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R-bounded approximating sequences and applications to semigroups ☆

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Abstract

It is shown that on certain Banach spaces, including $C[0,1]$ and $L_1[0,1]$, there is no strongly continuous semigroup $(T_t)_{0<t<1}$ consisting of weakly compact operators such that $(T_t)_{0<t<1}$ is an R-bounded family. More general results concerning approximating sequences are included and some variants of R-boundedness are also discussed.

Keywords: Strongly continuous semigroups; R-boundedness; Approximation property

1. Introduction

Recent work on semigroup theory [13,24] has highlighted the importance of the concept of R-boundedness. Let us recall the definition of R-bounded families of operators (cf. [2, 7,9]).

Definition 1.1. A family $\mathcal{T}$ of operators in $\mathcal{L}(X,Y)$ is called *R-bounded* with R-boundedness constant $C>0$ if letting $(\epsilon_k)_{k=1}^{\infty}$ be a sequence of independent Rademachers on some probability space then for every $x_1,\dots,x_n \in X$ and $T_1,\dots,T_n \in \mathcal{T}$ we have

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$$\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k T_k x_k\right\|^2\right)^{1/2} \leqslant C\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|^2\right)^{1/2}. \tag{1.1}$$

By the Kahane–Khintchine inequality we can replace 2 above by any other exponent $1 \leqslant p < \infty$ to obtain an equivalent definition. We will also need the following definition introduced in [13].

Definition 1.2. A family $\mathcal{T}$ of operators in $\mathcal{L}(X,Y)$ is called *WR-bounded* with WR-boundedness constant $C > 0$ if for every $x_1, \dots, x_n \in X$, $y_1^*, \dots, y_n^* \in Y^*$ and $T_1, \dots, T_n \in \mathcal{T}$ we have

$$\sum_{k=1}^{n}\left|\left\langle T_k x_k, y_k^*\right\rangle\right| \leqslant C\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|^2\right)^{1/2}\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k y_k^*\right\|^2\right)^{1/2}. \tag{1.2}$$

It is clear by the Cauchy–Schwarz inequality that R-boundedness implies WR-boundedness. The converse is not true in general, but it holds for spaces with non-trivial type [13, 20].

In [13] it was shown that no reasonable differential operator on L_1 can have an H^∞-calculus. In this note we consider the related question whether a differential-type operator on L_1 can generate an R-bounded semigroup. Note that if A is an R-sectorial operator (cf. [13]) with R-sectoriality angle less than $\pi/2$ then the semigroup $(e^{-tA})_{0<t<1}$ is necessarily R-bounded. In general, one expects a semigroup generated by a differential operator on a bounded domain to consist of weakly compact operators. We are thus led to consider the question whether one can have a strongly continuous semigroup $(T_t)_{0<t<1}$ on L_1 such that each T_t is weakly compact (or equivalently compact, since L_1 has the Dunford–Pettis property) and such that the family $(T_t)_{0<t<1}$ is R-bounded. In fact this leads to considering versions of the approximation property; the only property of the semigroup needed is commutativity. We consider the general question whether on a given separable Banach space one can find an R-bounded sequence $(T_n)_{n\in\mathbb{N}}$ of commuting weakly compact operators such that $\lim_{n\to\infty} T_n x = x$ for all $x \in X$. Our main results show that for the spaces $L_1[0,1]$, $C(K)$ (except c_0) and the disk algebra $A(\mathbb{D})$ this is impossible. These results may be regarded as extensions of classical results that the spaces $L_1, C(K)$ do not have unconditional bases [15].

In the case of L_1 we are led to consider a natural weakening of R-boundedness, where we use the definition (1.1) but only for single vectors.

Definition 1.3. A family $\mathcal{T}$ of operators in $\mathcal{L}(X,Y)$ is called *semi-R-bounded* if there is a constant $C > 0$ such that for every $x \in X$, $a_1, \dots, a_n \in \mathbb{C}$ and $T_1, \dots, T_n \in \mathcal{T}$ we have

$$\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k a_k T_k x\right\|^2\right)^{1/2} \leqslant C\left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2}\|x\|. \tag{1.3}$$

We note that semi-R-boundedness is equivalent to R-boundedness for operators on L_1. In Theorem 2.2 we actually characterize all spaces where semi-R-boundedness is equivalent to R-boundedness as spaces which are either Hilbert spaces or GT-spaces of cotype 2 in the terminology of Pisier [19].

2. R-boundedness and WR-boundedness

In this section, we make some remarks about R-boundedness and related notions.

