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London Mathematical Society Lecture Note Series. 158

Geometry of Banach Spaces

Proceedings of the Conference held in Strobl, Austria, 1989

Edited by
P.F.X. Müller
W. Schachermayer
Johannes Kepler Universität, Linz

The right of the University of Cambridge to print and sell all manner of books was granted by Henry VIII in 1534. The University has printed and published continuously since 1584.

CAMBRIDGE UNIVERSITY PRESS
Cambridge
New York Port Chester Melbourne Sydney

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
40 West 20th Street, New York, NY 10011, USA
10, Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1990

Printed in Great Britain at the University Press, Cambridge

Library of Congress cataloguing in publication data available

British Library cataloguing in publication data available

ISBN 0 521 40850 4

Notes on approximation properties in separable Banach spaces

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(*): The research of the first author was supported by NSF-grant DMS 8702329 and the research of the second author was supported by NSF-grant DMS 8901636

1. Introduction, definitions and discussion of results.

Although the example given by Enflo in 1973 [5] settled the approximation problem and the basis problem for Banach spaces, a number of closely related problems have continued to arouse interest. If X is a separable Banach space, there are a number of natural properties intermediate between X having the approximation property and having a basis.

Let us first make some definitions. Suppose X is a separable Banach space. Then X has the *approximation property (AP)* if there is a net of finite-rank operators T_α so that $T_\alpha x \to x$ for $x \in X$, uniformly on compact sets. X is said to have the *bounded approximation property (BAP)* if this net can be replaced by a sequence T_n; alternatively X has (BAP) if there is a sequence of finite-rank operators, T_n, such that $\sup \|T_n\| < \infty$ and $T_n x \to x$ for $x \in X$. A sequence T_n with these properties will be called an *approximating sequence.* If X has an approximating sequence T_n with $\lim_{n\to\infty} \|T_n\| = 1$ then X has the *metric approximation property (MAP).*

An important principle [15] that we will use frequently is that if T_n is any approximating sequence for X then there is an approximating sequence S_n satisfying $S_m S_n = S_n$ whenever $m > n$ and such that for some subsequence T_{k_n} of T_n then $\lim_{n\to\infty} \|T_{k_n} - S_n\| = 0$. (See Lemma 2.4 of [15]).

A slight weakening of the basis property is to require that X has a *finite-dimensional decomposition (FDD)* i.e. that X has an approximating sequence T_n satisfying $T_m T_n = T_{\min(m,n)}$ for $m, n \in \mathbb{N}$. Szarek [24] has given an example to show that a space with an (FDD) need not have a basis. Between (FDD) and (BAP) we can isolate two other natural properties. We say X has the *$\pi-$property* if X has an approximating sequence of projections, and the *commuting bounded approximation property (CBAP)* if it has a commuting approximating sequence. We may add to both these properties the corresponding metric properties (π_1) and (CMAP) where we also have $\lim_{n\to\infty} \|T_n\| = 1$. In general if X has a commuting approximating sequence T_n with $\liminf \|T_n\| = \lambda$ we say X has λ-CBAP.

Johnson [12] showed that a space with the (π_1)-property has an (FDD). However it is not known whether every $\pi-$space has an (FDD).

The (CBAP) property was first isolated by Rosenthal and Johnson [14] in the early seventies and has most recently been studied by the first author [2]. Let us note that X has $\lambda-$CBAP if and only if it has an approximating sequence T_n such that

$T_mT_n = T_{\min(m,n)}$ for $m \neq n$, and $\limsup \|T_n\| \leq \lambda$. (This is doubtless well-known; it is proved in Proposition 2.1 below.) This suggests that it is quite close to the (FDD) property. However very recently Read [22] gave an example of a Banach space with (CBAP) but failing (FDD). Conversely Casazza [2] showed that a space with both π and (CBAP) has an (FDD), so Read's space is not a $\pi-$space.

It is in general not known if (BAP) implies (CBAP). However certain hypotheses on X do give this implication: it holds if X is reflexive or is a separable dual space [13]. Coincidentally, the same hypotheses give that (AP) implies (MAP) (Grothendieck [11]). It has also been shown by Johnson [14] that any space with (CBAP) can be renormed to have (CMAP). These results suggest a close relationship between the properties (CBAP) and (MAP). Our main result is that X has (CBAP) if and only if it can be equivalently normed to have (MAP), so that (CBAP) is the isomorphic version of (MAP). Further if X has (MAP) it has (CMAP). The proofs of these results (Theorem 2.4 and Corollary 2.5) are quite simple modifications of techniques from the study of approximate identities in Banach algebras (due to Sinclair [23]; see also [4]). We also give another condition, the reverse metric approximation property, which implies (CBAP).

Pelczynski [21] and Johnson, Rosenthal and Zippin [15] showed that any space with (BAP) is isomorphic to a complemented subspace of a space with a basis. Johnson [14] has shown that if X has (BAP) then there is a reflexive space Y so that $X \oplus Y$ is a $\pi-$space. In fact Y can be taken to be the space $C_p\,(1 < p < \infty)$ defined in Section 3. He conjectures that in fact $X \oplus C_p$ has an (FDD) and hence a basis. However, we show that $X \oplus C_p$ has a basis if and only if X has (CBAP). This shows that several possible conjectures are equivalent.

