R-bounded approximating sequences and applications to semigroups

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Abstract

It is shown that on certain Banach spaces, including $C[0,1]$ and $L_1[0,1]$, there is no strongly continuous semigroup $(T_t)_{0<t<1}$ consisting of weakly compact operators such that $(T_t)_{0<t<1}$ is an $R$-bounded family. More general results concerning approximating sequences are included and some variants of $R$-boundedness are also discussed.

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1. Introduction

Recent work on semigroup theory [13,24] has highlighted the importance of the concept of $R$-boundedness. Let us recall the definition of $R$-bounded families of operators (cf. [2, 7,9]).

Definition 1.1. A family $\mathcal{T}$ of operators in $\mathcal{L}(X, Y)$ is called $R$-bounded with $R$-boundedness constant $C > 0$ if letting $(\epsilon_k)_{k=1}^{\infty}$ be a sequence of independent Rademachers on some probability space then for every $x_1, \ldots, x_n \in X$ and $T_1, \ldots, T_n \in \mathcal{T}$ we have

\[\left\| \sum_{k=1}^{n} \epsilon_k T_k x_k \right\| \leq C \left\| (T_1 x_1, \ldots, T_n x_n) \right\|_{\mathcal{L}}\]
By the Kahane–Khintchine inequality we can replace 2 above by any other exponent \(1 \leq p < \infty\) to obtain an equivalent definition. We will also need the following definition introduced in [13].

Definition 1.2. A family \(T\) of operators in \(L(X,Y)\) is called WR-bounded with WR-boundedness constant \(C > 0\) if for every \(x_1, \ldots, x_n \in X\), \(y_1^*, \ldots, y_n^* \in Y^*\) and \(T_1, \ldots, T_n \in T\) we have

\[
\left( \mathbb{E} \left\| \sum_{k=1}^n \epsilon_k T_k x_k \right\|^2 \right)^{1/2} \leq C \left( \mathbb{E} \left\| \sum_{k=1}^n \epsilon_k x_k \right\|^2 \right)^{1/2} \cdot (1.1)
\]

It is clear by the Cauchy–Schwarz inequality that R-boundedness implies WR-boundedness. The converse is not true in general, but it holds for spaces with non-trivial type [13, 20].

In [13] it was shown that no reasonable differential operator on \(L_1\) can have an \(H^\infty\)-calculus. In this note we consider the related question whether a differential-type operator on \(L_1\) can generate an R-bounded semigroup. Note that if \(A\) is an R-sectorial operator (cf. [13]) with R-sectoriality angle less than \(\pi/2\) then the semigroup \((e^{-tA})_{0 < t < 1}\) is necessarily R-bounded. In general, one expects a semigroup generated by a differential operator on a bounded domain to consist of weakly compact operators. We are thus led to consider the question whether one can have a strongly continuous semigroup \((T_t)_{0 < t < 1}\) on \(L_1\) such that each \(T_t\) is weakly compact (or equivalently compact, since \(L_1\) has the Dunford–Pettis property) and such that the family \((T_t)_{0 < t < 1}\) is R-bounded. In fact this leads to considering versions of the approximation property; the only property of the semigroup needed is commutativity. We consider the general question whether on a given separable Banach space one can find an R-bounded sequence \((T_n)_{n \in \mathbb{N}}\) of commuting weakly compact operators such that \(\lim_{n \to \infty} T_n x = x\) for all \(x \in X\). Our main results show that for the spaces \(L_1[0,1], C(K)\) (except \(c_0\)) and the disk algebra \(A(\mathbb{D})\) this is impossible. These results may be regarded as extensions of classical results that the spaces \(L_1, C(K)\) do not have unconditional bases [15].

In the case of \(L_1\) we are led to consider a natural weakening of R-boundedness, where we use the definition (1.1) but only for single vectors.

Definition 1.3. A family \(T\) of operators in \(L(X,Y)\) is called semi-R-bounded if there is a constant \(C > 0\) such that for every \(x \in X\), \(a_1, \ldots, a_n \in \mathbb{C}\) and \(T_1, \ldots, T_n \in T\) we have

\[
\left( \mathbb{E} \left\| \sum_{k=1}^n \epsilon_k a_k T_k x \right\|^2 \right)^{1/2} \leq C \left( \sum_{k=1}^n |a_k|^2 \right)^{1/2} \left\| x \right\|. (1.3)
\]

We note that semi-R-boundedness is equivalent to R-boundedness for operators on \(L_1\). In Theorem 2.2 we actually characterize all spaces where semi-R-boundedness is equivalent to R-boundedness as spaces which are either Hilbert spaces or GT-spaces of cotype 2 in the terminology of Pisier [19].
2. R-boundedness and WR-boundedness

In this section, we make some remarks about R-boundedness and related notions.

Note that in a space of type 2, any uniformly bounded collection $T \subset L(X, X)$ is semi-R-bounded. The converse is also true:

Proposition 2.1. A Banach space $X$ has type 2 if and only if uniform boundedness is equivalent to semi-R-boundedness.

