(\theta_2).$$

In order to estimate these we choose τ_k for $0 \le k \le n$ to minimize

$$\int \sin^2(2^{k-1}(\theta - \tau_k))d\mu(\theta).$$

It is clear that such a minimizer exists and further if $1 \le k \le n$, we can choose τ_k from amongst at least 2^k possibilities so that if $A_k = \{\theta : \delta(2^{k-1}\theta, 2^{k-1}\tau_k) > \pi/2\}$ and $a_k = \mu A_k$ then $a_k \le \frac{1}{2}$. We further introduce $B_k = \{\theta : \delta(2^k\theta, 2^k\tau_k) > \pi/4\}$ and $b_k = \mu B_k$ for $0 \le k \le n$. Let $C_k = (\mathbf{T} \setminus A_k) \cap (\mathbf{T} \setminus B_k)$ and $D_k = (\mathbf{T} \setminus B_k) \cap A_k$. Then $\mu C_k \ge 1 - a_k - b_k$ and $\mu D_k \ge a_k - b_k$.

We next relate a_k, b_k to α_k. Clearly by integrating over B_k we have

$$\sin^2(\frac{\pi}{8})b_k \le \int \sin^2(2^{k-1}(\theta - \tau_k))d\mu(\theta) \le \alpha_k.$$

Thus

$$b_k \le 10\alpha_k.$$

Thus as long as $\alpha_k \le 1/40$ we have $b_k \le 1/4$. Hence in this case we have $\mu C_k \ge 1/4$. Now if $k \ge 1$ and $\theta_1 \in C_k$ and $\theta_2 \in D_k$ we have $\delta(2^{k-1}\theta_1, 2^{k-1}\tau_k) \le \pi/8$ but $\delta(2^{k-1}\theta_1, 2^{k-1}\tau_k) \ge 3\pi/8$. Thus $\delta(2^{k-1}\theta_1, 2^{k-1}\theta_2) \ge \pi/4$ and $\sin^2(2^{k-2}\theta_1, 2^{k-2}\theta_2) \ge \sin^2(\pi/8) \ge 1/10$. Integrating over $C_k \times D_k \cup D_k \times C_k$ gives

$$\mu(C_k)\mu(D_k) \le 5\alpha_{k-1}.$$

Thus if $\alpha_k \le 1/40$ we obtain $\mu D_k \le 20\alpha_{k-1}$ and hence $a_k \le b_k + 20\alpha_{k-1} \le 20\alpha_{k-1} + 10\alpha_k$. Since $a_k \le \frac{1}{2}$ we can say in general that for $k \ge 1$,

$$a_k \le 20(\alpha_{k-1} + \alpha_k).$$

Now suppose $\theta_1, \theta_2 \in C_k$. Then, if $k \ge 1$, we clearly have $\delta(2^{k-1}\theta_1, 2^{k-1}\theta_2) \le \frac{1}{2}\delta(2^k\theta_1, 2^k\theta_2)$. Hence by integration we obtain

$$\begin{aligned}\beta_{k-1} &\le \frac{1}{2}\beta_k + \pi(1 - \mu(C_k)^2)\\ &\le \frac{1}{2}\beta_k + 2\pi(1 - \mu(C_k))\\ &\le \frac{1}{2}\beta_k + 2\pi(a_k + b_k).\end{aligned}$$

By induction, we obtain

$$\beta_l \le 2^{l-n}\beta_n + 2\pi \sum_{j=l+1}^{n} 2^{l+1-j}(a_j + b_j)$$

and hence

$$\begin{aligned}\sum_{l=0}^{n}\beta_l &\le 2(\beta_n + 2\pi(\sum_{j=1}^{n}(a_j + b_j)))\\ &\le 2\pi(1 + \sum_{j=1}^{n}(20\alpha_{j-1} + 30\alpha_j))\\ &\le 2\pi(1 + 50\sum_{j=0}^{n}\alpha_j).\end{aligned}$$

Now returning to the original statement of the theorem we observe that if the hypotheses on μ hold then:

$$\left(\int\int |g_n(\theta_1+\theta_0)-g_n(\theta_2+\theta_0)|^2 d\mu(\theta_1)d\mu(\theta_2)\right)^{\frac{1}{2}} \leq 2K$$

for every $0 \leq \theta_0 < 2\pi$. Hence integrating again and using Fubini's theorem,

$$\int\int\left(\int_0^{2\pi} |g_n(\theta_1+\theta_0)-g_n(\theta_2+\theta_0)|^2 \frac{d\theta_0}{2\pi}\right) d\mu(\theta_1)d\mu(\theta_2) \leq 4K^2.$$

However,

$$g_n(\theta_1+\theta_0)-g_n(\theta_2+\theta_0) = 2\sum_{k=0}^{n} \sin(2^{k-1}(\theta_1-\theta_2))\sin(2^k\theta_0+2^{k-1}(\theta_1+\theta_2))$$

so that the integral can be rewritten as

$$\int\int\sum_{k=0}^{n} \sin^2(2^{k-1}(\theta_1-\theta_2))\, d\mu(\theta_1)d\mu(\theta_2) \leq 2K^2$$

or

$$\sum_{k=0}^{n} \alpha_k \leq 2K^2.$$

This in turn yields

$$\sum_{k=0}^{n} \beta_k \leq 2\pi(1+100K^2).$$

It follows that we can pick $\sigma \in \mathbf{T}$ so that

$$\int\sum_{k=0}^{n} \delta(2^k\theta, 2^k\sigma)d\mu(\theta) \leq 2\pi(1+100K^2).$$

For this choice of σ notice that, for $0 \leq \theta_0 < 2\pi$,

$$\int\left|\sum_{k=0}^{n}(\cos 2^k(\theta+\theta_0)-\cos 2^k(\sigma+\theta_0))\right| d\mu(\theta) \leq 2\pi(1+100K^2)$$

and so, for $0 \leq \theta_0 < 2\pi$,

$$\int |g_n(\theta+\theta_0)-g_n(\sigma+\theta_0)|d\mu(\theta) \leq 2\pi(1+100K^2),$$

It now follows easily that

$$|h(\theta_0)-(g_n(\sigma+\theta_0)-g_n(\sigma))| \leq 4\pi(1+100K^2)+K$$

and the theorem is proved.∎

PROOF OF THEOREM 3.4, CONTINUED: This is almost immediate. If $G(x) = x^p\exp(\psi(-\log x))$ is $\lambda-$represented in F for some λ then setting $h_n(\theta) = \psi(2^n(\frac{\theta}{2\pi}))$ for $0 \leq \theta < 2\pi$ we certainly obtain that there exists a probability measure μ_n on $\mathbf{T}$ so that

$$\int |g_n(\theta+\theta_0)-g_n(\theta)-h_n(\theta_0)|^2 d\mu_n(\theta) \leq K^2$$

where K is independent of n. Thus there exists σ_n with

$$|h_n(\theta)-(g_n(\theta+\sigma_n)-g_n(\sigma_n))| \leq C$$

for $0 \leq \theta < 2\pi$ where C is independent of n. But then if $\alpha_n = 2^n(\sigma_n/2\pi)$ we have that

$$|\psi(u)-(\phi(u+\alpha_n)-\phi(u))| \leq C'$$

for $0 \leq u \leq 2^n$ where C' is independent of n. It quickly follows that G is equivalent to a function in E_F.∎

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