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Some remarks on interpolation of families of quasi-Banach spaces

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Abstract. We study some questions raised in theory of complex interpolation of quasi-Banach spaces. In particular we give a criterion for the interpolated space to be locally convex.

1. Introduction. In [1] and [2], Coifman, Cwikel, Rochberg, Saghar and Weiss introduced and studied complex interpolation of families of Banach spaces. Recently, Tabacco [11],[12] and Rochberg [10] have studied the extension of these ideas to the non-locally convex quasi-Banach case.

We let $\mathbf{T}$ denote the unit circle in the complex plane and λ denote normalized Haar measure on $\mathbf{T}$, i.e. $d\lambda = (2\pi)^{-1}d\theta$. Δ denotes the unit disk, $\{z : |z| < 1\}$. We then suppose that we are given a family of quasinormed spaces X_w for $w \in \mathbf{T}$ and define interpolation spaces X_z for $z \in \Delta$. The precise details of the construction are given in Section 2.

In this paper we prove two main results on interpolation of analytic families of quasi-Banach spaces. In Theorem 4, we answer a question of Rochberg [10] by giving a condition for the interpolated space to be locally convex. We use here the notion of (Rademacher) type. A quasi-Banach space X is of type p where $0 < p \le 2$ if there is a constant C so that if $x_1, \ldots, x_n \in X$ then

$$\mathcal{E}(\|\epsilon_1 x_1 + \cdots + \epsilon_n x_n\|^p) \le C^p(\|x_1\|^p + \cdots + \|x_n\|^p)$$

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where the signs $\epsilon_k = \pm 1$ are chosen at random. In fact if $p < 1$ then type p is equivalent to p-normability [5], but there are type one spaces which are not locally convex (e.g. the Lorentz spaces $L(1,p)$ where $1 < p < \infty$, or the Ribe space [5]). Now if X_w is type $p(w)$ for $w \in \mathbf{T}$, and a mild separability assumption holds, then the interpolated space at the origin X_0 is locally convex provided

$$\int p(w)^{-1}d\lambda(w) < 1.$$

Simple examples show that equality does not guarantee the conclusion, although under mild hypotheses it will force X_0 to be type one. By conformal transformation, similar results can be given for any interpolated space X_z where $|z| < 1$.

In Theorem 9 we show, again under some mild additional assumptions, that the interpolated space X_0 is unchanged if we replace the quasinorm on each X_w by the largest plurisubharmonic (semi-)quasinorm it dominates. It follows that the interpolated space X_0 collapses to the trivial zero space if X_w is A-trivial in the sense of [7] on a set of positive measure; an example of an A-trivial space is L_p/H_p where $p < 1$. The proof of Theorem 9 is similar to arguments used in [7].

Theorem 9 suggests that the interpolated space always has a plurisubharmonic quasi-norm. This, however, is false as we show by example. We also show that the Iteration Theorem of [1] has no non-locally convex extension, at least in an isometric sense. We leave open, however, the question of whether it is true isomorphically.

Let us note that we have attempted to avoid unnecessary continuity assumptions on the given quasinorms, since in many naturally arising spaces these quasinorms are not continuous. This does result in some tedious measurability problems.

2. Some basic interpolation results. Suppose X is a complex linear space and that we are given a function $H : \mathbf{T} \times X \to [0, \infty]$ with the property that the restriction of H to

$\mathbf{T}\times V$ is a Borel function for each finite-dimensional subspace V of X. Suppose further that for each $x\in X$ we have that $\log^+ H(w,x)\in L_1(\mathbf{T})$.

We define $\mathcal{A}$ to be the set of all analytic maps $g:\Delta\to X$ of the form

$$g(z)=\sum_{j=1}^{N}\phi_j(z)b_j$$

where $\phi_j\in N^+$ for $1\le j\le N$. Here N^+ denotes the Smirnov class (see, for example, [3] p. 25).

Then for $|z|<1$ we define $H_z:X\to[0,\infty)$ by

$$H_z(x)=\inf\{\int_{\mathbf{T}}H(w,g(w))\,d\lambda(w):\ g\in\mathcal{A},\ g(z)=x\}$$

We shall also write $H(z,x)$ for $H_z(x)$.