We then give some results on renormings of spaces with (CBAP) and conclude by studying spaces X which have an approximating sequence T_n with $\lim \|I - 2T_n\| = 1$. This condition is closely related to unconditional forms of the approximation property. We say that X has the *unconditional approximation property (UnAP)* if there is an approximating sequence T_n such that if $A_n = T_n - T_{n-1}$ $(T_0 = 0)$ then

$$\sup \| \sum_{i=1}^{N} \eta_i A_i \| < \infty$$

where the supremum is taken over all N and all $\eta_i = \pm 1$, $i = 1, 2, \ldots, N$. We introduce the metric version of (UnAP) and relate our work to recent results of Cho, Johnson, Godefroy, P. Saab and Li ([3],[8],[9] and [18]).

2. Equivalent formulations of (CBAP).

We will write $[A, B] = AB - BA$ and $\prod_{j=a}^{b} T_j = T_a T_{a+1} \ldots T_b$.

PROPOSITION 2.1. *Suppose X is a separable Banach space and T_n is an approximating sequence for X with $T_mT_n = T_n$ for $m > n$. Let $\lambda = \liminf \|T_n\|$. If $\sum \|[T_n, T_{n+1}]\| < \infty$, then X has $\lambda-$CBAP (and, further has an approximating sequence R_n for which $R_mR_n = R_{\min(m,n)}$ for $m \neq n$ and $\limsup \|R_n\| \leq \lambda$.)*

PROOF: We first show that we can suppose that $T_n(X) = T_n^2(X)$. Let P_n be any bounded projection of X onto $T_n(X)$ and choose a sequence $0 < \alpha_n < 1$, so that $\sum \alpha_n \|P_n\| < \infty$ and $-\alpha_n/(1-\alpha_n)$ is not an eigenvalue of T_n. Then we may replace T_n by $(1-\alpha_n)T_n + \alpha_n P_n$ and the hypotheses of the Proposition will still hold with the additional constraint that $T_n(X) = T_n^2(X)$.

Now let $\epsilon_n = \|[T_n, T_{n+1}]\|$. For $n \in \mathbb{N}$ and $k \geq 1$ we define $A(n,k) = \prod_{j=n}^{n+k-1} T_j$. Let $A(n, 0) = I$. Then for $k \geq 1$,

$$A(n, k+1) = A(n,k) + A(n, k-1)[T_{n+k-1}, T_{n+k}].$$

Thus if $M_n(k) = \max_{1 \leq l \leq k} \|A(n,l)\|$ then $M_n(1) = \|T_n\|$ and $M_n(k+1) \leq M_n(k)(1 + \epsilon_{n+k-1})$. Hence

$$\|A(n,k)\| \leq \|T_n\| \prod_{j=n}^{\infty} (1 + \epsilon_j).$$

It now follows that $A(n,k)$ is norm-convergent and we can define $S_n = \prod_{j=n}^{\infty} T_j = \lim_{k\to\infty} A(n,k)$. Clearly the operators S_n are finite-rank with $S_n(X) = S_n^2(X) = T_n(X)$. Further if $m > n$ we have $S_mS_n = S_n$ and S_n is an approximating sequence with $\|S_n\| \leq \beta_n \|T_n\|$ where $\lim_{n\to\infty} \beta_n = 1$.

A simple calculation also shows that if $m > n$ then

$$S_nS_m = \left(\prod_{j=n}^{m-1} T_j \right) T_m^2 S_{m+1}$$

so that

$$[S_m, S_n] = S_n(I - S_m) = A(n, m-n-1)[T_{m-1}, T_m]S_m.$$

Thus for a suitable constant K, independent of m, n we have

$$\|[S_m, S_n]\| \leq K\epsilon_{m-1}$$

whenever $m > n$. In particular we have $\lim_{m\to\infty} \|[S_m, S_n]\| = 0$ for each fixed n.

We now pass to a subsequence V_n of S_n so that $\limsup \|V_n\| \leq \lambda$. We will also have $V_mV_n = V_n$ for $m > n$ and $\lim_{m\to\infty} \|[V_m, V_n]\| = 0$ for each n.

Finally, by induction we will choose an increasing sequence of positive integers (n_k) and operators R_k so that:

(1) $R_k^2(X) = R_k(X) = V_{n_k}(X)$.
(2) R_k is a polynomial in $V_{n_1}, \ldots, V_{n_k}$.
(3) For $1 \leq l < k$, $R_kR_l = R_lR_k = R_l$.
(4) $\|R_k\| \leq \|S_{n_k}\| + 2^{-k}$.

To start let $n_1 = 1$ and $R_1 = V_1$. Now suppose $n_1, \ldots, n_k$ and $R_1, \ldots, R_k$ have been determined to satisfy (1-4). Since $R_k(X) = R_k^2(X)$ we can find an operator W_k which is a polynomial in R_k so that $W_kR_kx = x$ for $x \in R_k(X)$. Now, $\lim_{m\to\infty} \|R_k(I - V_m)\| = 0$ since R_k is a polynomial in $V_{n_1}, \ldots, V_{n_k}$. Thus we may pick n_{k+1} so that

$$\|R_k(I - V_{n_{k+1}})\| < 2^{-(k+1)} \|W_k\|^{-1}.$$

We then define

$$R_{k+1} = V_{n_{k+1}} + W_kR_k(I - V_{n_{k+1}}).$$

Clearly conditions (2) and (4) above hold. For condition (3) note that $I - R_{k+1} = (I - W_k R_k)(I - V_{n_{k+1}})$ from which it follows that $R_k(I - R_{k+1}) = (I - R_{k+1})R_k = 0$ or $R_k R_{k+1} = R_{k+1} R_k = R_k$. Now if $1 \le l < k$ then $R_l R_{k+1} = R_l R_k R_{k+1} = R_l R_k = R_l$ and similarly $R_{k+1} R_l = R_l$. Thus (3) is verified.