Proof. Suppose that every uniformly bounded family of operators is already semi-R-bounded. Pick any $x \in X$ and $x^* \in X^*$ such that $\|x\| = \|x^*\| = 1$ and $x^*(x) = 1$. Notice that the family $T = \{x^* \otimes u : \|u\| = 1\}$ is uniformly bounded with constant one and hence semi-R-bounded by assumption. Let $C$ be the semi-R-boundedness constant of $T$. Select any $x_1, \ldots, x_n \in X$ and write $x_k = \|x_k\| u_k$, where $\|u_k\| = 1$. Then $\{x^* \otimes u_k : k = 1, \ldots, n\} \subset T$ and

$$\left( \mathbb{E} \left\| \sum_{k=1}^n \epsilon_k x_k \right\|^2 \right)^{1/2} = \left( \mathbb{E} \left\| \sum_{k=1}^n \epsilon_k \|x_k\|(x^* \otimes u_k)x \right\|^2 \right)^{1/2} \leq C \left( \mathbb{E} \left\| \sum_{k=1}^n \epsilon_k \|x_k\|^2 \right\|^2 \right)^{1/2} \leq C \left( \sum_{k=1}^n \|x_k\|^2 \right)^{1/2}.
$$

Thus, $X$ has type 2. 

For some spaces, semi-R-boundedness is equivalent to R-boundedness and we are able to completely characterize these spaces in the next theorem. Let us recall that a Banach space $X$ is called a GT-space if every bounded operator $T : X \to \ell_2$ is absolutely summing.

Examples of GT-spaces of cotype 2 are $L_1$, the quotient of $L_1$ by a reflexive subspace [14, 19], and $L_1/H_1$ [8]. It is unknown whether every GT-space has cotype 2.

Theorem 2.2. Suppose $X$ is separable. Then the following are equivalent:

(i) Every semi-R-bounded family of operators on $X$ is R-bounded.
(ii) $X$ is isomorphic to $\ell_2$ or $X$ is a GT-space of cotype 2.

Proof. First we prove that (i) implies (ii). Suppose that every semi-R-bounded family of operators on $X$ is R-bounded. Let us note that this implies the existence of a constant $K$ so that if $T$ has semi-R-boundedness constant $C$ then it has R-boundedness constant $KC$; for otherwise we could find a sequence $T_n$ of families with semi-R-boundedness constant one and R-boundedness constant at least $4^n$; then the family $\bigcup_{n \geq 1} 2^{-n} T_n$ contradicts our
assumption. Fix $M > 1$ and take $x \in X$. Choose $n \in \mathbb{N}$. By Dvoretzky’s theorem [16] we can find $e_1, \ldots, e_n \in X$ such that for any $a_1, \ldots, a_n \in \mathbb{C}$ we have
\[
M^{-1} \left( \sum_{k=1}^{n} |a_k|^2 \right)^{1/2} \leq \left\| \sum_{k=1}^{n} a_k e_k \right\| \leq M \left( \sum_{k=1}^{n} |a_k|^2 \right)^{1/2}.
\]
Consider the family of operators $T_n = \{ u^* \otimes e_k : \|u^*\| = 1, \; k = 1, \ldots, n \}$. Then each $T_n$ is semi-R-bounded with constant $M$ as follows. A finite subfamily of $T_n$ is of the form $\{ u^*_{kj} \oplus e_k : 1 \leq k \leq n, \; 1 \leq j \leq m_k \}$ for some $m_1, \ldots, m_n \in \mathbb{N}$. Then for every $a_{11}, \ldots, a_{nm_n} \in \mathbb{C}$ we have (letting $e_{kj}$ denote independent Rademachers)
\[
\left( \mathbb{E} \left[ \left\| \sum_{k=1}^{n} \sum_{j=1}^{m_k} e_{kj} a_{kj} u^*_{kj}(x)e_k \right\|^2 \right] \right)^{1/2} \leq M \left( \mathbb{E} \left[ \left\| \sum_{k=1}^{n} \sum_{j=1}^{m_k} e_{kj} u^*_{kj}(x)a_{kj} \right\|^2 \right] \right)^{1/2}
\leq M \left( \sum_{k=1}^{n} \mathbb{E} \left[ \left\| \sum_{j=1}^{m_k} e_{kj} u^*_{kj}(x)a_{kj} \right\|^2 \right] \right)^{1/2} \leq M \left( \sum_{k=1}^{n} \sum_{j=1}^{m_k} |a_{kj}|^2 \right)^{1/2} \|x\|.
\]
Our assumption implies that each $T_n$ is $R$-bounded with constant $KM$. Let $x_1, \ldots, x_n \in X$ and write $x_k = \|x_k\| u_k$, where $\|u_k\| = 1$. Choose $u^*_k \in X^*$ such that $u^*_k(u_k) = 1$ and $\|u^*_k\| = 1$. Now we have
\[
\left( \sum_{k=1}^{n} \|x_k\|^2 \right)^{1/2} \leq M \left( \mathbb{E} \left[ \left\| \sum_{k=1}^{n} e_k \|x_k\| e_k \right\|^2 \right] \right)^{1/2}
\leq M \left( \mathbb{E} \left[ \left\| \sum_{k=1}^{n} e_k \|x_k\| u^*_k(a_k)e_k \right\|^2 \right] \right)^{1/2}
\leq M \left( \mathbb{E} \left[ \left\| \sum_{k=1}^{n} e_k \|x_k\| (u^*_k \otimes e_k)(u_k) \right\|^2 \right] \right)^{1/2}
\leq KM^2 \left( \mathbb{E} \left[ \left\| \sum_{k=1}^{n} e_k \|x_k\| u_k \right\|^2 \right] \right)^{1/2} = KM^2 \left( \mathbb{E} \left[ \sum_{k=1}^{n} e_k x_k \right]^2 \right)^{1/2}.
\]
This shows that $X$ has cotype 2.