Now suppose in addition that H satisfies the condition:

$$H(w,\alpha x)=|\alpha|H(w,x)\qquad \alpha\in\mathbf{C}$$

Under these conditions we say that H is homogeneous. It then follows as in the work of Tabacco [12] (see Theorem 1.8) that H_z can also be obtained from the formulas:

$$H(z,x)=\inf\{\left(\int_{\mathbf{T}}P(z,w)H(w,g(w))^p\,d\lambda(w)\right)^{1/p}:\ g\in\mathcal{A},\ g(z)=x\}$$
$$H(z,x)=\inf\{\text{ess sup}\,H(w,g(w)):\ g\in\mathcal{A},\ g(z)=x\}$$
$$H(z,x)=\inf\{\exp\int_{\mathbf{T}}P(z,w)\log(H(w,g(w))d\lambda(w):\ g\in\mathcal{A},\ g(z)=x\}$$

where $0<p<\infty$ and $|z|<1$. Here P denotes the Poisson kernel

$$P(z,w)=\frac{1-|z|^2}{|w-z|^2}.$$

Note now that our hypotheses force $H(z,x)<\infty$ for $|z|<1$ and $x\in X$.

Next we further suppose the existence of a Borel function $c=c_H$ so that

$$H(w,x_1+x_2)\le c(w)(H(w,x_1)+H(w,x_2))\qquad x_1,x_2\in X.$$

Then for each w the space $X_w=\{x\in X:\ H(w,x)<\infty\}$ is a linear space and $\|x\|_w=H(w,x)$ is a semi-quasinorm on X. In future we will use the term quasinorm with the understanding that positive-definiteness is not assumed. If, in addition, we assume that $\log c\in L_1(\mathbf{T})$ then for $|z|<1$, $\|x\|_z=H_z(x)$ defines a quasinorm on X and indeed

$$\|x_1+x_2\|_z\le c(z)(\|x_1+x_2\|_z)$$

where

$$c(z)=\exp\left(\int_{\mathbf{T}}P(z,w)\log c(w)\,d\lambda(w)\right).$$

For details, see Tabacco [12], Propositions 1.9 and 2.7. Under these hypotheses, we refer to X_w as an analytic family of quasinormed spaces, and say that H defines the analytic family X_w. The interpolated space X_z is the completion of the Hausdorff quotient of $(X,\|\ \|_z)$.

We shall concentrate on the case $z=0$ since a conformal transformation then gives the corresponding results for all $|z|<1$.

Suppose then as above H defines an analytic family of quasinormed spaces. Let $p:\mathbf{T}\to(0,\infty)$ be a measurable function such that $1/p\in L_1(\mathbf{T})$. Then we may consider H^p in place of H and Rochberg [10] shows that

$$H_0^{\bar p}=e^{\bar p\Lambda}(H_0)^{\bar p}$$

where

$$\bar p^{-1}=\int_{\mathbf{T}}p(w)^{-1}\,d\lambda(w)$$

and

$$\Lambda=\bar p^{-1}\log(\bar p^{-1})-\int_{\mathbf{T}}p^{-1}\log(p^{-1})\,d\lambda.$$

Rochberg calls this the Power Theorem for Complex Interpolation.

Let us derive two simple conclusions from these results. In order to state our results economically we shall sometimes impose a further natural conditon on the analytic family

X_w. We shall say this family is *uniformly separable* if there is a countable set D in X such that for each $w \in \mathbf{T}$ the set $D \cap X_w$ is dense in X_w. This will avoid certain cumbersome measurability assumptions, and is satisfied in most examples.

PROPOSITION 1. *Let $H : \mathbf{T} \times X \to [0,\infty]$ and $K : \mathbf{T} \times Y \to [0,\infty]$ induce analytic families of quasinormed spaces X_w and Y_w. Suppose $T : X \to Y$ is a linear map and define for $w \in \mathbf{T}$,*

$$\|T\|_w = \sup\{\|Tx\|_w : \|x\|_w \le 1\}.$$

Then:

(i) If η is a Borel measurable function such that $\eta(w) \ge \|T\|_w$ for all w and $\log_+ \eta \in L_1$ then

$$\|T\|_0 \le \exp\left(\int_{\mathbf{T}} \log \eta(w) d\lambda(w)\right)$$

(ii) If X is uniformly separable and both X_w and Y_w are continuously quasinormed then $w \to \|T\|_w$ is Borel measurable.