Note that in a space of type 2, any uniformly bounded collection $\mathcal{T} \subset L(X, X)$ is semi-R-bounded. The converse is also true:

Proposition 2.1. *A Banach space X has type 2 if and only if uniform boundedness is equivalent to semi-R-boundedness.*

Proof. Suppose that every uniformly bounded family of operators is already semi-R-bounded. Pick any $x \in X$ and $x^* \in X^*$ such that $\|x\| = \|x^*\| = 1$ and $x^*(x) = 1$. Notice that the family $\mathcal{T} = \{x^* \otimes u\colon \|u\| = 1\}$ is uniformly bounded with constant one and hence semi-R-bounded by assumption. Let C be the semi-R-boundedness constant of $\mathcal{T}$. Select any $x_1, \dots, x_n \in X$ and write $x_k = \|x_k\| u_k$, where $\|u_k\| = 1$. Then $\{x^* \otimes u_k\colon k = 1, \dots, n\} \subset \mathcal{T}$ and

$$\begin{aligned}
\left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k x_k\right\|^2\right)^{1/2} &= \left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k \|x_k\| (x^* \otimes u_k) x\right\|^2\right)^{1/2} \\
&\leqslant C\left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k \|x_k\| x\right\|^2\right)^{1/2} = C\|x\|\left(\sum_{k=1}^{n} \|x_k\|^2\right)^{1/2} \\
&= C\left(\sum_{k=1}^{n} \|x_k\|^2\right)^{1/2}.
\end{aligned}$$

Thus, X has type 2. □

For some spaces, semi-R-boundedness is equivalent to R-boundedness and we are able to completely characterize these spaces in the next theorem. Let us recall that a Banach space X is called a *GT-space* if every bounded operator $T : X \to \ell_2$ is absolutely summing. Examples of GT-spaces of cotype 2 are L_1, the quotient of L_1 by a reflexive subspace [14,19], and L_1/H_1 [8]. It is unknown whether every GT-space has cotype 2.

Theorem 2.2. *Suppose X is separable. Then the following are equivalent*:

(i) *Every semi-R-bounded family of operators on X is R-bounded.*
(ii) *X is isomorphic to ℓ_2 or X is a GT-space of cotype* 2.

Proof. First we prove that (i) implies (ii). Suppose that every semi-R-bounded family of operators on X is R-bounded. Let us note that this implies the existence of a constant K so that if $\mathcal{T}$ has semi-R-boundedness constant C then it has R-boundedness constant KC; for otherwise we could find a sequence $\mathcal{T}_n$ of families with semi-R-boundedness constant one and R-boundedness constant at least 4^n; then the family $\bigcup_{n \geqslant 1} 2^{-n} \mathcal{T}_n$ contradicts our assumption. Fix $M > 1$ and take $x \in X$. Choose $n \in \mathbb{N}$. By Dvoretzky's theorem [16] we can find $e_1, \dots, e_n \in X$ such that for any $a_1, \dots, a_n \in \mathbb{C}$ we have

$$M^{-1}\left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2} \leqslant \left\|\sum_{k=1}^{n} a_k e_k\right\| \leqslant M\left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2}.$$

Consider the family of operators $\mathcal{T}_n = \{u^* \otimes e_k\colon\ \|u^*\| = 1,\ k = 1, \dots, n\}$. Then each $\mathcal{T}_n$ is semi-R-bounded with constant M as follows. A finite subfamily of $\mathcal{T}_n$ is of the form $\{u_{kj}^* \oplus e_k\colon\ 1 \leqslant k \leqslant n,\ 1 \leqslant j \leqslant m_k\}$ for some $m_1, \dots, m_n \in \mathbb{N}$. Then for every $a_{11}, \dots, a_{nm_n} \in \mathbb{C}$ we have (letting ϵ_{kj} denote independent Rademachers)

$$\begin{aligned}\left(\mathbb{E}\left\|\sum_{k=1}^{n}\sum_{j=1}^{m_k}\epsilon_{kj}a_{kj}u_{kj}^*(x)e_k\right\|^2\right)^{1/2} &\leqslant M\left(\mathbb{E}\sum_{k=1}^{n}\left|\sum_{j=1}^{m_k}\epsilon_{kj}u_{kj}^*(x)a_{kj}\right|^2\right)^{1/2}\\ \leqslant M\left(\sum_{k=1}^{n}\mathbb{E}\left|\sum_{j=1}^{m_k}\epsilon_{kj}u_{kj}^*(x)a_{kj}\right|^2\right)^{1/2} &\leqslant M\left(\sum_{k=1}^{n}\sum_{j=1}^{m_k}|a_{kj}|^2\right)^{1/2}\|x\|.\end{aligned}$$

Our assumption implies that each $\mathcal{T}_n$ is R-bounded with constant KM. Let $x_1, \dots, x_n \in X$ and write $x_k = \|x_k\|u_k$, where $\|u_k\| = 1$. Choose $u_k^* \in X^*$ such that $u_k^*(u_k) = 1$ and $\|u_k^*\| = 1$. Now we have

$$\begin{aligned}\left(\sum_{k=1}^{n}\|x_k\|^2\right)^{1/2} &\leqslant M\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\|x_k\|e_k\right\|^2\right)^{1/2}\\ &= M\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\|x_k\|u_k^*(u_k)e_k\right\|^2\right)^{1/2}\\ &= M\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\|x_k\|\big(u_k^*\otimes e_k\big)(u_k)\right\|^2\right)^{1/2}\\ &\leqslant KM^2\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\|x_k\|u_k\right\|^2\right)^{1/2} = KM^2\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|^2\right)^{1/2}.\end{aligned}$$

This shows that X has cotype 2.