Let us assume that $X$ has non-trivial type. Then by results of Pisier [19] and also by Figiel and Tomczak-Jaegermann [12], $E^d$ is uniformly complemented in $X$. Thus, for some constant $C$, for every $n \in \mathbb{N}$ we can choose a biorthogonal system $\{(e_k, e_k') : k = 1, \ldots, n\}$ in $X \times X^*$ such that
\[
\left\| \sum_{k=1}^{n} a_k e_k \right\| \leq C \left( \sum_{k=1}^{n} |a_k|^2 \right)^{1/2}
\]
and
\[
\left\| \sum_{k=1}^{n} a_k e_k^* \right\| \leq C \left( \sum_{k=1}^{n} |a_k|^2 \right)^{1/2}.
\]
for all \(a_1, \ldots, a_n \in \mathbb{C}\). Note that for any \(x \in X\) and \(a_1, \ldots, a_n \in \mathbb{C}\),
\[
\left| \sum_{k=1}^{n} a_k e_k^*(x) \right| \leq C \left( \sum_{k=1}^{n} |a_k|^2 \right)^{1/2} \|x\|
\]
and so
\[
\left( \sum_{k=1}^{n} |e_k^*(x)|^2 \right)^{1/2} \leq C \|x\|.
\]
Consider the family of operators \(T_n = \{e_k^* \otimes u: \|u\| = 1, \ k = 1, \ldots, n\}\). Let \(x \in X\). Then for any \(a_1, \ldots, a_n \in \mathbb{C}\) and every \(u_1, \ldots, u_n \in X\) of norm one we have
\[
E \left| \sum_{k=1}^{n} a_k e_k^* (u_k \otimes e_k) (x) \right| = E \left| \sum_{k=1}^{n} a_k e_k^* (x) u_k \right| \leq \left( \sum_{k=1}^{n} |a_k|^2 \right)^{1/2} \left( \sum_{k=1}^{n} |e_k^*(x)|^2 \right)^{1/2} \leq C \left( \sum_{k=1}^{n} |a_k|^2 \right)^{1/2} \|x\|.
\]
We conclude that \(T_n\) is semi-R-bounded with constant \(C\) and hence \(T_n\) is R-bounded for constant \(KC\) independent of \(n\). This implies that \(X\) has type 2 as follows. Choose any \(x_1, \ldots, x_n \in X\) and write \(x_k = \|x_k\| u_k\), where \(\|u_k\| = 1\). Then
\[
\left( E \left| \sum_{k=1}^{n} e_k x_k \right|^2 \right)^{1/2} \leq KC \left( \sum_{k=1}^{n} \|x_k\|^2 \right)^{1/2} . \tag{2.1}
\]
Now, \(X\) has type 2 and cotype 2 and is therefore isomorphic to \(\ell_2\) by Kwapień’s theorem [25].

Now suppose on the contrary that \(X\) has trivial type. We will show that \(X\) is a GT-space, i.e., any \(T: X \to \ell_2\) is 1-summing. Fix \(T: X \to \ell_2\) of norm one. Since \(X\) has cotype 2 we can equivalently show that any such \(T\) is 2-summing [11]. It suffices to check that for any \(n \in \mathbb{N}\) and operator \(S: \ell_2^n \to X\) such that \(\|S\| \leq 1\) we have \(\pi_2(TS) \leq C\), where \(C\) does not depend on \(n\) [25]. One can assume that \(TS: \ell_2^n \to \ell_2^{n}\) and that \(TS\) is diagonal with respect to the canonical orthonormal basis \((e_k)\) in \(\ell_2^n\), i.e., \(TSe_k = \lambda_k e_k\) for some \(\lambda_1, \ldots, \lambda_n\). Then it suffices to show uniform boundedness of the Hilbert–Schmidt norms \(\|TS\|_{HS} = (\sum_{k=1}^{n} \|TSe_k\|^2)^{1/2}\). Write \(f_k^* = T^* e_k^* \in X^*\) and \(f_k = S e_k \in X\). Consider \(\{f_k^* \otimes u: k = 1, \ldots, n, \|u\| = 1\}\). We will show that this family is semi-R-bounded.
with constant one. Take \( u_1, \ldots, u_n \in X \) of norm one and \( a_1, \ldots, a_n \in \mathbb{C} \). Then for \( x \in X \) we have
\[
\mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k a_k f_k^*(x) u_k \right\| \leq \left( \sum_{k=1}^{n} |a_k|^2 \right)^{1/2} \left( \sum_{k=1}^{n} |\epsilon_k|^2 |T x_k|^2 \right)^{1/2} \leq \left( \sum_{k=1}^{n} |a_k|^2 \right)^{1/2} \left\| T x \right\| \leq \left( \sum_{k=1}^{n} |a_k|^2 \right)^{1/2} \left\| x \right\|.
\]

Therefore, \( \{ f_k^* \otimes u; k = 1, \ldots, n, \|u\| = 1 \} \) is R-bounded with constant \( K \).