(iii) If X is uniformly separable, but we do not assume that the families are continuously quasinormed, then there exists a Borel function η such that

$$\|T\|_w \le \eta(w) \le c_H(w)^6 c_K(w)^6 \|T\|_w$$

for $w \in \mathbf{T}$.

PROOF: (i) is a special case of a theorem of Tabacco [12], Theorem 1.13. (ii) is an immediate consequence of uniform separability. For (iii) let (x_n) be any sequence whose intersection with each X_w is dense. Define $\eta(w) = c_H(w)^2 c_K(w)^4 \sup\{\|Tx_n\|_w : \|x_n\|_w \le 1\}$. Note that on X_w there is an equivalent continuous quasinorm $|\ |$ with $\|x\|_w \le |x| \le c_H(w)^2 \|x\|_w$ (see [8], Lemma 1.1 and Theorem 1.2). Similarly there is an continuous equivalent quasinorm $|\ |$ on Y_w with $\|y\|_w \le |y| \le c_K(w)^2 \|y\|_w$. Now if $|x_n| \le 1$ then $|x_n|_w \le 1$ and so $|T| \le \sup\{|Tx_n| : \|x_n\|_w \le 1\}$. Thus $\|T\|_w \le c_H(w)^2 c_K(w)^2 |T| \le \eta(w)$. On the other hand, $\eta(w) \le c_H(w)^2 c_K(w)^4 \sup\{|Tx_n| : |x_n| \le c_H(w)^2\} \le c_H(w)^4 c_K(w)^4 |T| \le c_H(w)^6 c_K(w)^6 \|T\|_w$.

Now let Ω be a finite set and let μ be a finite strictly positive measure on Ω. Denote by X^Ω the space of all maps $f : \Omega \to X$.

PROPOSITION 2. *Let $H : \mathbf{T} \times X \to [0,\infty]$ induce an analytic family of quasinormed spaces on X. Let $p : \mathbf{T} \to (0,\infty)$ be a measurable map such that $1/p \in L_1$. Define $K : \mathbf{T} \times X^\Omega \to [0,\infty]$ by*

$$K(w,f) = \left(\int_\Omega H(w,f)^{p(w)}\, d\mu\right)^{1/p(w)}.$$

Then K also induces an analytic family with

$$c_K(w) \le \gamma(w) c_H(w)$$

where $\gamma(w) = \max(2^{\frac{1}{p}-1}, 1)$ and further

$$\|f\|_0 = K_0(f) = \left(\int_\Omega \|f(\omega)\|_0^{\bar p}\, d\mu(\omega)\right)^{1/\bar p}$$

where $\bar p = \int p^{-1}\, d\lambda$.

PROOF: We omit the simple proof that K satisfies the hypotheses to induce an analytic family, with c_K as given. Observe then that K^p is given by

$$K^p(w,f) = \int_\Omega H^p(w,f)\, d\mu$$

and since Ω is finite it then easily follows that

$$K_0^{\bar p}(f) = \int_\Omega H_0^{\bar p}(f(\omega))\, d\mu.$$

The result now follows from the Power Theorem of Rochberg quoted above.

Either as a limiting case or by a simple direct argument we can also easily show that:

PROPOSITION 3. *Under the hypotheses of Proposition 2 define* $J : \mathbf{T} \times X^{\Omega} \to [0,\infty]$ *by*

$$J(w,f) = \max_{\omega} H(w, f(\omega)).$$

Then J *also defines an analytic family with* $c_J = c_H$ *and*

$$\|f\|_0 = J_0(f) = \max_{\omega} \|f(\omega)\|_0 .$$

3. Convexity of the interpolated space. We now prove a theorem which provides an answer to a question of Rochberg [10].

THEOREM 4. *Suppose* $H : \mathbf{T} \times X \to [0,\infty]$ *induces a uniformly separable analytic family of quasinormed spaces. Suppose that* $p : \mathbf{T} \to (0,\infty)$ *is a measurable function such that each* X_w *is of type* $p(w)$ *and*

$$\int p(w)^{-1}\, d\lambda < 1.$$

Then X_0 *is locally convex, i.e.* $\|\;\|_0$ *is equivalent to a seminorm on* X.