Let us assume that X has non-trivial type. Then by results of Pisier [19] and also by Figiel and Tomczak-Jaegermann [12], ℓ_2^n is uniformly complemented in X. Thus, for some constant C, for every $n \in \mathbb{N}$ we can choose a biorthogonal system $\{(e_k, e_k^*)\colon\ k = 1, \dots, n\}$ in $X \times X^*$ such that

$$\left\|\sum_{k=1}^{n}a_k e_k\right\| \leqslant C\left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2}$$

and

$$\left\|\sum_{k=1}^{n}a_k e_k^*\right\| \leqslant C\left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2}$$

for all $a_1,\dots,a_n\in\mathbb{C}$. Note that for any $x\in X$ and $a_1,\dots,a_n\in\mathbb{C}$,

$$\left|\sum_{k=1}^n a_k e_k^*(x)\right| \leqslant C\left(\sum_{k=1}^n |a_k|^2\right)^{1/2}\|x\|$$

and so

$$\left(\sum_{k=1}^n \left|e_k^*(x)\right|^2\right)^{1/2} \leqslant C\|x\|.$$

Consider the family of operators $\mathcal{T}_n=\{e_k^*\otimes u\colon\ \|u\|=1,\ k=1,\dots,n\}$. Let $x\in X$. Then for any $a_1,\dots,a_n\in\mathbb{C}$ and every $u_1,\dots,u_n\in X$ of norm one we have

$$\begin{aligned}\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k a_k\big(e_k^*\otimes u_k\big)(x)\right\| &= \mathbb{E}\left\|\sum_{k=1}^n \epsilon_k a_k e_k^*(x)u_k\right\| \leqslant \sum_{k=1}^n \left\|a_k e_k^*(x)u_k\right\| \\ &= \sum_{k=1}^n |a_k|\left|e_k^*(x)\right| \leqslant \left(\sum_{k=1}^n |a_k|^2\right)^{1/2}\left(\sum_{k=1}^n \left|e_k^*(x)\right|^2\right)^{1/2} \leqslant C\left(\sum_{k=1}^n |a_k|^2\right)^{1/2}\|x\|.\end{aligned}$$

We conclude that $\mathcal{T}_n$ is semi-R-bounded with constant C and hence $\mathcal{T}_n$ is R-bounded for constant KC independent of n. This implies that X has type 2 as follows. Choose any $x_1,\dots,x_n\in X$ and write $x_k=\|x_k\|u_k$, where $\|u_k\|=1$. Then

$$\begin{aligned}\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k x_k\right\|^2\right)^{1/2} &= \left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k \|x_k\|u_k\right\|^2\right)^{1/2} \\ &= \left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k \|x_k\|e_k^*(e_k)u_k\right\|^2\right)^{1/2} \\ &= \left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k \|x_k\|\big(e_k^*\otimes u_k\big)(e_k)\right\|^2\right)^{1/2} \\ &\leqslant KC\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k \|x_k\|e_k\right\|^2\right)^{1/2} \leqslant KC^2\left(\sum_{k=1}^n \|x_k\|^2\right)^{1/2}. \end{aligned}\tag{2.1}$$

Now, X has type 2 and cotype 2 and is therefore isomorphic to ℓ_2 by Kwapien's theorem [25].

Now suppose on the contrary that X has trivial type. We will show that X is a GT-space, i.e., any $T: X\to\ell_2$ is 1-summing. Fix $T: X\to\ell_2$ of norm one. Since X has cotype 2 we can equivalently show that any such T is 2-summing [11]. It suffices to check that for any $n\in\mathbb{N}$ and operator $S:\ell_2^n\to X$ such that $\|S\|\leqslant 1$ we have $\pi_2(TS)\leqslant C$, where C does not depend on n [25]. One can assume that $TS:\ell_2^n\to\ell_2^n$ and that TS is diagonal with respect to the canonical orthonormal basis (e_k) in ℓ_2^n, i.e., $TSe_k=\lambda_k e_k$ for some $\lambda_1,\dots,\lambda_n$. Then it suffices to show uniform boundedness of the Hilbert–Schmidt norms $\|TS\|_{\mathrm{HS}}=(\sum_{k=1}^n\|TSe_k\|^2)^{1/2}$. Write $f_k^*=T^*e_k^*\in X^*$ and $f_k=Se_k\in X$. Consider $\{f_k^*\otimes u\colon\ k=1,\dots,n,\ \|u\|=1\}$. We will show that this family is semi-R-bounded with constant one. Take $u_1,\dots,u_n \in X$ of norm one and $a_1,\dots,a_n \in \mathbb{C}$. Then for $x \in X$ we have

$$\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k a_k f_k^*(x) u_k\right\| \leqslant \sum_{k=1}^n |a_k|\left|f_k^*(x)\right| \leqslant \left(\sum_{k=1}^n |a_k|^2\right)^{1/2}\left(\sum_{k=1}^n \left|e_k^*(Tx)\right|^2\right)^{1/2}$$
$$= \left(\sum_{k=1}^n |a_k|^2\right)^{1/2}\|Tx\| \leqslant \left(\sum_{k=1}^n |a_k|^2\right)^{1/2}\|x\|.$$

Therefore, $\{f_k^* \otimes u\colon\ k = 1,\dots,n,\ \|u\| = 1\}$ is R-bounded with constant K.