For (1) we clearly have $R_{k+1}(X) \subset V_{n_{k+1}}(X)$. It suffices to show that R_{k+1} is injective on $V_{n_{k+1}}(X)$. Indeed suppose $x \in V_{n_{k+1}}(X)$ and $R_{k+1}x = 0$. Then since $W_k R_k (I - V_{n_{k+1}})x \in R_k(X) = V_{n_k}(X)$ we have $V_{n_{k+1}} x \in V_{n_k}(X)$. Thus $V_{n_{k+1}}^2 x = V_{n_{k+1}} x$ and by the fact that $V_{n_{k+1}}$ is injective on $V_{n_{k+1}}(X)$ we have $x = V_{n_{k+1}} x$ and hence $x = 0$ as required.

It is now immediate that X has $\lambda-$CBAP.∎

COROLLARY 2.2. *Suppose X has an approximating sequence T_n for which*

$$\lim_{m,n\to\infty} \|[T_m, T_n]\| = 0$$

and $\liminf_{n\to\infty} \|T_n\| = \lambda$. Then X has $\lambda-$CBAP.

REMARK: By $\lim_{m,n\to\infty} a_{mn} = 0$ we mean that given $\epsilon > 0$ there exists N so that if $m, n \ge N$ then $|a_{mn}| < \epsilon$.

PROOF: We may find a subsequence T_{n_k} and finite rank operators S_k so that $\lim_k \|T_{n_k}\| = \lambda$, $\|S_k - T_{n_k}\| \to 0$, and $S_l S_k = S_k$ for $l > k$. Then $\lim_{m,n\to\infty} \|[S_m, S_n]\| = 0$ and so by passing to a further subsequence we can apply the Proposition.

COROLLARY 2.3. *Suppose X has an approximating sequence T_n for which*

$$\lim_{m\to\infty} \|[T_m, T_n]\| = 0$$

for each fixed n and $\liminf \|T_n\| = \lambda$. Then X has $\lambda-$CBAP.

This Corollary is immediate from the Proposition. The same result without the precise estimate on the constant was shown in [2].

We now come to our main result. The argument in the next theorem is a simple modification of a theorem of Sinclair [23] on approximate identities in Banach algebras. Sinclair shows that if A is a Banach algebra with a bounded two-sided sequential approximate identity then it has a commuting approximate identity with the same bound. This result can be applied directly to the algebra $\mathcal{K}(X)$ of compact operators on X when X^* is separable and has (AP), and hence (MAP). Under these circumstances $\mathcal{K}(X)$ has a norm-one two-sided approximate identity, and we can recover Theorem 2 of [2]. In general, however, some modification of Sinclair's approach is necessary.

THEOREM 2.4. *Suppose X is a separable Banach space with (MAP). Then X has (CMAP).*

PROOF: We shall suppose that X has an approximating sequence T_n with $T_m T_n = T_n$ for $m > n$ and $\|T_n\| \le 1 + \epsilon_n$ where $\sum \epsilon_n = \beta < \infty$. For $t > 0$ define the operators

$$V_n(t) = e^{-nt} \exp(t \sum_{k=1}^{n} T_k) = e^{-nt} \sum_{j=0}^{\infty} \frac{t^j}{j!} (T_1 + \cdots + T_n)^j.$$

Then

$$\|V_n(t)\| \le e^{-nt} \exp(t \sum_{k=0}^{n} \|T_k\|) \le e^{\beta t}.$$

Let $E_n = T_n(X)$. Then each E_n is an invariant subspace for every T_m and hence also for every $V_m(t)$. Rewriting $V_m(t)$ as $\exp(t \sum_{k=1}^{m} (T_k - I))$ it is clear that if $x \in E_n$ and $m > n$ then $V_m(t)x = V_n(t)x$. It follows therefore from the bound on the norms of $V_n(t)$ that we can define $S(t)$ by $S(t)x = \lim_{n\to\infty} V_n(t)x$ for all $x \in X$. Clearly $\|S(t)\| \le e^{\beta t}$. Furthermore $S(t)$ has the semigroup property $S(t_1 + t_2) = S(t_1)S(t_2)$ since each $V_n(t)$ is a semigroup and the property is preserved by strong limits.

We further claim that each $S(t)$ is compact for $t > 0$. Indeed suppose $l \in \mathbb{N}$ and that $x \in E_n$ where $n > l$. Then $d(S(t)x, E_l) = d(V_n(t)x, E_l)$. It is then easy to see, by expansion, that the operator $\exp t(T_1 + \cdots + T_n) - \exp t(T_{l+1} + \cdots + T_n)$ has range contained in E_l. Thus

$$\begin{aligned} d(S(t)x, E_l) &= e^{-nt} d(\exp t(T_{l+1} + \cdots + T_n)x, E_l) \\ &\le e^{-nt} \|\exp t(T_{l+1} + \cdots + T_n)\| \|x\| \\ &\le e^{-nt} \exp t(\|T_{l+1}\| + \cdots + \|T_n\|) \|x\| \\ &\le e^{\beta t} e^{-lt} \|x\|. \end{aligned}$$

Hence for all $x \in X$,

$$d(S(t)x, E_l) \le e^{\alpha t} e^{-lt} \|x\|$$

and hence $S(t)$ is compact.

Note that as $t \to 0$ $\|S(t)\| \to 1$. Also if $x \in E_n$ we have $S(t)x = V_n(t)x \to x$. Hence for all $x \in X$, we have $\lim_{t\to 0} S(t)x = x$.