Since \( X \) has trivial type, it contains \( \ell^n_1 \) uniformly [19]. Hence, for fixed \( M > 1 \) and every \( n \in \mathbb{N} \) there are \( y_1, \ldots, y_n \in X \) with \( \|y_k\| = 1 \) for \( 1 \leq k \leq n \) such that
\[
\sum_{k=1}^{n} |a_k| \leq M \left\| \sum_{k=1}^{n} a_k y_k \right\|. \tag{2.2}
\]

Choose any scalars \( b_1, \ldots, b_n \). Now using R-boundedness and Kahane’s inequality for \( p = 1 \) with constant \( A \) we have
\[
\sum_{k=1}^{n} |b_k| \|\lambda_k\| = \sum_{k=1}^{n} |b_k| \|f_k^*(f_k)\| \leq M \left( \mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k b_k f_k^*(f_k) y_k \right\| \right)^{1/2} \leq KM \left( \mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k b_k f_k \right\| \right)^{1/2} \leq KM \left( \mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k b_k e_k \right\| \right)^{1/2} \leq KM \left( \sum_{k=1}^{n} |b_k|^2 \right)^{1/2}.
\]

Thus,
\[
\left( \sum_{k=1}^{n} |\lambda_k|^2 \right)^{1/2} \leq KM,
\]
and so \( \|TS\|_{\text{HS}} \leq KM \). Therefore, any operator \( T : X \to \ell_2 \) is 2-summing. This completes the proof of (i) implies (ii).

Now we will show that (ii) implies (i). Suppose that \( X \) is a GT-space of cotype 2, and that \( T \) is a family of semi-R-bounded operators. We will show that \( T \) is R-bounded. Since \( X \) is separable, there is a quotient map \( Q : \ell_1 \to X \). First, we show that any semi-R-bounded family of operators from \( \ell_1 \) into \( X \) is already R-bounded. Let \( S \) be such a family with semi-R-boundedness constant one. Suppose \( S_1, \ldots, S_n \in S \) and \( x_1, \ldots, x_n \in \ell_1 \). Then \( x_k = \sum_{j=1}^{\infty} \xi_{jk} e_j \), where \( (e_j) \) is the canonical basis of \( \ell_1 \).

Let us denote by \( C \) the constant in the Kahane–Khintchine inequality for any Banach space:
\[
\left( \mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k x_k \right\| \right)^{2} \leq CE \left\| \sum_{k=1}^{n} \epsilon_k x_k \right\|.
\]
Thus
\[
\left( \mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k S_k x_k \right\|^2 \right)^{1/2} \leq C \mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k S_k x_k \right\|.
\]

Then
\[
\mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k S_k x_k \right\| = \mathbb{E} \left\| \sum_{k=1}^{n} \sum_{j=1}^{\infty} \epsilon_k \xi_{jk} e_j \right\| \leq \sum_{k=1}^{\infty} \mathbb{E} \left\| \sum_{j=1}^{\infty} \epsilon_k \xi_{jk} S_k e_j \right\| \leq \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} |\xi_{jk}|^2 \right)^{1/2}.
\]

Combining and using the Khintchine inequality again we obtain
\[
\left( \mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k S_k x_k \right\|^2 \right)^{1/2} \leq C^2 \sum_{j=1}^{\infty} \mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k \xi_{jk} e_j \right\| = C^2 \mathbb{E} \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{n} \epsilon_k \xi_{jk} e_j \right\| = C^2 \mathbb{E} \left\| \sum_{k=1}^{\infty} \sum_{j=1}^{n} \epsilon_k \xi_{jk} e_j \right\| = C^2 \mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k x_k \right\|.
\]

Combining the previous two computations gives that \( S \) is R-bounded.

Now let \( T \) be a family of operators on \( X \) with semi-boundedness constant one. Let \( Q : \ell_1 \rightarrow X \) be a quotient map and note that the family \( S = \{ TQ : T \in T \} \) is R-bounded with some constant \( B \) by the above calculation.

We will apply a characterization of GT-spaces of cotype 2 due to Pisier [19].

