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On a Class of Nonlinear Functionals on Spaces of Continuous Functions

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1. INTRODUCTION

This paper is motivated by certain ideas which arise in the theory of differential games (see [3, 4]). We abstract the particular features of differential games which are relevant in the notion of a *positional game*. Suppose S is a topological space, and g is a continuous real-valued function on S. We consider a game between two players, whom we call the maximizer and the minimizer, which falls into two phases.

Phase 1. According to some given process, subject to control by both players and possibly random moves, a point $s \in S$ is determined.

Phase 2. The pay-off is computed as $g(s)$. The aim of the maximizer is to maximize $g(s)$, and similarly the minimizer aims to minimize $g(s)$.

This is best illustrated by a differential game. Consider a game with dynamics

$$dx/dt = f(t, x, y, z) \qquad (1)$$

and initial condition

$$x(t_0) = x_0,$$

where $x \in R^m$, $f: R \times R^m \times Y \times Z \to R$ is continuous and Lipschitz in the x-variable, and Y and Z are compact metric spaces. Let $U \subset R \times R^m$ be an open set containing (t_0, x_0), such that there exists $T > t_0$ and if $(t, x) \in U$ then $t \leqslant T$.

We shall assume that the maximizer controls the y-variable and the minimizer the z-variable in Eq. (1).

Suppose now the pay-off is given by

$$P = g(t^*, s(t^*)),$$

where g is a continuous function on ∂U and t^* is the first time for which $(t^*, x(t^*)) \notin U$.

The game we have described is a game of survival ([4, Chap. 5]). This is an example of a positional game in which $S = \partial U$. The outcome of Phase 1 is a point of the topological space ∂U. In this particular game, there is no randomness involved. However, we can consider a stochastic differential game of survival (cf. [2, 5]) with dynamics given by

$$dx = f(t, x, y, z)\, dt + \sigma(t, x)\, dw \tag{2}$$

in place of (1), where w is an m-dimensional Brownian motion and $\sigma(t, x)$ is an $m \times m$-matrix-valued function satisfying certain measurability conditions. With the same form of the pay-off this is again a positional game.

A much simpler example is a two-person zero-sum matrix game. Let S be the discrete space $\{(i, j): 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$. Then a matrix game can be considered as a positional game in which Phase 1 consists of the determination of a pair of pure strategies (i, j), i.e., a point of S, and then in Phase 2 the pay-off $g(i, j)$ is computed.

The object is to study the way in which the value of such a game varies with the pay-off function g. Of course, there may be several notions of value attached to a given positional game. For example, in a matrix game we may consider the upper value

$$V^+(g) = \min_{1 \leqslant j \leqslant n} \max_{1 \leqslant i \leqslant m} g(i, j)$$

the lower value

$$V^-(g) = \max_{1 \leqslant i \leqslant m} \min_{1 \leqslant j \leqslant n} g(i, j)$$

or the von Neumann value

$$V(g) = \max_{\substack{p_i \geqslant 0 \\ \Sigma p_i = 1}} \min_{\substack{q_j \geqslant 0 \\ \Sigma q_j = 1}} \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j g(i, j).$$

Recently Moulin [8] described another notion of value for matrix games, which is rather more difficult to formulate explicitly.

Equally, for differential games, there are many different concepts of value (see [3], for example); in particular we would like to draw attention to the Danskin σ-value [1].

Suppose S is compact and $V: C(S) \to R$ is a value function for some positional game. Then it is clear that V must satisfy certain conditions:

(i) Positivity. If $f \geqslant g$ then $V(f) \geqslant V(g)$,

(ii) Positive homogeneity. If $\alpha \in R$, $\alpha \geqslant 0$, then $V(\alpha f) = \alpha V(f)$ for $f \in C(S)$,

(iii) Translation invariance. If $\alpha \in R$ then $V(f + \alpha) = V(f) + \alpha$ for $f \in C(S)$.

We call a functional on $C(S)$ satisfying these three conditions a *gamonic* functional. The main result of this paper is that any gamonic functional can be realized as the upper value of a positional game. Thus, V has a representation in the form

$$V(f) = \min_{C \in \mathscr{C}} \max_{\mu \in C} \int f \, d\mu,$$

where $\mathscr{C}$ is a collection of subsets of the space $P(S)$ of Radon probability measures on S. In the first phase of the positional game the minimizer selects $C \in \mathscr{C}$, then the maximizer selects $\mu \in C$, and then a point $s \in S$ is selected according to the probability measure μ.