Since X has trivial type, it contains ℓ_1^n uniformly [19]. Hence, for fixed $M > 1$ and every $n \in \mathbb{N}$ there are $y_1,\dots,y_n \in X$ with $\|y_k\| = 1$ for $1 \leqslant k \leqslant n$ such that

$$\sum_{k=1}^n |a_k| \leqslant M\left\|\sum_{k=1}^n a_k y_k\right\|. \tag{2.2}$$

Choose any scalars $b_1,\dots,b_n$. Now using R-boundedness and Kahane's inequality for $p = 1$ with constant A we have

$$\sum_{k=1}^n |b_k||\lambda_k| = \sum_{k=1}^n |b_k|\left|f_k^*(f_k)\right| \leqslant M\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k b_k f_k^*(f_k) y_k\right\|^2\right)^{1/2}$$
$$= M\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k b_k \big(f_k^* \otimes y_k\big)(f_k)\right\|^2\right)^{1/2} \leqslant KM\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k b_k f_k\right\|^2\right)^{1/2}$$
$$\leqslant KM\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k b_k e_k\right\|^2\right)^{1/2} \leqslant KM\left(\sum_{k=1}^n |b_k|^2\right)^{1/2}.$$

Thus,

$$\left(\sum_{k=1}^n |\lambda_k|^2\right)^{1/2} \leqslant KM,$$

and so $\|TS\|_{\mathrm{HS}} \leqslant KM$. Therefore, any operator $T : X \to \ell_2$ is 2-summing. This completes the proof of (i) implies (ii).

Now we will show that (ii) implies (i). Suppose that X is a GT-space of cotype 2, and that $\mathcal{T}$ is a family of semi-R-bounded operators. We will show that $\mathcal{T}$ is R-bounded. Since X is separable, there is a quotient map $Q : \ell_1 \to X$. First, we show that any semi-R-bounded family of operators from ℓ_1 into X is already R-bounded. Let $\mathcal{S}$ be such a family with semi-R-boundedness constant one. Suppose $S_1,\dots,S_n \in \mathcal{S}$ and $x_1,\dots,x_n \in \ell_1$. Then $x_k = \sum_{j=1}^\infty \xi_{jk} e_j$, where (e_j) is the canonical basis of ℓ_1.

Let us denote by C the constant in the Kahane–Khintchine inequality for any Banach space:

$$\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k x_k\right\|^2\right)^{1/2} \leqslant C\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k x_k\right\|.$$

Thus

$$\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k S_k x_k\right\|^2\right)^{1/2} \leqslant C\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k S_k x_k\right\|.$$

Then

$$\begin{aligned}\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k S_k x_k\right\| &= \mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k S_k\sum_{j=1}^{\infty}\xi_{jk}e_j\right\| \leqslant \sum_{j=1}^{\infty}\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\xi_{jk}S_k e_j\right\| \\ &\leqslant \sum_{j=1}^{\infty}\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\xi_{jk}S_k e_j\right\|^2\right)^{1/2} \leqslant \sum_{j=1}^{\infty}\left(\sum_{k=1}^{n}|\xi_{jk}|^2\right)^{1/2}.\end{aligned}$$

Combining and using the Khintchine inequality again we obtain

$$\begin{aligned}\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k S_k x_k\right\|^2\right)^{1/2} &\leqslant C^2\sum_{j=1}^{\infty}\mathbb{E}\left|\sum_{k=1}^{n}\epsilon_k\xi_{jk}\right| = C^2\mathbb{E}\left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{n}\epsilon_k\xi_{jk}\right|\right) \\ &= C^2\mathbb{E}\left\|\sum_{j=1}^{\infty}\sum_{k=1}^{n}\epsilon_k\xi_{jk}e_j\right\|_{\ell_1} = C^2\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\sum_{j=1}^{\infty}\xi_{jk}e_j\right\| \\ &= C^2\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|.\end{aligned}$$

Combining the previous two computations gives that $\mathcal{S}$ is R-bounded.

Now let $\mathcal{T}$ be a family of operators on X with semi-boundedness constant one. Let $Q:\ell_1 \to X$ be a quotient map and note that the family $\mathcal{S} = \{TQ\colon\ T \in \mathcal{T}\}$ is R-bounded with some constant B by the above calculation.

We will apply a characterization of GT-spaces of cotype 2 due to Pisier [19].