**Proposition 2.3** (Pisier). \( X \) is a GT-space of cotype 2 if and only if there is a constant \( C > 0 \) such that for any \( n \in \mathbb{N} \), \( x_1, \ldots, x_n \in X \), there are \( y_1, \ldots, y_n \in \ell_1 \) such that \( Q y_k = x_k \), \( k = 1, \ldots, n \), and
\[
\mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k y_k \right\| \leq C \mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k x_k \right\|. \tag{2.3}
\]

Now take \( n \in \mathbb{N} \), \( T_1, \ldots, T_n \in T \) and \( x_1, \ldots, x_n \in X \). Choose \( y_1, \ldots, y_n \in \ell_1 \) according to Proposition 2.3. Then
\[
\mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k T_k x_k \right\| \leq \mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k T_k Q y_k \right\| \leq B \mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k y_k \right\| \leq C B \mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k x_k \right\|.
\]

Thus, \( T \) is R-bounded. The proof is complete. \( \Box \)

For a set \( T \) of bounded linear operators we will use the notation \( T^* = \{ T^* : T \in T \} \).
Lemma 2.4.

(i) If $T$ is $R$-bounded then $T^{**}$ is $R$-bounded (with the same constant).

(ii) If $T$ is WR-bounded then $T^*$ and $T^{**}$ are WR-bounded (with the same constant).

(iii) If $T$ is semi-R-bounded then $T^{**}$ is semi-R-bounded (with the same constant).

Proof. The proofs of (i) and (iii) are similar. For (i) suppose $T_1, \ldots, T_n \in T$ and that $T$ has $R$-boundedness constant one. Let $\Omega = \{-1, 1\}^n$ with $\mathbb{P}$ normalized counting measure on $\Omega$. Let $\epsilon_k$ be the sequence of coordinate maps on $\Omega$. Let $Rad(\Omega; X)$ be the subspace of $L_2(\Omega, \mathbb{P}; X)$ generated by the functions $\epsilon_k \otimes x$ for $1 \leq k \leq n$ and $x \in X$ (this space is isomorphic to $X^n$). Then $Rad(\Omega; X^{**})$ can be identified naturally with a subspace of $Rad(\Omega; X)^{**}$. Consider the map $T: Rad(\Omega; X) \to Rad(\Omega; X)$ defined by

$$T \left( \sum_{k=1}^n \epsilon_k \otimes x_k \right) = \sum_{k=1}^n \epsilon_k \otimes T_k x_k.$$

Then $\|T\| \leq 1$ and so $\|T^{**}\| \leq 1$ and (i) follows.

Let us now prove (ii). Suppose $T$ is WR-bounded with constant one and $T_1, \ldots, T_n \in T$. Suppose $x_1^*, \ldots, x_n^* \in X^*$ are such that

$$\left( \mathbb{E} \left\| \sum_{k=1}^n \epsilon_k x_k^* \right\|^2 \right)^{1/2} \leq 1.$$

Then, using the identification of $Rad(\Omega, X^{**})$ as the bidual of $Rad(\Omega, X)$ we observe that the set of functions of the form $\sum_{k=1}^n \epsilon_k x_k^*$ in $Rad(\Omega, X^{**})$ such that

$$\sum_{k=1}^n \left| \langle T_k^* x_k^*, x_k^* \rangle \right| \leq 1$$

is weak*-closed and contains the unit ball of $Rad(\Omega, X)$. By Goldstine’s theorem it contains the unit ball of $Rad(\Omega, X^{**})$ and this implies that $T^*$ is WR-bounded with constant one.

Now it is time to give an example of a family of operators that is uniformly bounded but not WR-bounded. The previous lemma will imply that the corresponding dual family is semi-R-bounded but not WR-bounded.

Example. Let $X = \ell_p$, $1 \leq p < 2$. Pick any non-zero element $x \in X$ and choose $u^* \in X^*$ of norm one such that $u^*(x) \neq 0$. Define $T_k = u^* \otimes e_k$, where $(e_k)$ is the canonical basis of $X$. The family $\{T_k\}$ is uniformly bounded, $\|T_k\| = 1$, but we will show that it is not WR-bounded. Consider the dual basis $(e_k^*)$ in $(\ell_p)^*$. Then

$$\sum_{k=1}^n \left| \langle T_k x, e_k^* \rangle \right| = \sum_{k=1}^n \left| \langle u^*(x) e_k, e_k^* \rangle \right| = n |u^*(x)|. \quad (2.4)$$
On the other hand, we have
\[
\left( \mathbb{E} \left| \sum_{k=1}^{n} \epsilon_k x_k \right|^2 \right)^{1/2} \left( \mathbb{E} \left| \sum_{k=1}^{n} \epsilon_k^* e_k^* \right|^2 \right)^{1/2} = \|x\| n^{1/2} n^{1/q}.
\] (2.5)

Here \( q \) satisfies \( 1/p + 1/q = 1 \). If \( p < 2 \) then \( q > 2 \) and \( 1/2 + 1/q < 1 \), so for \( 1 \leq p < 2 \) the family \( \{T_k\} \) cannot be WR-bounded.

We have \( T_k^* = e_k^* \otimes u^* \) on \( X^* = \ell_q \), where \( 2 < q \leq \infty \). Consider \( q \neq \infty \). Since by reflexivity \( T_k^* = T_k \) and using Lemma 2.4 we see that \( \{T_k^*\} \) is not WR-bounded. However, \( X^* \) has type 2 and hence \( \{T_k^*\} \) is semi-R-bounded by Proposition 2.1.