The other main results are concentrated on characterizing those functionals V which arise from purely deterministic positional games, i.e., have a representation

$$V(f) = \min_{E \in \mathscr{E}} \max_{s \in E} f(s)$$

where $\mathscr{E}$ is a collection of subsets of S. By a utility transformation we mean a continuous increasing map $\Phi: R \to R$. If V arises from a deterministic game then it is clear that for any utility transformation Φ

$$V(\Phi \circ f) = \Phi(V(f)), \qquad f \in C(S).$$

Here our main result is that if this equation holds for just one nonlinear utility transformation then V arises from a deterministic game.

The results in this paper were announced in [6]. The author intends to use these results to develop an abstract theory of differential games [7].

2. Positively Homogeneous Functionals on Locally Convex Spaces

Let X be a real locally convex topological vector space and let $w: X \to R$ be a uniformly continuous positively homogeneous functional on X. In this section we shall give a general representation theorem for w.

Lemma 2.1. *Let* $p(x) = \sup_{y \in X}[w(x + y) - w(y)]$. *Then* p *is a continuous sublinear functional on* X.

Proof. Clearly p is positively homogeneous, and $p(x + y) \leqslant p(x) + p(y)$

for $x, y \in X$. Hence p is sublinear. As w is uniformly continuous there exists a neighborhood U of 0 such that $p(x) \leqslant 1$ for $x \in U$. Hence p is continuous.

DEFINITION 1. The *support* of w is the subset Supp w of X' of all linear functionals φ such that $\varphi(x) \leqslant p(x)$ for $x \in X$.

We remark that Supp w is a $\sigma(X', X)$-compact convex subset of X', which is equicontinuous.

LEMMA 2.2. *Suppose $x, y \in X$. Then there exists $\varphi \in \operatorname{Supp} w$ such that $\varphi(x) \geqslant w(x)$ and $\varphi(y) \leqslant w(y)$.*

Proof. For $\alpha \in R_+$,

$$w(\alpha y) + p(x - \alpha y) \geqslant w(x).$$

We define

$$a = \inf_{\alpha \in R_+} [w(\alpha y) + p(x - \alpha y)].$$

Then clearly $w(x) \leqslant a \leqslant p(x)$. Thus for $\beta \in R_+$,

$$\beta a - p(\beta x - y) \leqslant p(y)$$

and

$$\begin{aligned} -\beta a - p(-\beta x - y) &\leqslant -\beta w(x) - p(-\beta x - y) \\ &\leqslant -w(-y). \end{aligned}$$

Hence we may define

$$b = \sup_{\lambda \in R} [\lambda a - p(\lambda x - y)].$$

Then for $\lambda \in R$,

$$\lambda a - b \leqslant p(\lambda x - y). \tag{3}$$

Now for $\beta \in R_+$,

$$\begin{aligned} \beta a &\leqslant p(\beta x), \\ -\beta a &\leqslant -\beta w(x) \\ &\leqslant p(-\beta x), \end{aligned}$$

so that for $\lambda, \mu \in R$,

$$\begin{aligned} (\lambda + \mu)\, a &\leqslant p[(\lambda + \mu)\, x] \\ &\leqslant p(\mu x + y) + p(\lambda x - y). \end{aligned}$$

Hence

$$\lambda a - p(\lambda x - y) \leqslant p(\mu x + y) - \mu a$$

and therefore

$$b \leqslant p(\mu x + y) - \mu a.$$

Hence

$$\mu a + b \leqslant p(\mu x + y). \tag{4}$$

Combining (3) and (4), we see that for $\mu, \lambda \in R$,

$$\mu a + \lambda b \leqslant p(\mu x + \lambda y),$$

and hence by the Hahn–Banach theorem there is a linear functional $\varphi \in X'$ such that $\varphi(z) \leqslant p(z)$, $z \in X$ and $\varphi(x) = a$, $\varphi(y) = b$.