PROOF: We set $p_0 = \int 1/p\, d\lambda$. For any $n \in \mathbf{N}$ let Ω be a finite set (depending on n), let μ be a probability measure on Ω and let $\epsilon_1, \dots, \epsilon_n$ be a sequence of independent random variables on (Ω, μ) such that $\mu(\epsilon_k = 1) = \mu(\epsilon_k = -1) = \frac{1}{2}$ for $k = 1, 2, \dots, n$. Let $d_n(w)$ be the least constant such that for every $x_1, \dots, x_n$ we have

$$\left(\int_{\Omega} \|\epsilon_1 x_1 + \cdots + \epsilon_n x_n\|_w^{p(w)}\, d\mu \right)^{1/p(w)} \le d_n(w) \max(\|x_1\|_w, \dots, \|x_n\|_w).$$

Similarly $d_n(0)$ is the least constant such that

$$\left(\int_{\Omega} \|\epsilon_1 x_1 + \ldots + \epsilon_n x_n\|_0^{p_0}\, d\mu \right)^{1/p_0} \le d_n(0) \max(\|x_1\|_0, \dots, \|x_n\|_0).$$

A crude estimate on d_n when $2^{m-1} < n \le 2^m$ is given by

$$d_n(w) \le (2c(w))^m$$

so that we have

$$\log d_n(w) \le \frac{\log n + \log 2}{\log 2} (\log 2 + \log c(w)).$$

Define $T_n : X^n \to X^{\Omega}$ by

$$T_n(x_1, \dots, x_n) = \epsilon_1 x_1 + \cdots + \epsilon_n x_n .$$

If we equip X^n with the quasinorm $\|(x_1, \dots, x_n)\|_w = \max \|x_k\|_w$ and X^{Ω} with the quasinorm $\|f\|_w = (\int \|f\|_w^{p(w)}\, d\mu)^{1/p(w)}$ then $d_n(w) = \|T_n\|_w$ and so it follows from Propositions 1, 2 and 3 that

$$d_n(0) = \|T_n\|_0 \le \exp\left(\int_{\mathbf{T}} \log \eta_n(w)\, d\lambda(w) \right),$$

where η is any Borel measurable function satisfying $\eta_n(w) \ge d_n(w)$. We may further suppose that $\eta_n(w) \le \gamma(w)^6 c(w)^{12} d_n(w)$ where $\gamma(w) = \max(2^{\frac{1}{p(w)} - 1}, 1)$.

Now from the calculation above we have

$$\frac{\log d_n(w)}{\log n} \le A(\log 2 + \log c(w))$$

for suitable constant A. Thus

$$\frac{\log \eta_n(w)}{\log n} \le A_1(\log 2 + \log c(w) + \log \gamma(w))$$

for suitable A_1. Furthermore for each w then exists a constant B_w such that

$$\left(\int_{\Omega} \|\epsilon_1 x_1 + \cdots + \epsilon_n x_n\|_w^{p(w)}\, d\mu \right)^{1/p(w)} \le B_w (\|x_1\|_w^{p(w)} + \cdots + \|x_n\|_w^{p(w)})^{1/p(w)}$$

whenever $x_1, \cdots, x_n \in X_w$. Thus

$$d_n(w) \le B_w n^{1/p}.$$

It follows that

$$\limsup \frac{\log d_n(w)}{\log n} \le \frac{1}{p(w)}$$

and so

$$\limsup \frac{\log \eta_n(w)}{\log n} \leq \frac{1}{p(w)}.$$

Now by the Dominated Convergence Theorem,

$$\limsup_n \int_{\mathbf{T}} \frac{\log \eta_n(w)}{\log n} d\lambda \leq \int_{\mathbf{T}} \frac{1}{p(w)} d\lambda$$

Hence

$$\limsup \frac{\log d_n(0)}{\log n} < 1.$$

It now follows from Theorem 2.5 of [4] that X_0 is locally convex.

In the above theorem we did not assume any bound on the type $p(w)$ constants B_w. If we assume some control on these constants then we can make more precise statements.