Since X has (MAP) there exist finite-rank operators R_n so that $\|R_n - S(1/n)\| \to 0$. Then R_n is an approximating sequence, $\lim \|R_n\| = 1$ and $\lim_{m,n\to\infty} \|[R_m, R_n]\| = 0$ since the operators $S(1/n)$ commute. Hence by Corollary 2.2, X has (CMAP).∎

COROLLARY 2.5. *Let X be a separable Banach space. Then X has (CBAP) if and only if X can be equivalently normed to have (MAP).*

Corollary 2.5 follows directly from Theorem 2.4 and the result of Johnson [14] that every space with (CBAP) can be renormed to have (MAP).

THEOREM 2.6. *Suppose X is a separable Banach space with an approximating sequence T_n satisfying $T_m T_n = T_n$ for $m > n$ and such that*

$$\sup_{i_1 < i_2 < \cdots < i_k} \|T_{i_1} T_{i_2} \ldots T_{i_k}\| = \lambda < \infty.$$

Then X has $\lambda-$CBAP.

PROOF: The proof is essentially that of Theorem 2.4. We first estimate $\|T_1 + \cdots + T_n\|^k$. On expanding this consists of n^k terms of the form $T_1^{l_1} T_2^{l_2} \ldots T_n^{l_n}$. It is clear that if we define the weight, $w(l_1, \ldots, l_n) = \sum_{l_j > 1} (l_j - 1)$ then by grouping terms,

$$\| \prod_{j=1}^{n} T_j^{l_j} \| \le \lambda^{(1 + w(l_1, \ldots, l_n))}.$$

Now $(\prod T_j^{l_j})(T_1+\cdots+T_n)$ consists of n terms of which $n-1$ have at most the same weight $w=w(l_1,\cdots,l_n)$ and one has weight $w+1$. Thus if if we define W_k by

$$W_k=\sum \lambda^{w(l_1,\ldots,l_n)}$$

where the sum is over all terms $\prod T_j^{l_j}$ in $(\sum T_j)^k$ then $W_{k+1}\le (n-1+\lambda)W_k$ for every k. As $W_1\le n-1+\lambda$ we obtain $W_k\le (n-1+\lambda)^k$ for $k\ge 1$. Thus

$$\|(T_1+\cdots+T_n)^k\|\le \lambda(n-1+\lambda)^k$$

and so

$$\|\exp t(T_1+\cdots+T_n)\|\le \lambda e^{t(n-1+\lambda)}.$$

Thus repeating the proof of Theorem 2.4 we obtain

$$\|V_n(t)\|\le \lambda e^{t(\lambda-1)}.$$

The proofs now goes through unchanged since $\|S(t)\|\le \lambda e^{t(\lambda-1)}$ and $\lim_{t\to 0}\|S(t)\|=\lambda$. Hence X has $\lambda-$CBAP.∎

We will use Theorem 2.6 to give another characterization of (CBAP). We say that X has the *reverse monotone approximation property* (RMAP) if there is an approximating sequence T_n with $\lim_{n\to\infty}\|I-T_n\|=1$. We first prove a simple equivalence for (RMAP).

PROPOSITION 2.7. *(i) X has (MAP) if and only if there exists $\alpha>0$ and an approximating sequence (T_n) with $\lim\|I+\alpha T_n\|=1+\alpha$.*

(ii) X has (RMAP) if and only if there exists $\alpha>0$ and an approximating sequence (T_n) with $\lim\|I-\alpha T_n\|=1$.

PROOF: The proof of (i) is again somewhat similar to the proof of Theorem 2.3. By passing to a subsequence one may suppose that $\|I+\alpha T_n\|\le 1+\alpha(1+\epsilon_n)$ where $\sum\epsilon_n=\beta<\infty$. Then, defining $V_n(t)$ as in Theorem 2.3 one obtains, by estimating $\|\exp(t\sum_{k=1}^n(I+\alpha T_k))\|$, that $\|V_n(\alpha t)\|\le e^{\alpha\beta t}$ and the proof goes through as before.

For (ii) it suffices to consider the case $\alpha<1$ by a simple convexity argument. We may further suppose $T_mT_n=T_n$ for $m>n$. Then pick a sequence of integers l_n so that $\lim l_n=\infty$ and $\lim\|(I-\alpha T_n)^{l_n}\|=1$. Then set $S_n=I-(I-\alpha T_n)^{l_n}$. Clearly (S_n) is an approximating sequence with $\lim\|I-S_n\|=1$.∎

THEOREM 2.8. *Let X be a separable Banach space with (RMAP). Then X has (CBAP).*

To prove Theorem 2.8 we require the following lemma:

LEMMA 2.9. *Suppose X has (RMAP). Then X has an approximating sequence T_n with $\lim\|I-T_n\|=1$ and $\limsup\|(T_n-T_n^2)^2\|\le\frac{15}{16}$.*

PROOF: We assume S_n is an approximating sequence with S_mS_n for $m>n$ and $\|I-S_n\|\le 1+\epsilon_n$ where $\epsilon_n\downarrow 0$. Put $T_n=\frac{1}{2n}\sum_{j=n+1}^{3n}S_j$. Then the properties of T_n specified in the lemma are clear except possibly for the last.