Now for $\alpha, \beta \in R_+$,

$$\begin{aligned}(\alpha + \beta)\, w(y) &= w((\alpha + \beta)\, y) - w(0)\\ &\geqslant -p(-(\alpha + \beta)\, y)\\ &\geqslant -p(x - \alpha y) - p(-x - \beta y),\end{aligned}$$

so that

$$\alpha w(y) + p(x - \alpha y) \geqslant -\beta w(y) - p(-x - \beta y).$$

Hence

$$a \geqslant -\beta w(y) - p(-x - \beta y),$$

i.e.,

$$\beta w(y) \geqslant -a - p(-x - \beta y), \qquad \beta > 0.$$

Hence for $\gamma > 0$

$$w(y) \geqslant -\gamma a - p(-\gamma x - y). \tag{5}$$

Also, for $\alpha \in R_+$,

$$a - p(x - \alpha y) \leqslant \alpha w(y),$$

so that for $\gamma > 0$,

$$\gamma a - p(\gamma x - y) \leqslant w(y). \tag{6}$$

Combining (5) and (6), and the case $\gamma = 0$ we obtain $b \leqslant w(y)$. Thus $a = \varphi(x) \geqslant w(x)$, while $b = \varphi(y) \leqslant w(y)$ as required.

DEFINITION 2. A subset C of X' is *w-admissible* if C is $\sigma(X', X)$-compact convex and

$$\max_{\varphi \in C} \varphi(x) \geqslant w(x), \qquad x \in X.$$

DEFINITION 3. $\mathscr{M}(w)$ is the set of minimal w-admissible sets in X'.

LEMMA 2.3. *If C is w-admissible, then there exists $C_0 \in \mathscr{M}(w)$ with $C_0 \subset C$.*

Proof. By Zorn's lemma, let C_α be a descending chain of w-admissible sets contained in C and let

$$p_0(x) = \inf_{\alpha} \max_{\varphi \in C_\alpha} \varphi(x).$$

Then p_0 is sublinear and continuous, since $p_0(x) \leqslant \max_{\varphi \in C} \varphi(x)$, and $p_0(x) \geqslant w(x)$. Hence if $C_{p_0} = \{\varphi: \varphi(x) \leqslant p_0(x), \forall x \in X\}$ then C_{p_0} is w-admissible and $C_{p_0} \subset \cap C_\alpha$.

THEOREM 2.4. *Let $w: X \to R$ be a uniformly continuous positively homogeneous map. Then*

$$w(x) = \min_{C \in \mathscr{M}(w)} \max_{\varphi \in C} \varphi(x).$$

Proof. Clearly $w(x) \leqslant \min_{C \in \mathscr{M}(w)} \max_{\varphi \in C} \varphi(x)$. Conversely for fixed y, let

$$C = \{\varphi \in \operatorname{Supp} w: \varphi(y) \leqslant w(y)\}.$$

Then by Lemma 2.2, for any $x \in X$,

$$\max_{\varphi \in C} \varphi(x) \geqslant w(x),$$

so that C is w-admissible. Now select $C_0 \in \mathscr{M}(w)$ such that $C_0 \subset C$ by Lemma 2.3. Then

$$\max_{\varphi \in C_0} \varphi(y) = w(y)$$

and the theorem follows.

COROLLARY 2.5. *The $\sigma(X', X)$-closed convex hull of $\cup(C: C \in \mathscr{M}(w))$ is* Supp w.

Proof. Let $D = \overline{\text{co}} \cup [C: C \in \mathscr{M}(w)]$. Then, by 2.4,

$$w(x + y) - w(x) \leqslant \max_{\varphi \in D} \varphi(y),$$

so that $D \supset \operatorname{Supp} w$. Conversely, suppose $C \in \mathscr{M}(w)$.

Then for $x \in X$,

$$w(x) \leqslant w(y) + p(x - y),$$

where $p(z) = \max_{\varphi \in \operatorname{Supp} w} \varphi(z)$. Thus

$$w(x) \leqslant \max_{\varphi \in C} \varphi(y) + p(x - y)$$

and so the sublinear functional q given by

$$q(x) = \inf_y [\max_{\varphi \in C} \varphi(y) + p(x - y)]$$

satisfies $q(x) \geqslant w(x)$, $\forall x \in X$. Now let $C_0 = \{\varphi : \varphi(x) \leqslant q(x)\ \forall x \in X\}$. Then C_0 is w-admissible and $C_0 \subset C \cap \operatorname{Supp} w$. Hence, as C is minimal, $C \subset \operatorname{Supp} w$. Therefore $D \subset \operatorname{Supp} w$ and the result is proved.