Proposition 2.3 (Pisier). *X is a GT-space of cotype 2 if and only if there is a constant $C > 0$ such that for any $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$, there are $y_1, \dots, y_n \in \ell_1$ such that $Qy_k = x_k$, $k = 1, \dots, n$, and*

$$\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k y_k\right\| \leqslant C\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|. \tag{2.3}$$

Now take $n \in \mathbb{N}$, $T_1, \dots, T_n \in \mathcal{T}$ and $x_1, \dots, x_n \in X$. Choose $y_1, \dots, y_n \in \ell_1$ according to Proposition 2.3. Then

$$\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k T_k x_k\right\| = \mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k T_k Q y_k\right\| \leqslant B\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k y_k\right\| \leqslant CB\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|.$$

Thus, $\mathcal{T}$ is R-bounded. The proof is complete. □

For a set $\mathcal{T}$ of bounded linear operators we will use the notation $\mathcal{T}^* = \{T^*\colon\ T \in \mathcal{T}\}$.

Lemma 2.4.

(i) *If $\mathcal{T}$ is R-bounded then $\mathcal{T}^{**}$ is R-bounded* (*with the same constant*).
(ii) *If $\mathcal{T}$ is WR-bounded then $\mathcal{T}^{*}$ and $\mathcal{T}^{**}$ are WR-bounded* (*with the same constant*).
(iii) *If $\mathcal{T}$ is semi-R-bounded then $\mathcal{T}^{**}$ is semi-R-bounded* (*with the same constant*).

Proof. The proofs of (i) and (iii) are similar. For (i) suppose $T_1, \dots, T_n \in \mathcal{T}$ and that $\mathcal{T}$ has R-boundedness constant one. Let $\Omega = \{-1, 1\}^n$ with $\mathbb{P}$ normalized counting measure on Ω. Let ϵ_k be the sequence of coordinate maps on Ω. Let $\mathrm{Rad}(\Omega; X)$ be the subspace of $L_2(\Omega, \mathbb{P}; X)$ generated by the functions $\epsilon_k \otimes x$ for $1 \leqslant k \leqslant n$ and $x \in X$ (this space is isomorphic to X^n). Then $\mathrm{Rad}(\Omega; X^{**})$ can be identified naturally with a subspace of $\mathrm{Rad}(\Omega; X)^{**}$. Consider the map $\mathbf{T} : \mathrm{Rad}(\Omega; X) \to \mathrm{Rad}(\Omega; X)$ defined by

$$\mathbf{T}\left(\sum_{k=1}^{n} \epsilon_k \otimes x_k\right) = \sum_{k=1}^{n} \epsilon_k \otimes T_k x_k.$$

Then $\|\mathbf{T}\| \leqslant 1$ and so $\|\mathbf{T}^{**}\| \leqslant 1$ and (i) follows.

Let us now prove (ii). Suppose $\mathcal{T}$ is WR-bounded with constant one and $T_1, \dots, T_n \in \mathcal{T}$. Suppose $x_1^*, \dots, x_n^* \in X^*$ are such that

$$\left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k x_k^*\right\|^2\right)^{1/2} \leqslant 1.$$

Then, using the identification of $\mathrm{Rad}(\Omega, X^{**})$ as the bidual of $\mathrm{Rad}(\Omega, X)$ we observe that the set of functions of the form $\sum_{k=1}^{n} \epsilon_k x_k^{**}$ in $\mathrm{Rad}(\Omega, X^{**})$ such that

$$\sum_{k=1}^{n} \left|\left\langle T_k^* x_k^*, x_k^{**}\right\rangle\right| \leqslant 1$$

is weak*-closed and contains the unit ball of $\mathrm{Rad}(\Omega, X)$. By Goldstine's theorem it contains the unit ball of $\mathrm{Rad}(\Omega, X^{**})$ and this implies that $\mathcal{T}^*$ is WR-bounded with constant one. $\square$

Now it is time to give an example of a family of operators that is uniformly bounded but not WR-bounded. The previous lemma will imply that the corresponding dual family is semi-R-bounded but not WR-bounded.

Example. Let $X = \ell_p$, $1 \leqslant p < 2$. Pick any non-zero element $x \in X$ and choose $u^* \in X^*$ of norm one such that $u^*(x) \neq 0$. Define $T_k = u^* \otimes e_k$, where (e_k) is the canonical basis of X. The family $\{T_k\}$ is uniformly bounded, $\|T_k\| = 1$, but we will show that it is not WR-bounded. Consider the dual basis (e_k^*) in $(\ell_p)^*$. Then

$$\sum_{k=1}^{n} \left|\left\langle T_k x, e_k^*\right\rangle\right| = \sum_{k=1}^{n} \left|\left\langle u^*(x) e_k, e_k^*\right\rangle\right| = n\left|u^*(x)\right|. \tag{2.4}$$

On the other hand, we have

$$\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k x\right\|^2\right)^{1/2}\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k e_k^*\right\|^2\right)^{1/2} = \|x\| n^{1/2} n^{1/q}. \tag{2.5}$$

Here q satisfies $1/p + 1/q = 1$. If $p < 2$ then $q > 2$ and $1/2 + 1/q < 1$, so for $1 \leqslant p < 2$ the family $\{T_k\}$ cannot be WR-bounded.