3. The main results

Suppose \( X \) is any Banach space. We shall say that a sequence \( T = (T_k)_{k=1}^{\infty} \) is an approximating sequence if \( \lim_{k \to \infty} \|x - T_k x\| = 0 \) for every \( x \in X \). We will say that \( T \) is compact (relatively, weakly compact) if each \( T_k \) is compact (relatively, weakly compact).

We will say that \( T \) is commuting if we have \( T_k T_l = T_l T_k \) for \( l, k \in \mathbb{N} \).

If \( T \) is a commuting approximating sequence, let us define the subspace \( E_T \) of \( X^* \) to be the closed linear span of \( \bigcup_{k} T_k^* (X^*) \). The following lemma is trivial.

Lemma 3.1. If \( T \) is a commuting approximating sequence then \( E_T \) is a norming subspace of \( X^* \), i.e., for some \( C \) we have
\[
\|x\| \leq C \sup_{x^* \in B_{E_T}} |x^*(x)|, \quad x \in X,
\]
and, if \( T \) is weakly compact, \( \lim_{n \to \infty} T_n^* x^* = x^* \) weakly for \( x^* \in E_T \).

Let us recall that a Banach space \( X \) has property (V) of Pełczyński if every unconditionally converging operator \( T : X \to Y \) is weakly compact. The spaces \( C(K) \) have property (V) [17] and more generally any \( C^* \)-algebra has property (V) [18]. The disk algebra \( A(\mathbb{D}) \) also has property (V) [10,14]; see also [23]. We also recall that a Banach space \( X \) is said to have property \((V^*)\) if whenever \( (x_n) \) is a bounded sequence in \( X \) then either

(i) \( (x_n) \) has a subsequence which is weakly Cauchy or
(ii) \( (x_n) \) has a subsequence \( (y_n) \) such that for some sequence \( (\gamma_n^*) \) in \( X^* \) and \( \delta > 0 \) we have \( |\gamma_n^*(y_n)| \geq \delta \) and
\[
\left| \sum_{k=1}^{n} a_k y_k^* \right| \leq \max_{1 \leq k \leq n} |a_k|, \quad a_1, \ldots, a_n \in \mathbb{C}, \quad n \in \mathbb{N}.
\]

Property \((V^*)\) was introduced by Pełczyński [17]. We note that Bombal [4] shows that every Banach lattice not containing \( c_0 \) has property \((V^*)\). Any subspace of a space with property \((V^*)\) also has property \((V^*)\).
Lemma 3.2. Let $X, Y$ be Banach spaces and let $T = (T_k)_{k=1}^\infty$ be any sequence of operators in $\mathcal{L}(X, Y)$. Suppose either

(i) $T$ is semi-R-bounded or
(ii) $T$ is WR-bounded and $Y$ has property $(V^*)$.

Then for every $x \in X$ the sequence $(T_kx)_{k=1}^\infty$ has a weakly Cauchy subsequence.

Proof. If not, by passing to a subsequence we can suppose $(T_kx)_{k=1}^\infty$ is equivalent to the canonical $\ell_1$-basis [21,22]. If $T$ is semi-R-bounded we observe that for some $C$ we have

$$E\left\|\sum_{k=1}^n \epsilon_k a_k T_k x\right\| \leq C \left(\sum_{k=1}^n |a_k|^2\right)^{1/2} \|x\|, \quad a_1, \ldots, a_n \in \mathbb{C}, \quad n \in \mathbb{N}.$$ 

This gives a contradiction.

In case (ii), we can pass to a subsequence and assume the existence of $y_n^* \in Y^*$ such that

$$\left\|\sum_{k=1}^n a_k y_k^*(T_k x)\right\| \leq \max_{1 \leq k \leq n} |a_k|, \quad a_1, \ldots, a_n \in \mathbb{C}, \quad n \in \mathbb{N},$$

and $|y_n^*(T_n x)| \geq \delta > 0$ for all $n$. Then

$$n \delta \leq \sum_{k=1}^n |y_k^*(T_k x)| \leq C \left(E\left\|\sum_{k=1}^n \epsilon_k x\right\|\right)^{1/2} \left(E\left\|\sum_{k=1}^n \epsilon_k y_k^*\right\|\right)^{1/2} \leq C \sqrt{n}.$$

This also yields a contradiction. \qeda

Theorem 3.3. Let $X$ be a Banach space with a commuting weakly compact approximating sequence $T$. Suppose either that

(i) $T$ is semi-R-bounded and $X$ is weakly sequentially complete or
(ii) $T$ is WR-bounded and $X$ has property $(V^*)$.

Then $X$ is isomorphic to a dual space.