THEOREM 5. *Suppose $H : \mathbf{T} \times X \to [0,\infty]$ induces a uniformly separable family of continuously quasinormed spaces. Suppose that $p : \mathbf{T} \to (0,\infty)$ and $B : \mathbf{T} \to (0,\infty)$ are measurable functions such that each X_w is of type $p(w)$ and further:*

(a) $$\int p(w)^{-1} d\lambda = p_0^{-1}$$

(b) $$\int \log B(w) d\lambda = \log B_0 < \infty$$

(c) $$\left(\int_\Omega \|\epsilon_1 x_1 + \cdots + \epsilon_n x_n\|_w^{p(w)} d\mu\right)^{1/p(w)} \leq B(w)(\|x_1\|_w^{p(w)} + \cdots + \|x_n\|_w^{p(w)})^{1/p(w)}$$

whenever $x_1, \ldots x_n \in X_w$ and $\epsilon_1, \ldots \epsilon_n$ is a sequence of independent random variables defined on a probability space (Ω, μ) with $\mu(\epsilon_k = 1) = \mu(\epsilon_k = -1) = \frac{1}{2}$ for $1 \leq k \leq n$.

Then X_0 is type p_0 with type constant at most B_0 i.e. for $x_1, \ldots, x_n \in X$ we have

$$\left(\int_\Omega \|\epsilon_1 x_1 + \cdots \epsilon_n x_n\|_0^{p_0} d\mu\right)^{1/p_0} \leq B_0(\|x_1\|_0^{p_0} + \cdots + \|x_n\|_0^{p_0})^{1/p_0}.$$

REMARK: If we relax the conditions that the quasinorms are continuous then it is still possible to conclude that X is type p_0, but without the precise bound on the type p_0 constant.

We omit the proof of Theorem 5 in view of its similarity to Theorem 4. Let us note that these results can be applied to complex interpolation of two spaces (E,F) by regarding this as a special case of a family. Let $H(w,x) = \|x\|_E$ if $0 < \arg w < 2\pi\theta$ and $H(w,x) = \|x\|_F$ otherwise; then $X_0 = (E,F)_\theta$. The separability assumptions now become unnecessary. Suppose E is p-normable where $p < 1$ and F is a Banach space of type q where $q > 1$. Then $(E,F)_\theta$ is isomorphic to a Banach space provided $\frac{\theta}{p} + \frac{1-\theta}{q} < 1$. If θ is chosen at the critical value i.e. $\frac{\theta}{p} + \frac{1-\theta}{q} = 1$ then $(E,F)_\theta$ is type one, but need not be locally convex. An easy example is given by interpolating between weak $L_{1/2}$ and L_2.

Let us note here that if we interpolate in the last example between $L_{1/2}$ and L_2 then the critical space is L_1 and is locally convex. This result is related to the lattice structure. The notion of lattice p-convexity studied in [6] interpolates nicely and a lattice 1-convex space is locally convex. The distinction here is that weak $L_{1/2}$ is $\frac{1}{2}$-normable but not lattice $\frac{1}{2}$-convex.

Finally let us note that if E is p-normable and F is a Hilbert space then if we choose θ to satisfy $\frac{\theta}{p} + \frac{1-\theta}{2} = 1$ then $(E,F)_\theta$ can be renormed to be type one with constant one. This point is discussed, for Banach spaces, in the context of the Clarkson inequalities by Pisier [9].

4. A-convexity. Let X be a quasinormed space with the property that the quasinorm is a Borel function on each finite-dimensional subspace. Then we can define $H : \mathbf{T} \times X \to [0,\infty]$ by $H(w,x) = \|x\|$ for all w. In this case the interpolated quasinorm $\| \ \|_0$ on X is given by

$$\|x\|_0 = \inf \int_{\mathbf{T}} \|g(w)\| d\lambda(w)$$

where $g \in \mathcal{A}$ runs through all maps with with $g(0) = x$. We shall denote the interpolated quasinorm $\| \ \|_0$ in this case by $\| \ \|_A$. This corresponds, when the original quasinorm is continuous, with the terminology introduced in [7]. It is shown in [7] that $\| \ \|_A$ is plurisubharmonic at least when the original quasinorm is p-subadditive for some $p > 0$. This result will be reproved, for separable spaces, and extended below. We say that X is A-convex if $\| \|_A$ is equivalent to the original quasinorm; A-convexity may be reformulated in terms of the Maximum Modulus Principle or the existence of an equivalent plurisubharmonic quasinorm, as in [7]. Conversely X is A-trivial if $\|x\|_A = 0$ for every $x \in X$. Tabacco [12] uses the A-trivial space L_p/H_p for $0 < p < 1$ to show that in general complex interpolation of quasi-Banach spaces can yield trivial spaces.