We have $T_k^* = e_k^{**} \otimes u^*$ on $X^* = \ell_q$, where $2 < q \leqslant \infty$. Consider $q \neq \infty$. Since by reflexivity $T_k^{**} = T_k$ and using Lemma 2.4 we see that $\{T_k^*\}$ is not WR-bounded. However, X^* has type 2 and hence $\{T_k^*\}$ is semi-R-bounded by Proposition 2.1.

3. The main results

Suppose X is any Banach space. We shall say that a sequence $\mathcal{T} = (T_k)_{k=1}^{\infty}$ is an *approximating sequence* if $\lim_{k\to\infty} \|x - T_k x\| = 0$ for every $x \in X$. We will say that $\mathcal{T}$ is compact (relatively, weakly compact) if each T_k is compact (relatively, weakly compact). We will say that $\mathcal{T}$ is commuting if we have $T_k T_l = T_l T_k$ for $l, k \in \mathbb{N}$.

If $\mathcal{T}$ is a commuting approximating sequence, let us define the subspace $E_{\mathcal{T}}$ of X^* to be the closed linear span of $\bigcup_k T_k^*(X^*)$. The following lemma is trivial.

Lemma 3.1. *If $\mathcal{T}$ is a commuting approximating sequence then $E_{\mathcal{T}}$ is a norming subspace of X^*, i.e., for some C we have*

$$\|x\| \leqslant C \sup_{x^* \in B_{E_{\mathcal{T}}}} \left|x^*(x)\right|, \quad x \in X,$$

and, if $\mathcal{T}$ is weakly compact, $\lim_{n\to\infty} T_n^* x^* = x^*$ *weakly for $x^* \in E_{\mathcal{T}}$.*

Let us recall that a Banach space X has property (V) of Pełczyński if every unconditionally converging operator $T : X \to Y$ is weakly compact. The spaces $C(K)$ have property (V) [17] and more generally any C^*-algebra has property (V) [18]. The disk algebra $A(\mathbb{D})$ also has property (V) [10,14]; see also [23]. We also recall that a Banach space X is said to have property (V*) if whenever (x_n) is a bounded sequence in X then either

(i) (x_n) has a subsequence which is weakly Cauchy or

(ii) (x_n) has a subsequence (y_n) such that for some sequence (y_n^*) in X^* and $\delta > 0$ we have $|y_n^*(y_n)| \geqslant \delta$ and

$$\left\|\sum_{k=1}^{n} a_k y_k^*\right\| \leqslant \max_{1\leqslant k\leqslant n} |a_k|, \quad a_1, \ldots, a_n \in \mathbb{C},\ n \in \mathbb{N}.$$

Property (V*) was introduced by Pełczyński [17]. We note that Bombal [4] shows that every Banach lattice not containing c_0 has property (V*). Any subspace of a space with property (V*) also has property (V*).

Lemma 3.2. *Let X, Y be Banach spaces and let $\mathcal{T} = (T_k)_{k=1}^{\infty}$ be any sequence of operators in $\mathcal{L}(X, Y)$. Suppose either*

(i) *$\mathcal{T}$ is semi-R-bounded or*
(ii) *$\mathcal{T}$ is WR-bounded and Y has property* (V^*).

Then for every $x \in X$ the sequence $(T_k x)_{k=1}^{\infty}$ has a weakly Cauchy subsequence.

Proof. If not, by passing to a subsequence we can suppose $(T_k x)_{k=1}^{\infty}$ is equivalent to the canonical ℓ_1-basis [21,22]. If $\mathcal{T}$ is semi-R-bounded we observe that for some C we have

$$\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k a_k T_k x\right\| \leqslant C\left(\sum_{k=1}^{n} |a_k|^2\right)^{1/2} \|x\|, \quad a_1, \dots, a_n \in \mathbb{C},\ n \in \mathbb{N}.$$

This gives a contradiction.

In case (ii), we can pass to a subsequence and assume the existence of $y_n^* \in Y^*$ such that

$$\left\|\sum_{k=1}^{n} a_k y_k^*\right\| \leqslant \max_{1 \leqslant k \leqslant n} |a_k|, \quad a_1, \dots, a_n \in \mathbb{C},\ n \in \mathbb{N},$$

and $|y_n^*(T_n x)| \geqslant \delta > 0$ for all n. Then

$$n\delta \leqslant \sum_{k=1}^{n} \left|y_k^*(T_k x)\right| \leqslant C\left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k x\right\|^2\right)^{1/2} \left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k y_k^*\right\|^2\right)^{1/2} \leqslant C\sqrt{n}.$$

This also yields a contradiction. □

Theorem 3.3. *Let X be a Banach space with a commuting weakly compact approximating sequence $\mathcal{T}$. Suppose either that*

(i) *$\mathcal{T}$ is semi-R-bounded and X is weakly sequentially complete or*
(ii) *$\mathcal{T}$ is WR-bounded and X has property* (V^*).

Then X is isomorphic to a dual space.