3. Gamonic Functionals

Let S be a compact Hausdorff space.

Definition 4. A functional $V: C(S) \to R$ is *gamonic* if

(i) $V(f) \geqslant V(g)$ whenever $f \geqslant g$ with $f, g \in C(S)$,

(ii) $V(f + \alpha) = V(f) + \alpha$ whenever $f \in C(S)$, $\alpha \in R$,

(iii) $V(\alpha f) = \alpha V(f)$ whenever $f \in C(S)$, $\alpha \in R_+$.

Lemma 3.1. *If $V: C(S) \to R$ is gamonic then*

$$V(f + g) - V(f) \leqslant \max_{s \in S} g(s) \qquad f, g \in C(S).$$

Proof. $f + g \leqslant f + \max g$ and by the positivity of V,

$$\begin{aligned} V(f + g) &\leqslant V(f + \max g) \\ &= V(f) + \max g. \end{aligned}$$

The sublinear functional $g \to \max g$ is generated by the set $P(S)$ of regular probability measures on S. Thus Lemma 3.1 implies that $\operatorname{Supp} V \subset P(S)$. We can now state formally our main theorem which is simply a rewording of Theorem 2.4.

Theorem 3.2. *Let S be a compact Hausdorff space and suppose $V: C(S) \to R$ is a gamonic functional. Then*

$$V(f) = \min_{C \in \mathscr{M}(V)} \max_{\mu \in C} \int f \, d\mu, \qquad f \in C(S),$$

where $\mathscr{M}(V)$ is the collection of closed convex subsets C of $P(S)$ which are minimal with respect to the condition

$$V(f) \leqslant \max_{\mu \in C} \int f \, d\mu, \qquad f \in C(S).$$

If we write $V^*(f) = -V(-f)$, then V^* is also gamonic, and hence

$$V^*(f) = \min_{C \in \mathscr{M}(V)} \max_{\mu \in C} \int f \, d\mu.$$

We deduce:

COROLLARY 3.3.

$$V(f) = \max_{C \in \mathscr{M}(V^*)} \min_{\mu \in C} \int f \, d\mu.$$

It is of interest to determine a similar representation for locally compact spaces S (particularly for application to stochastic differential games). Here we consider functionals defined on $C_0(S)$ and note that $P(S)$ is no longer a weak*-closed subset of $C_0(S)^*$.

THEOREM 3.4. *Let S be a locally compact Hausdorff space and suppose $V: C_0(S) \to R$ is a functional satisfying:*

(i) *if $f \in C_0(S)$ and $\alpha \in R_+$ then $V(\alpha f) = \alpha V(f)$;*

(ii) *for $f, g \in C_0(S)$,*

$$\inf_{s \in S}(f(s) - g(s)) \leqslant V(f) - V(g) \leqslant \sup_{s \in S}(f(s) - g(s)).$$

Then there is a collection $\mathscr{A}$ of weak-closed subsets of $P(S)$ such that*

$$V(f) = \inf_{C \in \mathscr{A}} \sup_{\mu \in C} \int f \, d\mu.$$

Proof. Let $\hat{S} = S \cup \{\infty\}$ be the one-point compactification of S and define $\tilde{V}: C(\hat{S}) \to R$ by

$$\tilde{V}(f) = V(f - f(\infty)) + f(\infty).$$

Then $\tilde{V}$ is gamonic and so

$$\tilde{V}(f) = \min_{C \in \mathscr{C}} \max_{\mu \in C} \int f \, d\mu,$$

where $\mathscr{C}$ is a collection of closed convex subsets of $P(\hat{S})$. Now $P(S)$ is weak*-

dense in $P(\hat{S})$ in $C(S)^*$. For each $C \in \mathscr{C}$ let $\mathscr{G}(C)$ be the set of all open neighborhoods of C in $P(\hat{S})$. Then

$$\tilde{V}(f) = \min_{C \in \mathscr{C}} \inf_{V \in \mathscr{G}(C)} \sup_{\mu \in V \cap P(S)} \int f \, d\mu$$

$$= \inf_{\substack{C \in \mathscr{C} \\ V \in \mathscr{G}(C)}} \sup_{\mu \in V \cap P(S)} \int f \, d\mu.$$

Restricting to $C_0(S)$ we obtain the result.