Let us now describe our general setup. Suppose $H : \mathbf{T} \times X \to [0, \infty]$ defines a uniformly separable family of quasinormed spaces, with the additional property that the associated quasinorms $\| \|_w$ are upper-semi-continuous . Then we define

$$H'(w,x) = \inf_{y \in X} \int_{\mathbf{T}} H(w, x + zy)\, d\lambda(z).$$

Then it is readily verified that H' also defines a uniformly separable family of quasinormed spaces (with the constants $c(w)$ unchanged) and has the property that the associated quasinorms $\| \|'_w$ are upper-semi-continuous on $(X_w, \| \|_w)$. Furthermore the spaces X'_w coincide with the spaces X_w as linear subspaces of X.

LEMMA 6. *Under the hypotheses above, we have $H'_0 = H_0$.*

PROOF: Suppose $x \in X$ and that $g \in \mathcal{A}$ is such that $g(0) = x$. Next suppose $y_1, \ldots, y_k \in X$ and that $\phi_1, \ldots, \phi_k$ are polynomials. Define $h(z) = \phi_1(z) y_1 + \cdots + \phi_k(z) y_k$. For any $m \in \mathbf{N}$ we have

$$H_0(x) \le \int_{\mathbf{T}} H(w, g(w) + w^m h(w))\, d\lambda(w).$$

Now for almost every fixed w we have for every continuous function ψ on $\mathbf{T}$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \psi(w^j) = \int_{\mathbf{T}} \psi\, d\lambda.$$

For such w let ψ be any continuous function on $V_w = [g(w), y_1, \ldots, y_k]$ with $\psi(u) \ge H(w,u)$ for $u \in X$. Then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \psi(g(w) + w^j h(w)) = \int_{\mathbf{T}} \psi(g(w) + z h(w))\, d\lambda(z).$$

By taking infima over all such ψ we conclude that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} H(w, g(w) + w^j h(w)) \le \int_{\mathbf{T}} H(w, g(w) + z h(w))\, d\lambda(z).$$

Now by integration we obtain that

$$H_0(x) \le \int \int H(w, g(w) + z h(w))\, d\lambda(z)\, d\lambda(w).$$

Now if $B_1, \ldots, B_k$ are any disjoint Borel sets we can find uniformly bounded sequences of polynomials $\phi_j^{(n)}$ so that the functions $|\phi_j^{(n)}|$ converge almost everywhere to the characteristic functions χ_{B_j}. Define $h^{(n)}(w) = \sum \phi_j^{(n)}(w) y_j$. Suppose $w \in B_j$; except for w in a set of measure zero, we may pick α_n so that $|\alpha_n| = 1$ and $\alpha_n \phi_j^{(n)}(w) \to 1$, while $\alpha_n \phi_r^{(n)}(w) \to 0$ for $r \ne j$. Then

$$\int H(w, g(w) + z h^{(n)}(w))\, d\lambda(z) = \int H(w, g(w) + \alpha_n z h^{(n)}(w))\, d\lambda(z).$$

Letting $n \to \infty$ we have (for almost every $w \in B_j$)

$$\limsup \int H(w, g(w) + z h^{(n)}(w))\, d\lambda(z) \le \int H(w, g(w) + z y_j)\, d\lambda(z).$$

Thus

$$H_0(x) \le \sum_{j=1}^{N} \int_{B_j} \int_{\mathbf{T}} H(w, g(w) + z y_j)\, d\lambda(z)\, d\lambda(w).$$

Note that equation holds for any choice of $y_1, \ldots, y_k$ and $B_1, \ldots, B_k$. In particular let $(y_n,\ n \in \mathbf{N})$ be any sequence in X whose intersection with each X_w is dense. Clearly we have for each $k \in \mathbf{N}$,

$$H_0(x) \le \int_{\mathbf{T}} \inf_{j \le k} \left(\int_{\mathbf{T}} H(w, g(w) + z y_j)\, d\lambda(z) \right) d\lambda(w).$$

Letting $k \to \infty$ since each $H(w,.)$ is upper-semi-continuous we obtain

$$H_0(x) \le \int_{\mathbf{T}} H'(w, g(w))\,d\lambda(w).$$

Now by allowing g to vary we have $H_0(x) \le H_0'(x)$.