Proof. In either case we consider the family $\mathcal{T}^{**} \subset \mathcal{L}(X^{**}, X)$. By Lemma 3.2 for each $x^{**} \in X^{**}$ we can find a subsequence $T_{k_n}^{**} x^{**}$ so that $T_{k_n}^{**}(x^{**})$ is weakly convergent to some $y \in X$. Then for $x^* \in X^*$,

$$x^*(T_k y) = \lim_{n \to \infty} x^*\left(T_k T_{k_n}^{**} x^{**}\right) = \lim_{n \to \infty} x^*\left(T_{k_n} T_k^{**} x^{**}\right)$$

so that $T_k y = T_k^{**} x^{**}$. Hence $\lim_{k \to \infty} \|y - T_k^{**} x^{**}\| = 0$.

We now show that $E_{\mathcal{T}}^*$ can be identified with X. Clearly X canonically embeds in $E_{\mathcal{T}}^*$ since $E_{\mathcal{T}}$ is norming. If $f^* \in E_{\mathcal{T}}^*$ then by the Hahn–Banach theorem there exists $x^{**} \in X^{**}$ with $\|x^{**}\| = \|f^*\|$ and $x^{**}(x^*) = f^*(x^*)$ for $x^* \in E_{\mathcal{T}}$. Let $y = \lim_{k \to \infty} T_k^{**} x^{**}$. Then for $x^* \in E_{\mathcal{T}}$,

$$x^*(y) = \lim_{k\to\infty} x^*\bigl(T_k^{**}x^{**}\bigr) = \lim_{k\to\infty} x^{**}\bigl(T_k^* x^*\bigr) = f^*(x^*).$$

Hence $E_{\mathcal{T}}^* = X$. □

Theorem 3.4. *The space $L_1(0,1)$ does not have a commuting weakly compact approximating sequence which is either semi-R-bounded or WR-bounded.*

Proof. L_1 is not a dual space [25]. □

Of course a semi-R-bounded sequence in L_1 is actually R-bounded.

Theorem 3.5. *Let X be a separable Banach space with property* (V)*. If X has a commuting weakly compact approximating sequence $(T_n)_{n=1}^\infty$ which is WR-bounded, then X^* is separable, and has a WR-bounded commuting weakly compact approximating sequence.*

Proof. Since X has (V), it follows that X^* has property (V^*). We show that $\lim_{n\to\infty} T_n^* x^* = x^*$ weakly for $x^* \in X^*$. Indeed $T_n^* x^*$ converges weak* to x^* and it must have a weakly convergent subsequence by Lemma 3.2. Hence $x^* \in E_{\mathcal{T}}$ so $X^* = E_{\mathcal{T}}$. Now $T_n^*(B_{X^*})$ is weakly compact by Gantmacher's theorem also and weak*-metrizable, hence norm separable. Thus X^* is separable, and so by Mazur's theorem, and a diagonal argument, we can find a sequence of convex combinations $(S_n^*)_{n=1}^\infty$ of $(T_n^*)_{n=1}^\infty$ which is an approximating sequence. □

Corollary 3.6. *If K is an uncountable compact metric space then $C(K)$ has no WR-bounded commuting weakly compact approximating sequence. The disk algebra has no WR-bounded weakly compact approximating sequence.*

We now consider $C(K)$ when K is countable. In this case $C(K)$ is homeomorphic to a space $C(\alpha) = C([1,\alpha])$, where α is a countable ordinal. There is a characterization of such $C(K)$ due to Bessaga and Pełczyński [3].

Theorem 3.7 (Bessaga–Pełczyński). *If $\alpha < \beta$, $C(\omega^\alpha \cdot k)$ is isomorphic to $C(\omega^\beta \cdot n)$ if and only if $\beta < \alpha \cdot \omega$. Consequently, $C(\omega^{\omega^\gamma})$, $0 \leqslant \gamma < \omega_1$, is a complete list of representatives of the isomorphism classes of $C(K)$ for K a countable compact metric space.*

The following lemma can be obtained as an applications of ℓ_1-indices [1,5,6]. However, for convenience of the reader we will give a direct proof by construction.

Lemma 3.8. *Let α be a countable ordinal with $\alpha \geqslant \omega^\omega$. Then there exists $f \in C(\alpha)^{**}$ so that whenever $f_n \in C(\alpha)$ converges to $f \in C(\alpha)^{**}$ weak* then for any $m \in \mathbb{N}$ there exist $n_1, \ldots, n_m \in \mathbb{N}$ such that*

$$\Biggl\| \sum_{k=1}^m \epsilon_k f_{n_k} \Biggr\| \geqslant \frac{1}{2} m, \quad \epsilon_k = \pm 1,\ k = 1, 2, \ldots, m.$$

Proof. In this case $C(\alpha)^{**}$ can be identified with $\ell_\infty(\alpha)$. It is easy to see that it suffices to consider the case $\alpha = \omega^\omega$.