A min-max theorem in this case is only apparently attainable under some additional assumptions.

THEOREM 3.5. *Under the same assumptions as Theorem* 3.4, *suppose in addition that for every* $\epsilon > 0$ *there is a compact subset* K_ϵ *of* S *such that if* $f \in C_0(S)$ *and* $h \in C_0(S)$ *with* $\| h \| \leqslant 1$; *then*

$$V(f + h) \leqslant V(f) + \max_{s \in K_\epsilon} h(s) + \epsilon.$$

Then there is a collection $\mathscr{A}$ *of weak*-closed subsets of* $P(S)$ *such that*

$$V(f) = \min_{C \in \mathscr{C}} \max_{\mu \in C} \int f \, d\mu.$$

Proof. Construct $\tilde{V}: C(\hat{S}) \to R$ as in Theorem 3.4. Now suppose $\lambda \in \operatorname{Supp} \tilde{V}$ and $h \in C_0(S)$ is such that $h = -1$ on K_ϵ and $-1 \leqslant h \leqslant 0$ everywhere. Then for $f \in C(\hat{S})$,

$$\begin{aligned} \tilde{V}(f + h) - \tilde{V}(f) &= V(f - f(\infty) + h) - V(f - f(\infty)) \\ &\leqslant \max_{s \in K_\epsilon} h(s) + \epsilon \\ &= \epsilon - 1. \end{aligned}$$

Hence as $\lambda \in \operatorname{Supp} \tilde{V}$,

$$\int_S h \, d\lambda \leqslant \epsilon - 1.$$

Therefore,

$$\int_S h \, d\lambda \leqslant \epsilon - 1$$

and $\lambda(S) \geqslant 1 - \epsilon$. Thus $\lambda(\infty) \leqslant \epsilon$, and as $\epsilon > 0$ is arbitrary, we have that if $\lambda \in \operatorname{Supp} \tilde{V}$, then $\lambda \in P(S)$. Thus the canonical representation of Theorem 3.2 restricts to $C_0(S)$ as required.

4. Convergence Theorems

Suppose again that S is compact and Hausdorff, and let $B(S)$ denote the space of bounded Borel functions on S. In view of the results of Section 3 a gamonic functional V on $C(S)$ may be extended to $B(S)$ in two different canonical ways:

$$V^+(f) = \inf_{C\in\mathscr{M}(V)} \sup_{\mu\in C} \int f\, d\mu,$$

$$V^-(f) = \sup_{C\in\mathscr{M}(V^*)} \inf_{\mu\in C} \int f\, d\mu.$$

The author does not know whether it is true in general that $V^+(f) = V^-(f)$ for all bounded Borel functions. In this section we show that this is the case for semicontinuous functions and also establish some monotone convergence theorems.

Proposition 4.1. *For any $g \in B(S)$, $V^-(g) \leqslant V^+(g)$.*

Proof. Suppose $C \in \mathscr{M}(V)$ and $C' \in \mathscr{M}(V^*)$. Suppose $C \cap C' = \varnothing$; then by the Hahn–Banach theorem, since both C and C' are weak*-compact, there exists $f \in C(S)$ such that

$$\max_{\mu\in C} \int f\, d\mu < \min_{\mu\in C'} \int f\, d\mu.$$

However, $V(f) \leqslant \max_{\mu\in C} \int f\, d\mu$ and $V(f) \geqslant \min_{\mu\in C'} \int f\, d\mu$. Hence it follows by contradiction that $C \cap C' \neq \varnothing$ and therefore if $g \in B(S)$,

$$\sup_{\mu\in C} \int g\, d\mu \geqslant \inf_{\mu\in C'} \int g\, d\mu$$

and the result follows.