Proof. In either case we consider the family $T^{**} \subset \mathcal{L}(X^{**}, X)$. By Lemma 3.2 for each $x^{**} \in X^{**}$ we can find a subsequence $T_{k_n}^{**}x^{**}$ so that $T_{k_n}^{**}(x^{**})$ is weakly convergent to some $y \in X$. Then for $x^{*} \in X^*$,

$$x^{*}(T_k y) = \lim_{n \to \infty} x^{*}\left(T_{k_n} T_{k_n}^{**} x^{**}\right) = \lim_{n \to \infty} x^{*}(T_{k_n} T_{k_n}^{**} x^{**})$$

so that $T_k y = T_{k_n}^{**} x^{**}$. Hence $\lim_{n \to \infty} \|y - T_{k_n}^{**} x^{**}\| = 0.$

We now show that $E_T'$ can be identified with $X$. Clearly $X$ canonically embeds in $E_T'$ since $E_T$ is norming. If $f^* \in E_T'$ then by the Hahn–Banach theorem there exists $x^{**} \in X^{**}$ with $\|x^{**}\| = \|f^*\|$ and $x^{**}(x^*) = f^*(x^*)$ for $x^* \in E_T$. Let $y = \lim_{k \to \infty} T_k x^{**}$. Then for $x^* \in E_T$,
\[ x^*(y) = \lim_{k \to \infty} x^*(T^*_k x^*) = \lim_{k \to \infty} x^{**}(T_k^{**} x^{**}) = f^*(x^*). \]

Hence \( E_T = X. \)

**Theorem 3.4.** The space \( L_1(0, 1) \) does not have a commuting weakly compact approximating sequence which is either semi-R-bounded or WR-bounded.

**Proof.** \( L_1 \) is not a dual space [25]. □

Of course a semi-R-bounded sequence in \( L_1 \) is actually R-bounded.

**Theorem 3.5.** Let \( X \) be a separable Banach space with property (V). If \( X \) has a commuting weakly compact approximating sequence \( (T_n)_{n=1}^\infty \) which is WR-bounded, then \( X^* \) is separable, and has a WR-bounded commuting weakly compact approximating sequence.

**Proof.** Since \( X \) has (V), it follows that \( X^* \) has property (V*). We show that \( \lim_{n \to \infty} T_n x^* = x^* \) weakly for \( x^* \in X^* \). Indeed \( T_n x^* \) converges weakly to \( x^* \) and it must have a weakly convergent subsequence by Lemma 3.2. Hence \( x^* \in E_T \) so \( X = E_T \). Now \( T_n (B_{X^*}) \) is weakly compact by Gantmacher’s theorem also and weak*-metrizable, hence norm separable. Thus \( X^* \) is separable, and so by Mazur’s theorem, and a diagonal argument, we can find a sequence of convex combinations \( (S_n^*)_{n=1}^\infty \) of \( (T_n^*)_{n=1}^\infty \) which is an approximating sequence. □

**Corollary 3.6.** If \( K \) is an uncountable compact metric space then \( C(K) \) has no WR-bounded commuting weakly compact approximating sequence. The disk algebra has no WR-bounded weakly compact approximating sequence.

We now consider \( C(K) \) when \( K \) is countable. In this case \( C(K) \) is homeomorphic to a space \( C(\alpha) = C([1, \alpha]) \), where \( \alpha \) is a countable ordinal. There is a characterization of such \( C(K) \) due to Bessaga and Pełczyński [3].

**Theorem 3.7** (Bessaga–Pełczyński). If \( \alpha < \beta \), \( C(\omega^\alpha \cdot k) \) is isomorphic to \( C(\omega^\beta \cdot n) \) if and only if \( \beta < \alpha \cdot \omega \). Consequently, \( C(\omega^{\omega^\gamma}) \), \( 0 \leq \gamma < \omega_1 \), is a complete list of representatives of the isomorphism classes of \( C(K) \) for \( K \) a countable compact metric space.

The following lemma can be obtained as an applications of \( \ell_1 \)-indices [1,5,6]. However, for convenience of the reader we will give a direct proof by construction.

**Lemma 3.8.** Let \( \alpha \) be a countable ordinal with \( \alpha \geq \omega^\rho \). Then there exists \( f \in C(\alpha)^{**} \) so that whenever \( f_n \in C(\alpha) \) converges to \( f \in C(\alpha)^{**} \) weak* then for any \( m \in \mathbb{N} \) there exist \( n_1, \ldots, n_m \in \mathbb{N} \) such that

\[
\left| \sum_{k=1}^{m} \epsilon_k f_{n_k} \right| \geq \frac{1}{2} m, \quad \epsilon_k = \pm 1, \quad k = 1, 2, \ldots, m.
\]
Proof. In this case $C(\alpha)^*$ can be identified with $\ell_\infty(\alpha)$. It is easy to see that it suffices to consider the case $\alpha = \omega^\alpha$.

Consider $f \in X^{**}$ defined by $f(\sum_{k=0}^{N}o^{k}l_{k}) = (-1)^{\sum_{k=0}^{N}l_{k}}$ and $f(\omega^{\alpha}) = 1$. Writing $K$ for the space $[1, \omega^{\alpha}]$ let $K^{(p)}$ denote the $p$th derived set of $K$. Then $K^{(p)}$ consists of all ordinals of the form $\sum_{k=p}^{\infty}o^{k}l_{k}$ together with $\omega^{\alpha}$. For each $p \in \mathbb{N}$, $K^{(p)}$ is non-empty.

Furthermore for each $\alpha \in K^{(p)}$ and every open neighborhood $V$ of $\alpha$ we have that $f$ takes both values $\pm 1$ on $V \cap K^{(p-1)}$.