Consider $f \in X^{**}$ defined by $f(\sum_{k=0}^N \omega^k l_k) = (-1)^{\sum_{k=0}^N l_k}$ and $f(\omega^\omega) = 1$. Writing K for the space $[1, \omega^\omega]$ let $K^{(p)}$ denote the pth derived set of K. Then $K^{(p)}$ consists of all ordinals of the form $\sum_{k=p}^n \omega^k l_k$ together with ω^ω. For each $p \in \mathbb{N}$, $K^{(p)}$ is non-empty. Furthermore for each $\alpha \in K^{(p)}$ and every open neighborhood V of α we have that f takes both values ± 1 on $V \cap K^{(p-1)}$.

Let $f_n \in C(K)$ be any sequence such that (f_n) converges to f weak*.

Fix $0 < \delta < 1/2$ and $m \in \mathbb{N}$. We construct $(f_{n_1}, \dots, f_{n_m})$ inductively. We start from $K^{(m)}$. By definition of f we can pick $\alpha_1^1, \alpha_2^1 \in K^{(m)}$ such that $f(\alpha_j^1) = (-1)^j$ for $j = 1, 2$. Then find $n_1 \in \mathbb{N}$ such that $|f_{n_1}(\alpha_j^1) - (-1)^j| < \delta$. Since f_{n_1} is continuous we can choose open neighborhoods U_j^1 of α_j^1 such that $|f_{n_1}(\alpha) - (-1)^j| < \delta$ for all $\alpha \in U_j^1$.

For the inductive step, suppose that $(n_j)_{j=1}^k$, $(\alpha_j^k)_{j=1}^{2^k}$ and open sets $(U_j^k)_{j=1}^{2^k}$ have been chosen so that $\alpha_j^k \in U_j^k$. Then for $i = 1, \dots, 2^k$ find points $\alpha_{2i-1}^{k+1}, \alpha_{2i}^{k+1} \in U_i^k \cap K^{(m-k+1)}$ with $f(\alpha_j^{k+1}) = (-1)^j$. By pointwise convergence, we can select $n_{k+1} > n_k$ such that $|f_{n_{k+1}}(\alpha_j^{k+1}) - (-1)^j| < \delta$. Since $f_{n_{k+1}}$ is continuous, there are neighborhoods $U_{2i-1}^{k+1}, U_{2i}^{k+1} \subset U_i^k$, $i = 1, \dots, 2^k$, such that for all $\alpha \in U_j^{k+1}$ we have $|f_{n_{k+1}}(\alpha) - (-1)^j| < \delta$.

In the mth iteration this will give 2^m neighborhoods and m functions $f_{n_1}, \dots, f_{n_m}$ so that for any $\epsilon_1, \dots, \epsilon_m \in \{-1, +1\}$ there is α contained in one of these neighborhoods such that $|f_k(\alpha) - \epsilon_k| < \delta$ for all $k = 1, \dots, m$. Hence

$$\left\| \sum_{k=1}^m \epsilon_k f_{n_k} \right\| \geqslant (1-\delta)m. \qquad \square$$

Theorem 3.9. *Let K be a compact metric space. Suppose there is an R-bounded commuting weakly compact approximating sequence in $C(K)$. Then $C(K)$ is isomorphic to c_0.*

Proof. By Corollary 3.6 we need only consider the case when K is countable. By Theorem 3.7 it suffices to consider the case when $K = [1, \alpha]$, where $\alpha \geqslant \omega^\omega$. Pick $f \in C(K)^{**}$ satisfying the hypotheses of Lemma 3.8.

Suppose (T_n) is an R-bounded weakly compact approximating sequence for $C(K)$. Then (T_n^*) is an approximating sequence for $C(K)^*$ by Theorem 3.5 and hence $T_n^{**} f$ converges to f weak*. It follows that for any m we can choose $n_1, \dots, n_m$ so that

$$\left\| \sum_{k=1}^m \epsilon_k T_{n_k}^{**} f \right\| \geqslant \frac{1}{2} m, \quad \epsilon_k = \pm 1.$$

Hence

$$\left(\mathbb{E} \left\| \sum_{k=1}^m \epsilon_k T_{n_k}^{**} f \right\|^2 \right)^{1/2} \geqslant \frac{1}{2} m.$$

This contradicts the fact that T_n is R-bounded (or even semi-R-bounded). $\square$

Remark. We can replace the assumption of R-boundedness by the assumption that (T_n) and (T_n^*) are both semi-R-bounded. By Theorem 2.2 this hypothesis would imply that (T_n^*) is actually R-bounded and hence that (T_n) is WR-bounded. We only used the fact that (T_n) is both semi-R-bounded and WR-bounded.

Let us conclude by stating our main result with respect to semigroups. (Actually our results are somewhat stronger than stated below.)

Theorem 3.10. *Let X be a separable Banach space with an R-bounded strongly continuous semigroup $(T_t)_{t>0}$ consisting of weakly compact operators. Then if*

(1) $X = L_1(\mu)$ *for some measure μ then X is isomorphic to ℓ_1 (i.e., μ is purely atomic).*
(2) $X = C(K)$ *then X is isomorphic to c_0.*

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