Consider

$$(I-T_n)T_n=\frac{1}{4n^2}\sum_{i=n+1}^{3n}\sum_{j=n+1}^{3n}(I-T_i)T_j.$$

$$(T_n-T_n^2)^2=\frac{1}{16n^4}\sum_{i=n+1}^{3n}\sum_{j=n+1}^{3n}\sum_{k=n+1}^{3n}\sum_{l=n+1}^{3n}(I-T_i)T_j(I-T_k)T_l.$$

Now $(I-T_i)T_j(I-T_k)T_l$ vanishes if either $i>j$ or $k>l$; this eliminates all but $n^2(2n+1)^2$ terms. Consider those remaining terms where $k<l\le 2n<i<j$. In this case $(I-T_i)T_jT_kT_l=(I-T_i)T_kT_l=0$ and hence the term becomes $(I-T_i)T_jT_l=0$. There are $\frac14 n^2(n-1)^2$ such terms. Thus there remain at most $n^2(2n+1)^2-\frac14 n^2(n-1)^2$ terms of norm at most $(2+\epsilon_n)^2(1+\epsilon_n)^2$. Hence

$$\|(T_n-T_n^2)^2\|\le\frac{1}{16}((2+\frac1n)^2-\frac14(1-\frac1n)^2)(2+\epsilon_n)^2(1+\epsilon_n)^2.$$

Thus

$$\limsup_{n\to\infty}\|(T_n-T_n^2)^2\|\le\frac{15}{16}.∎$$

PROOF OF THEOREM 2.8: We may suppose X has an approximating sequence T_n with $\|I-T_n\|\le 1+\epsilon_n$ where $\prod(1+\epsilon_n)\le 2$, $T_mT_n=T_n$ for $m>n$ and $\|(T_n-T_n^2)^2\|<19/20$ for all n. This is possible by the lemma. We then define a new approximating sequence (S_n) by

$$I-S_n=\left(\prod_{k=1}^{n-1}(I-T_k)^2\right)(I-T_n).$$

Thus $\|I-S_n\|\le 4$ and so $\|S_n\|\le 5$.

Now if $A_n=I-S_n$ we have, provided $l_n\ge 1$,

$$\prod_{k=1}^{n}A_k^{l_k}=\left(\prod_{k=1}^{n-1}(I-T_k)^{2+l_k}\right)(I-T_n)^{l_n}.$$

Thus if $p_1,\ldots,p_{n-1}$ are any polynomials,

$$\left(\prod_{k=1}^{n-1}p_k(A_k)\right)A_n=\left(\prod_{k=1}^{n-1}[p_k(I-T_k)](I-T_k)^2\right)(I-T_n).$$

In particular if $i_1<i_2<\cdots<i_m=n$,

$$\left(\prod_{k=1}^{m-1}S_{i_k}^2\right)(I-S_{i_m})=\left(\prod_{k=1}^{n-1}(I-T_k)^2T_k^{2\beta_k}\right)(I-T_n)^2,$$

where $\beta_k=1$ if $k\in\{i_1,\ldots,i_{m-1}\}$ and $\beta_k=0$ otherwise. Hence

$$\|\left(\prod_{k=1}^{m-1}S_{i_k}^2\right)(I-S_{i_m})\|\le 4(\frac{19}{20})^{m-1}.$$

This implies an estimate, since $\|I + S_{i_m}\| \leq 6$,

$$\|\prod_{k=1}^{m} S_{i_k}^2 - \prod_{k=1}^{m-1} S_{i_k}^2\| \leq 24(\frac{19}{20})^{m-1}$$

from which we obtain

$$\|\prod_{k=1}^{m} S_{i_k}^2\| \leq 1 + 24\sum_{k=0}^{\infty}(\frac{19}{20})^k = 481.$$

Notice that $S_n(X) \subset T_n(X)$; hence if $m > n$, $(I - S_m)S_n = 0$ so that $S_mS_n = S_n$ Hence $S_m^2S_n^2 = S_n^2$ and so we may apply Theorem 2.6 to to the approximating sequence S_n^2 to deduce that X has (CBAP).■

3. Complementation and renormings.

PROPOSITION 3.1. *Suppose X is a separable Banach space and Y is a separable reflexive Banach space so that $X \oplus Y$ has (CBAP). Then X has (CBAP).*

PROOF: Suppose S_n is an approximating sequence for $X \oplus Y$ such that $S_mS_n = S_{\min(m,n)}$ for $m \neq n$. Let P be the projection onto X. Consider the operators $PS_n : Y \to X$. Then for every $y \in Y$ we have $\lim\|PS_ny\| = 0$. Hence since Y is reflexive we have $\lim(PS_n)^*x^* = 0$ weakly for every $x^* \in X^*$. Thus ([17]) PS_n converges weakly to zero in $\mathcal{L}(Y,X)$. Hence if $Q = I - P$, PS_nQ converges weakly to zero in $\mathcal{L}(X \oplus Y)$. Now we may pass to a sequence of convex combinations R_n which is still an approximating sequence for $X \oplus Y$ and so that $\lim\|PR_nQ\| = 0$. Define $T_n : X \to X$ by $T_nx = PR_nx$. Then T_n is an approximating sequence for X and

$$[T_m, T_n] = PR_nQR_mP - PR_mQR_nP\mid_X.$$

Hence $\lim_{m,n\to\infty}\|[T_m,T_n]\| = 0$ and the result follows by Corollary 2.2.■

Our next result is a slight modification of an argument in [2].