Let $f_{n} \in C(K)$ be any sequence such that $(f_{n})$ converges to $f$ weak$^*$. Fix $0 < \delta < 1/2$ and $m \in \mathbb{N}$. We construct $(f_{n_{1}}, \ldots, f_{n_{m}})$ inductively. We start from $K^{(m)}$. By definition of $f$ we can pick $\alpha_{1}^{1}, \alpha_{1}^{2} \in K^{(m)}$ such that $f(\alpha_{1}^{j}) = (-1)^{j}$ for $j = 1, 2$. Then find $n_{1} \in \mathbb{N}$ such that $|f_{n_{1}}(\alpha_{1}^{j}) - (-1)^{j}| < \delta$. Since $f_{n_{1}}$ is continuous we can choose open neighborhoods $U_{j}^{1}$ of $\alpha_{1}^{j}$ such that $|f_{n_{1}}(\alpha) - (-1)^{j}| < \delta$ for all $\alpha \in U_{j}^{1}$.

For the inductive step, suppose that $(n_{j}^{1})_{j=1}^{k}, (\alpha_{j}^{k})_{j=1}^{2^{k}}$ and open sets $(U_{j}^{k})_{j=1}^{2^{k}}$ have been chosen so that $\alpha_{j}^{k} \in U_{j}^{k}$. Then for $i = 1, \ldots, 2^{k}$ find points $\alpha_{2i-1}^{k+1}, \alpha_{2i}^{k+1} \in U_{i}^{k} \cap K^{(m-k+1)}$ with $f(\alpha_{2i}^{k+1}) = (-1)^{j}$. By pointwise convergence, we can select $n_{k+1} > n_{k}$ such that $|f_{n_{k+1}}(\alpha_{2i}^{k+1}) - (-1)^{j}| < \delta$. Since $f_{n_{k+1}}$ is continuous, there are neighborhoods $U_{2i-1}^{k+1}, U_{2i}^{k+1} \subset U_{i}^{k}$, $i = 1, \ldots, 2^{k}$, such that for all $\alpha \in U_{j}^{k+1}$ we have $|f_{n_{k+1}}(\alpha) - (-1)^{j}| < \delta$.

In the $m$th iteration this will give $2^{m}$ neighborhoods and $m$ functions $f_{n_{1}}, \ldots, f_{n_{m}}$ so that for any $\epsilon_{1}, \ldots, \epsilon_{m} \in [-1, +1]$ there is $\alpha$ contained in one of these neighborhoods such that $|f_{k}(\alpha) - \epsilon_{k}| < \delta$ for all $k = 1, \ldots, m$. Hence

$$\left| \sum_{k=1}^{m} \epsilon_{k} f_{n_{k}} \right| \geq (1 - \delta)m.$$ 

□

Theorem 3.9. Let $K$ be a compact metric space. Suppose there is an $R$-bounded commuting weakly compact approximating sequence in $C(K)$. Then $C(K)$ is isomorphic to $c_{0}$.

Proof. By Corollary 3.6 we need only consider the case when $K$ is countable. By Theorem 3.7 it suffices to consider the case when $K = [1, \omega]$, where $\alpha \geq \omega^{\alpha}$. Pick $f \in C(K)^{**}$ satisfying the hypotheses of Lemma 3.8.

Suppose $(T_{n})$ is an $R$-bounded weakly compact approximating sequence for $C(K)$. Then $(T_{n}^{*})$ is an approximating sequence for $C(K)^{*}$ by Theorem 3.5 and hence $T_{n}^{*} f$ converges to $f$ weak$^*$. It follows that for any $m$ we can choose $n_{1}, \ldots, n_{m}$ so that

$$\left| \sum_{k=1}^{m} \epsilon_{k} T_{n_{k}}^{*} f \right| \geq \frac{1}{2} m, \quad \epsilon_{k} = \pm 1.$$ 

Hence

$$\left( \mathbb{E} \left| \sum_{k=1}^{m} \epsilon_{k} T_{n_{k}}^{*} f \right|^{2} \right)^{1/2} \geq \frac{1}{2} m.$$ 

This contradicts the fact that $T_{n}$ is $R$-bounded (or even semi-$R$-bounded). □
Remark. We can replace the assumption of R-boundedness by the assumption that \((T_n)\) and \((T^*_n)\) are both semi-R-bounded. By Theorem 2.2 this hypothesis would imply that \((T^*_n)\) is actually R-bounded and hence that \((T_n)\) is WR-bounded. We only used the fact that \((T_n)\) is both semi-R-bounded and WR-bounded.

Let us conclude by stating our main result with respect to semigroups. (Actually our results are somewhat stronger than stated below.)

**Theorem 3.10.** Let \(X\) be a separable Banach space with an R-bounded strongly continuous semigroup \((T_t)_{t>0}\) consisting of weakly compact operators. Then if

1. \(X = L_1(\mu)\) for some measure \(\mu\) then \(X\) is isomorphic to \(\ell_1\) (i.e., \(\mu\) is purely atomic).
2. \(X = C(\mathbb{K})\) then \(X\) is isomorphic to \(c_0\).

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**References**