We then define a sequence by $H^{(n)} = H^{(n-1)\prime}$ for $n > 0$ with $H^{(0)} = H$. This sequence is monotone decreasing, and so we can define a limit by $H^a(w,x) = \lim H^{(n)}(w,x)$. It is again the case that H^a defines a uniformly separable family of quasinorms. Also each $H^{(n)}(w,.)$ and $H^a(w,.)$ is upper-semi-continuous on $(X_w, \| \|_w)$. It is easy to verify that $H_0^a(x) = \lim H_0^{(n)}(x)$ and hence by the preceding lemma we also have:

LEMMA 7. $H_0^a = H_0$.

THEOREM 8. *Let X be a separable quasinormed space such that the quasinorm is upper-semi-continuous . Then $\| \|_A$ is plurisubharmonic on X. Furthermore if the sequence of quasinorms $\| \|^{(n)}$ is defined by $\| \|^{(0)} = \| \|$ and*

$$\|x\|^{(n)} = \inf_{y \in X} \int_{\mathbf{T}} \|x + zy\|^{(n-1)}\,d\lambda(z)$$

then for each x we have $\lim_{n\to\infty} \|x\|^{(n)} = \|x\|_A$.

PROOF: Define $H(w,x) = \|x\|$ for all $w \in \mathbf{T}$. Then $H_0(x) = \|x\|_A$ for all $x \in X$. Furthermore $H^{(n)}(w,x) = \|x\|^{(n)}$ for all w, x. Thus $\|x\|_A = H_0(x) = H_0^a(x) \le H_0^{(n)}(x) \le \|x\|^{(n)}$ for all n. Now for any x, y we have

$$\lim_{n\to\infty} \|x\|^{(n)} \le \lim_{n\to\infty} \int_{\mathbf{T}} \|x + zy\|^{(n-1)}\,d\lambda(z)$$
$$= \int_{\mathbf{T}} \lim_{n\to\infty} \|x + zy\|^{(n)}\,d\lambda(z).$$

Since the function $\lim \| \|^{(n)}$ is upper-semi-continuous we conclude that it is plurisubharmonic. Hence if $g \in \mathcal{A}$ with $g(0) = x$ then

$$\lim_{n\to\infty} \|x\|^{(n)} \le \int_{\mathbf{T}} \|g(w)\|\,d\lambda(w)$$

so that $\lim \|x\|^{(n)} \le \|x\|_A$ and the theorem is proved.

THEOREM 9. *Let $H : \mathbf{T} \times X \to [0,\infty]$ induce a uniformly separable analytic family o upper-semi-continuously quasi-normed spaces. Define $H^*(w,x) = \|x\|_{w,A}$ for $w \in \mathbf{T}$, $x \in X$. Then H^* also induces a uniformly separable analytic family of quasinormed spaces anc $H_0^* = H_0$.*

PROOF: In fact by the above theorem, $H^* = H^a$ and so this reduces to Lemma 7.

COROLLARY 10. *Let H induce a uniformly separable analytic family of upper-semi-continuously quasinormed spaces X_w such that X_w is A-trivial on a set of positive measure. Then $H_0 = 0$.*

PROOF: This is immediate from the preceding theorem.

Theorems 8 and 9 might suggest that we can expect the interpolated quasinorm H_0 itself to be plurisubharmonic. However, this is false and we now present a simple three-dimensional example to show that this is not the case. In fact, our example also shows that there is no iteration theorem in the quasinormed setting, in contrast to the normed setting (see [1], Corollary 4.2).

We will consider $\mathbf{C}^3$ with the ℓ_∞-norm which we denote by $\| \|$. Fix $\kappa > 1$ and $0 < p < 1$. Denote by $(e_j, j = 1,2,3)$ the standard basic vectors in $\mathbf{C}^3$ and then for $u, w \in \mathbf{T}$ define $\xi_{u,w} \in \mathbf{C}^3$ by $\xi_{u,w} = e_1 + ue_2 + w\bar{u}e_3$. We then define $\| \|_w$ to be the greatest p-subadditive quasinorm such that $\|x\|_w \le \kappa\|x\|$ for $x \in \mathbf{C}^3$ and $\xi_{u,w} \le 1$ for all $u \in \mathbf{T}$. It is immediate from the definition that we have $\|x\| \le \|x\|_w \le \kappa\|x\|$ for all x.