PROPOSITION 3.2. *Let X be a separable Banach space with an approximating sequence T_n such that $T_mT_n = T_{\min(m,n)}$ for $m \neq n$. Let $E_n = (T_n - T_n^2)(X)$. Then for $1 < p < \infty$, $X \oplus \ell_p(E_n)$ has an FDD. Furthermore if we denote by S_n the associated partial sum projections, we have*

$$\lim_{n\to\infty}\|\frac{1}{n}\sum_{k=1}^{n}(T_n - S_n\mid_X)\| = 0.$$

PROOF: We use an argument which dates back to Johnson [13] and is exploited in [2]. Define projections S_n on $X \oplus \ell_p(E_n)$ by

$$S_n(x, y_1, y_2, \ldots) = (T_nx + y_n, y_1, \ldots, y_{n-1}, (T_n - T_n^2)x + (I - T_n)y_n, 0, 0, \ldots).$$

Then $S_mS_n = S_{\min(m,n)}$ and S_n is an approximating sequence so that $X \oplus \ell_p(E_n)$ has an (FDD). For $x \in X$

$$T_nx - S_nx = (0, 0, \ldots, 0, (T_n - T_n)^2x, 0, 0, \ldots)$$

where the only non-zero entry is in the position of E_n. Since $p > 1$ the last part follows easily.■

Let C_p denote the space $\ell_p(F_n)$ where F_n is a sequence of finite-dimensional Banach spaces dense in the Banach-Mazur sense in the collection of all finite-dimensional Banach spaces (we may assume each space is repeated infinitely often). This space has been studied extensively by Johnson and Zippin (see [16]); it is noted by Johnson [13] that $X \oplus C_p$ has an (FDD) if and only if $X \oplus C_p$ has a basis. The next Corollary combines the two preceding results.

COROLLARY 3.3. *Let X be a separable Banach space and suppose $1 < p < \infty$. Then X has (CBAP) if and only if $X \oplus C_p$ has a basis.*

We remark that Lusky [20] has shown that if C_∞ denotes $c_0(F_n)$ then X has (BAP) if and only if $X \oplus C_\infty$ has a basis. Let us note a brief proof of Lusky's theorem. Let T_n be an approximating sequence for X with $T_mT_n = T_n$ for $m > n$ and let $E_n = T_n(X)$. We define $S_n : X \oplus c_0(E_n) \to X \oplus c_0(E_n)$ by

$$S_n(x, y_1, y_2, \ldots) = (T_nx + y_n, z_1, z_2, \ldots)$$

where

$$z_k = T_k(I - T_n)x - T_ky_n + y_k$$

for $1 \leq k \leq n$ and $z_k = 0$ for $k \geq n+1$. Then $S_mS_n = S_{\min(m,n)}$ and so $X \oplus c_0(E_n)$ has an (FDD). Thus $X \oplus C_\infty$ has an (FDD) and hence also a basis.

We now apply the complementation results to give a renorming theorem. We require first the following lemma.

LEMMA 3.4. *Suppose X has an (FDD) with partial sum operators S_n, and suppose $0 < \alpha < 2$. Then X can be equivalently renormed so that $\|S_n\| = \|I - \alpha S_n\| = 1$ for all n.*

PROOF: It suffices to show that the semigroup of operators generated by S_n, $I - \alpha S_n$, $n \in \mathbb{N}$ is bounded. To do this it suffices to consider a product

$$T = \prod_{k=1}^{n}(I - \alpha S_{i_k})$$

where $i_1 \leq i_2 \leq \cdots \leq i_n$. We can rewrite $I - \alpha S_m = I - S_m + \beta S_m$ where $\beta = 1 - \alpha$. Then

$$T = (I - S_{i_n}) + \sum_{k=1}^{n-1}\beta^{n-k}(S_{i_k} - S_{i_{k-1}})$$

where S_{i_0} is defined to be zero. Thus

$$\|T\| \leq M + 1 + 2M|\beta|(1 - |\beta|)^{-1}$$

where $M = \sup_n \|S_n\|$.∎

THEOREM 3.5. *Suppose X has (CBAP) and $1 \le \alpha < 2$. Then X can renormed so it has a commuting approximating sequence (T_n) with*

$$\lim_{n\to\infty} \|T_n\| = \lim_{n\to\infty} \|I - \alpha T_n\| = 1$$

and

$$\limsup_{n\to\infty} \|T_n - T_n^2\| \le \frac{1}{4}.$$

In particular X can be renormed to have both (MAP) and (RMAP).

PROOF: We use Proposition 3.2. We can find an (FDD) of a space $X \oplus Y$ with partial sum projections S_n and a commuting approximating sequence T_n for X so that

$$\lim_{n\to\infty} \|\frac{1}{n}\sum_{k=1} S_n|_X - T_n\| = 0.$$

Here we have replaced the original sequence T_n by its sequence of Cesaro means. By Lemma 3.4 we can renorm $X \oplus Y$ so that $\|S_n\| = \|I - \alpha S_n\| = 1$ for all n. Then under the same renorming restricted to X we easily get that the first equation holds for the sequence T_n. Further more if $R_n = \frac{1}{n}\sum_{k=1}^n S_k$,

$$\begin{aligned} R_n(I - R_n) &= \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n S_i(I - S_j) \\ &= \frac{1}{n^2}\sum_{j<i}(S_i - S_j) \\ &= \frac{1}{n^2}\sum_{i=1}^n(2i - n)S_i \\ &= \frac{1}{n^2}\sum_{2i<n}(n - 2i)S_{n-i}(I - S_i) \end{aligned}$$

so that

$$\|R_n(I - R_n)\| \le \frac{1}{n^2}\sum_{2i<n}(n - 2i) \to \frac{1}{4}.$$

Thus $\limsup \|T_n - T_n^2\| \le \frac{1}{4}$.∎

We now demonstrate the limits of this renorming by using a simple modification of an argument of Beauzamy [1] and Esterle [6].