Theorem 4.2. *Let f be a bounded lower-semicontinuous function on S and let $\{f_\alpha\}$ be any increasing net of bounded lower-semicontinuous functions such that $f = \sup_\alpha f_\alpha$. Then*

(i) $V^+(f) = \min_{C\in\mathscr{M}(V)} \sup_{\mu\in C} \int f\, d\mu$;

(ii) $V^+(f) = V^-(f)$;

(iii) $\lim_\alpha V^+(f_\alpha) = V^+(f)$.

Conversely, if g is a bounded upper-semicontinuous function and g is any decreasing net of bounded upper-semicontinuous functions such that $g = \inf_\alpha g_\alpha$, then

(i)′ $V^-(g) = \max_{C\in\mathscr{M}(V)} \inf_{\mu\in C} \int g\, d\mu$;

(ii)′ $V^+(g) = V^-(g)$;

(iii)′ $\lim_\alpha V^-(g_\alpha) = V^-(g)$.

Proof. Clearly (i)′, (ii)′, and (iii)′ are proved in precisely the same way as (i), (ii), and (iii), and so we shall only prove the first half of the theorem.

Let us suppose, to begin with, that the net $\{f_\alpha\}$ consists of continuous functions and let $v = \sup_\alpha V(f_\alpha)$. We have immediately that $v \leqslant V^+(f)$.

For each α, let $C_\alpha = \{\mu \in P(S)\colon \int f_\alpha\, d\mu \leqslant v\}$; then C_α is certainly V-admissible and hence, if p_α is the associated sublinear functional,

$$p_\alpha(h) = \max_{\mu \in C_\alpha} \int h\, d\mu \geqslant V(h).$$

The nets (C_α) and (p_α) decrease and hence, if

$$q(h) = \inf_\alpha p_\alpha(h),$$

then q is sublinear and $q(h) \geqslant V(h)$. Let $C_q = \{\nu\colon \int h\, d\nu \leqslant q(h)\ \forall h \in C(S)\}$. Then C_q is V-admissible and hence contains some $D \in \mathscr{M}(V)$. If $\mu \in D$ then $\mu \in C_\alpha$ for every α.

Then

$$\int f\, d\mu = \sup_\alpha \int f_\alpha\, d\mu$$

(since each f_α is lower-semicontinuous and $\{f_\alpha\}$ converges monotonically to f), so that

$$\int f\, d\mu \leqslant v.$$

Hence

$$\sup_{\mu \in D} \int f\, d\mu \leqslant v$$

and $V^+(f) \leqslant v$. We have thus proved (i) and (iii) for increasing nets of continuous functions. To prove (iii) for each $\{f_\alpha\}$ lower-semicontinuous we now observe that the net $\{h\colon h \in C(S)\ h \leqslant f_\alpha \text{ some } \alpha\}$ directed by the ordering of $C(S)$ converges pointwise to f. Hence

$$\begin{aligned} V^+(f) &= \sup\{V(h)\colon h \leqslant f_\alpha \text{ some } \alpha\} \\ &= \sup_\alpha V^+(f_\alpha), \end{aligned}$$

proving (iii) in general.

Finally, we observe that (ii) follows, since

$$\begin{aligned} V^+(f) &= \sup\{V(h)\colon h \leqslant f\} \\ &\leqslant V^-(f) \leqslant V^+(f) \qquad \text{by 4.1.} \end{aligned}$$

5. Deterministic Functionals

If S is compact and Hausdorff and $V: C(S) \to R$ is a gamonic functional, then we define $\mathscr{E}(V)$ as the collection of subsets E of S minimal with respect to the condition

$$\max_{s \in E} f(s) \geqslant V(f).$$

Definition 5. V is *deterministic* if $V(f) = \min_{E \in \mathscr{E}(V)} \max_{s \in E} f(s)$.