Now define $H(w,x) = \|x\|_w$. It may be checked that H is continuous on $\mathbf{T} \times X$. Thus we may interpolate and we will have for all z with $|z| < 1$, $\|x\| \le H_z(x) = \|x\|_z \le \kappa\|x\|$.

If we set $g(z) = e_1 + ue_2 + z\bar{u}e_3$ where $u \in \mathbf{T}$ is fixed we can see that $\|e_1 + ue_2 + z\bar{u}e_3\|_z \le 1$ and hence since each interpolated quasinorm is p-subadditive,

$$\|e_1 + ue_2\|_z \le (1 + \kappa^p|z|^p)^{1/p}.$$

Now suppose that H_0 is plurisubharmonic. Then using the above equation for $z=0$ we have $H_0(e_1) = \|e_1\|_0 = 1$. Equally, if the iteration theorem is valid then we would have for any r, $0<r<1$,

$$\|e_1\|_0 \le \int_{\mathbf{T}} \|e_1 + we_2\|_{rw}\, d\lambda(w)$$
$$\le (1+\kappa^p r^p)^{1/p}$$

so that $\|e_1\|_0 \le 1$.

We will show however that $\|e_1\|_0 > 1$. Let us suppose $0 < \epsilon < 1$. In the ensuing argument we will adopt the convention that $\delta = \delta(\epsilon)$ represents a function such that $\delta(\epsilon) \to 0$ as $\epsilon \to 0$; δ may depend on p, κ, and is allowed to vary from line to line.

Suppose $\|e_1\|_0 \le 1$. Then there exists $g \in \mathcal{A}$ such that $g(0) = (1-\epsilon^3)e_1$ and $\|g(w)\|_w < 1$ for all $w \in \mathbf{T}$. We may write $g(z) = \phi_1(z)e_1 + \phi_2(z)e_2 + \phi_3(z)e_3$ where each ϕ_j is in H_∞ with $\|\phi_j\|_\infty \le 1$. First we observe that

$$\int_{\mathbf{T}} |1-\phi_1(w)|^2 d\lambda(w) \le 2 - 2\Re \int_{\mathbf{T}} \phi_1(w) d\lambda(w)$$
$$\le 2\epsilon^3.$$

Let A denote the set where $|1-\phi_1(w)| \le \epsilon$. then $\lambda(A) \ge 1-2\epsilon$. If $w \in A$ then we can write

$$g(w) = y + \sum_{n=1}^{\infty} \alpha_n \xi_{u_n,w}$$

where

$$\sum_{n=1}^{\infty} |\alpha_n|^p + \kappa^p \|y\|^p < 1.$$

Thus,

$$\phi_1(w) = y_1 + \sum_{n=1}^{\infty} \alpha_n.$$

Hence

$$1-\epsilon \le |y_1| + \sum_{n=1}^{\infty} |\alpha_n|.$$

Now, since $p < 1 < \kappa$ we conclude that

$$\max_n |\alpha_n| \ge 1 - \delta(\epsilon)$$

and so there exists m such that

$$\|g(w) - \alpha_m \xi_{u_m,w}\| \le \delta(\epsilon).$$

Considering the first co-ordinate we have

$$|1-\alpha_m| \le \delta(\epsilon)$$

and so

$$\|g(w) - \xi_{u_m,w}\| \le \delta(\epsilon).$$

This in turn implies, for $w \in A$,

$$|\phi_2(w)\phi_3(w) - w| \le \delta(\epsilon).$$

Hence

$$\int_{\mathbf{T}} |1 - \bar{w}\phi_2(w)\phi_3(w)| d\lambda(w) \le \delta(\epsilon).$$

However ϕ_2, ϕ_3 both have zeros at the origin and so

$$\int_{\mathbf{T}} \bar{w}\phi_2(w)\phi_3(w) d\lambda(w) = 0.$$

This contradiction completes the example.

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