PROPOSITION 3.6. *Suppose X is a Banach space and that T is a bounded operator on X. Suppose $\|T - T^2\| = \theta < \frac{1}{4}$. Then there is a projection P on X such that $\{x : Tx = x\} \subset P(X) \subset T(X)$ and*

$$\|P\| \le \frac{1}{2}\left(1 + \frac{1 + 2\|T\|}{(1 - 4\theta)^{1/2}}\right).$$

PROOF: Define

$$S = \sum_{m=0}^{\infty}\binom{2m}{m}(T - T^2)^m$$

and

$$P = \frac{1}{2}(I - (I - 2T)S).$$

Since $(1 - 4z)^{-\frac{1}{2}}$ has a power series expansion $\sum_{m=0}^{\infty}\binom{2m}{m}z^m$ valid for $|z| < \frac{1}{4}$, it is clear that $\|S\| \le (1 - 4\theta)^{-\frac{1}{2}}$ and by a power series manipulation that $(I - 2T)^2S^2 = I$. Hence P is a projection on X and the estimate on $\|P\|$ follows. Note also that

$$P = 3T^2 - 2T^3 + \frac{1}{2}(I - 2T)\sum_{m=2}^{\infty}\binom{2m}{m}(T - T^2)^m.$$

The remaining properties of P follow easily.∎

A result of Casazza [2] asserts that a separable Banach space has an (FDD) if and only if it has (CBAP) and the π-property. In view of Proposition 3.6 we obtain:

THEOREM 3.7. *Let X be a separable Banach space. Suppose X has an approximating sequence T_n for which $\limsup_{n\to\infty} \|T_n - T_n^2\| < \frac{1}{4}$. Then X has the π-property, and if X has (CBAP) then X has an (FDD).*

We remark that Read [22] gives an example of a reflexive Banach space with (CBAP) but having no (FDD). Thus this corollary shows that Read's space cannot be renormed to have an approximating sequence T_n for which $\limsup \|T_n - T_n^2\| < \frac{1}{4}$.

Motivated by Theorem 3.5 we introduce the *unconditional metric approximation property* (UMAP). We shall say that X has (UMAP) provided it has an approximating sequence T_n for which $\lim_{n\to\infty} \|I - 2T_n\| = 1$. The justification for this terminology lies in the following:

THEOREM 3.8. *A separable Banach space X has (UMAP) if and only if for every $\epsilon > 0$ there exists an approximating sequence (T_n) so that if $A_n = T_n - T_{n-1}$ for $n \in \mathbb{N}$ (with $T_0 = 0$) then for every $N \in \mathbb{N}$ and $\eta_i = \pm 1$, $i = 1, 2, \ldots, N$ then*

$$\|\sum_{i=1}^{N}\eta_i A_i\| \le 1 + \epsilon.$$

PROOF: First suppose X has (UMAP), and $\epsilon > 0$. Then X has an approximating sequence T_n for which $T_mT_n = T_n$ for $m > n$ and $\|I - 2T_n\| = 1 + \delta_n$ where $\prod(1 + \delta_n) < 1 + \epsilon$. Defining $A_n = T_n - T_{n-1}$ as above with $T_0 = 0$ we have for $N \in \mathbb{N}$ and $\eta_i = \pm 1$,

$$\begin{aligned} \eta_N \prod_{i=1}^{N-1}(I - (1 - \eta_{N-i}\eta_{N-i-1})T_{N-i}) &= \eta_N \prod_{i=1}^{N-1}(I - T_{N-i} + \eta_{N-i}\eta_{N-i-1}T_{N-i}) \\ &= \eta_N(I - T_{N-1}) + \sum_{i=1}^{N-1}\eta_i A_i. \end{aligned}$$

Thus

$$\|\sum_{i=1}^{N}\eta_iA_i\| = \|T_N\prod_{i=1}^{N-1}(I-(1-\eta_{N-i}\eta_{N-i-1})T_{N-i})\|$$
$$\leq (1+\frac{1}{2}\delta_N)\prod_{i=1}^{N-1}(1+\delta_i)$$
$$\leq 1+\epsilon.$$

For the converse direction suppose (T_n) is an approximating sequence for which for every N and $\eta_i = \pm 1$ we have

$$\|\sum_{i=1}^{N}\eta_iA_i\| \leq 1+\epsilon.$$

Then for any n and $m > n$

$$T_m - 2T_n = \sum_{i=n+1}^{m} A_i - \sum_{i=1}^{n} A_i$$

so that $\|I-2T_n\| \leq \liminf_{m\to\infty}\|T_m - 2T_n\| \leq 1+\epsilon$. It follows easily that X has an approximating sequence S_n for which $\|I-2S_n\| \to 1$.∎

Let us say that X has *UCMAP* if for every $\epsilon > 0$ there is a commuting approximating sequence T_n for which if $A_n = T_n - T_{n-1}$ and $\eta_i = \pm 1$, $i = 1,2,\ldots,N$ then $\|\sum_{i=1}^{N}\eta_iA_i\| \leq 1+\epsilon$.

We also introduce the notion of *u-ideal*. We say that if X is a subspace of Y then X is a u-ideal in Y if there is a projection Π of Y^* onto $X^\perp$ satisfying $\|I-2\Pi\| = 1$. Equivalently there is a complementary subspace M for $X^\perp$ such that if $\phi \in M$ and $\psi \in X^\perp$ then $\|\phi+\psi\| = \|\phi-\psi\|$. For example, if X is an M-ideal in Y then X is a u-ideal. The following result is suggested by results of Feder [7], Godefroy-P. Saab [9] and Godefroy-Li [8] and Li [18]. We are grateful to Gilles Godefroy for correcting an error in the original proof of Theorem 3.9.

THEOREM 3.9. *Let X be a separable reflexive Banach space with the approximation property. Then the following conditions are equivalent:*
(i) X has (UMAP).
(ii) X has (UCMAP).
(iii) X is isometric to a subspace of a Banach space with (UMAP).
(iv) $\mathcal{K}(X)$ is a u-ideal in $L(X)$.

PROOF: Plainly we have $(ii) \Rightarrow (i) \Rightarrow (iii)$. Let us prove $(iii) \Rightarrow (ii)$. Suppose X is (isometric to) a subspace of Y where Y has (UMAP). Since X is reflexive it has (MAP) and hence (CMAP). Let T_n be an approximating sequence for Y such that $\lim\|I-2T_n\| = 1$ and let S_n be a commuting approximating sequence for X. Then $T_n - S_n \in \mathcal{K}(X,Y)$ and $T_n - S_n \to 0$ in the strong operator topology. Hence as X is reflexive, $T_n^* - S_n^* \to 0$ in the weak-operator topology and thus [17] $T_n - S_n \to 0$ weakly. It follows that we can find $V_n \in \text{co}(T_n, T_{n+1},\ldots)$ and $R_n \in \text{co}(S_n, S_{n+1},\ldots)$ so that $\|R_n - V_n\| \to 0$. Then R_n is a commuting approximating sequence for which $\|I-2R_n\| \to 1$ and the argument of Theorem 3.8 shows that X has (UCMAP).

Now let us show $(i) \Rightarrow (iv)$. Let T_n be an approximating sequence for X such that $\lim\|I-2T_n\| = 1$. For $\phi \in \mathcal{L}(X)^*$ we define $\Pi(\phi) \in \mathcal{L}(X)^*$ by $\Pi(\phi)(A) = \lim_{\mathcal{U}}\phi(A-T_nA)$ where $\mathcal{U}$ is a non-principal ultrafilter on $\mathbf{N}$. Clearly $(\phi-2\Pi(\phi))(A) = \lim_{n\in\mathcal{U}}\phi((2T_n-I)A)$ so that $\|I-2\Pi\| = 1$. It is also easy to check that Π is a projection onto $\mathcal{K}(X)^\perp$.

Conversely, to show $(iv) \Rightarrow (i)$, suppose $\Pi : \mathcal{L}(X)^* \to \mathcal{K}(X)^\perp$ is a projection satisfying $\|I-2\Pi\| = 1$. Since X is reflexive with (AP) we can identify $\mathcal{L}(X)$ with $\mathcal{K}(X)^{**}$; thus if $j : \mathcal{K}(X) \to \mathcal{L}(X)$ denotes the natural inclusion, then j^* induces a projection of $\mathcal{L}(X)^*$ onto its subspace M of weak* continuous linear functionals. Now, according to a result of Godefroy and Saphar [10], Corollary 5.4, $\mathcal{K}(X)$ has the unique extension property (U.E.P.) so that j^* is the unique projection of norm one on $\mathcal{L}(X)^*$ with kernel $\mathcal{K}(X)^\perp$ (see the remark on p. 681 of [10]). It follows that $j^* = I - \Pi$.

Now since X has (AP) it also has (BAP) and so there is an approximating sequence (T_n). It will suffice to show that for any n, $\inf\|I-2A\| = 1$ where A runs through all convex combinations of $(T_n, T_{n+1},\ldots)$. If this fails, then by the Hahn-Banach theorem there exists $\phi \in \mathcal{L}(X)^*$ with $\|\phi\| = 1$, and $\phi(I-2T_n) \geq 1+\delta$ where $\delta > 0$. Then since $T_n \to I$ for the weak*-topology, we have $j^*\phi(I-2T_n) \to -j^*\phi(I)$. However $\Pi(\phi)(I-2T_n) = \Pi(\phi)(I)$. Hence $\lim\phi(I-2T_n) = (\Pi - j^*)(\phi)(I) = (2\Pi(\phi)-\phi)(I)$. This is a contradiction since $\|\phi - 2\Pi(\phi)\| = \|\phi\| = 1$.∎

4. Concluding remarks.

We first make some comments on the example of Read [22]. Read shows that there is a subspace X of C_2 so that X has (BAP) but no (FDD). This answers a question raised by Johnson and Zippin [16], (cf. also [19], p. 86) as to whether every subspace of C_2 with the approximation property is of the form $\ell_2(E_n)$ with E_n finite-dimensional. Clearly X is reflexive and has (CBAP) and even (UCMAP) by Theorem 3.9. It follows by Corollary 3.3 that $X \oplus C_2$ has a basis. Hence by the results of Johnson-Zippin [16] ([19] p.85), $X \oplus C_2$ is of the form $\ell_2(E_n)$ and hence $X \oplus C_2$ is isomorphic to C_2. Hence X is isomorphic to a complemented subspace of C_2. Thus we have an example of a complemented subspace of a space with a (UFDD) which fails to have an (FDD).

The major remaining open problem here seems to be whether (BAP) implies (CBAP). Let us note that this is equivalent to the problem of whether a π−space has an (FDD). Indeed if X has (BAP) but not (CBAP) then for any $1 < p < \infty$ $X \oplus C_p$ is a π−space (cf. Johnson [13]) but must fail (CBAP) by Theorem 3.1. For the converse, Casazza [2] shows that a π−space with (CBAP) has an (FDD). Yet another form of this problem [13] is whether for any X with (BAP) $X \oplus C_p$ has a basis (when $1 < p < \infty$.) A related problem is whether (UnAP) implies (CBAP).

We also may raise the question of whether (UnAP) and (CBAP) together will imply the existence of a commuting approximating family (T_n) for which the series $\sum(T_n - T_{n-1})$ is weakly unconditionally Cauchy. In a similar vein, does (UMAP) imply (UCMAP) in general? This is proved for reflexive spaces in Theorem 3.9; obviously (UMAP) implies (CBAP).

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