Remarks. For any gamonic V the collection $\mathscr{E}(V)$ is nonempty. For if $\{E_\alpha\}$ is a descending chain of closed subsets such that

$$\max_{s \in E_\alpha} f(s) \geqslant V(f), \qquad f \in C(S),$$

for each α, then $E_0 = \bigcap_\alpha E_\alpha$ is nonempty, and by the compactness of S

$$\max_{s \in E_0} f(s) \geqslant V(f), \qquad f \in C(S).$$

The existence of sets in $\mathscr{E}(V)$ follows by Zorn's lemma. Furthermore, the minimum in Definition 5 is attained. For if $E_n \in \mathscr{E}(V)$ are such that

$$\max_{s \in E_n} f(s) \leqslant \inf_{E \in \mathscr{E}(V)} \max_{s \in E} f(s) + (1/n),$$

then let $E_\infty = \limsup E_n = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m \geqslant n} E_m}$. For any $h \in C(S)$,

$$\max_{s \in E_\infty} h(s) \geqslant V(h)$$

(again by the compactness of S) and hence E_∞ contains a set F in $\mathscr{E}(V)$ and

$$\max_{s \in F} f(s) = \inf_{E \in \mathscr{E}(V)} \max_{s \in E} f(s).$$

Lemma 5.1. *If V is deterministic and $E \in \mathscr{E}(V)$, then $P(E) \in \mathscr{M}(V)$.*

Proof. From the definition

$$P(E) = \{\mu \in P(S) : \mu(E) = 1\}$$

is a weak*-closed convex subset of $P(S)$. If $E \in \mathscr{E}(V)$, then $P(E)$ is V-admissible and so there exists $D \in \mathscr{M}(V)$ with $D \subset P(E)$. Suppose $\bar{s} \in E$ and let $\delta_{\bar{s}}$ be the unit mass at $\bar{s}$.

If U is an open neighborhood of $\bar{s}$ in E, then since E is minimal with respect to the condition $\max_{s \in E} f(s) \geqslant V(f)$, there exists $f \in C(S)$ such that

$$\max_{s \in E - U} f(s) < V(f).$$

Let $g(s) = \min(f(s), V(f))$ for $s \in S$. Then since V is deterministic $V(g) = V(f)$.

Now there exists $\mu \in D$ such that

$$\int g \, d\mu \geqslant V(g)$$

and hence

$$\mu(E - U) \max_{s \in E-U} g(s) + \mu(U) \, V(g) \geqslant V(g).$$

Hence $\mu(U) = 1$. Such a $\mu \in D$ can be constructed for any open neighborhood of $\bar{s}$, and so, as D is weak*-closed, $\delta_{\bar{s}} \in D$. This is true for any $\bar{s}$ and so $\overline{\text{co}}\{\delta_{\bar{s}} \colon \bar{s} \in E\} \subset D$; i.e., $P(E) = D$.

For any gamonic functional V, we define the indicator set function I_V of V by

$$I_V(K) = V^+(\chi_K)$$

whenever K is a closed subset of S. The indicator function does not determine V in general (unlike the linear case when the induced measure determines the linear functional on $C(S)$). However, we can use the indicator set function to determine deterministic functionals.

Definition 6. A *utility transformation* is a continuous increasing map $\Phi \colon R \to R$.

We can now state our main theorem on deterministic functionals.

Theorem 5.2. *Let S be a compact Hausdorff space and $V \colon C(S) \to R$ be a gamonic functional. Then the following conditions are equivalent.*

(i) *V is deterministic.*

(ii) *I_V takes only the values* 0 *and* 1.

(iii) *For every utility transformation Φ, we have*

$$V(\Phi \circ f) = \Phi(V(f)), \qquad f \in C(S).$$

(iv) *For a single nonlinear utility transformation Φ we have*

$$V(\Phi \circ f) = \Phi(V(f)), \qquad f \in C(S).$$

Proof.

(i) $\Rightarrow$ (iii): This is immediate from Definition 5.

(iii) $\Rightarrow$ (iv): Obvious.

(iv) $\Rightarrow$ (ii): First we observe that if u is bounded and upper-semi-continuous then

$$V^+(\Phi \circ u) = \lim_\alpha V(\Phi \circ g_\alpha),$$

where $\{g_\alpha\}$ is decreasing net of continuous functions converging to u pointwise. Thus

$$\begin{aligned} V^+(\Phi \circ u) &= \lim_\alpha \Phi(V(g_\alpha)) \\ &= \Phi(V^+(u)). \end{aligned}$$

Now suppose E is closed and $E \subset S$ with $I_V(E) = \delta$, where $0 < \delta < 1$. Since Φ is monotonic, Φ is differentiable on a dense subset of points of R. Suppose c is a point of differentiability of Φ. and $t > 0$. Then

$$\begin{aligned} \Phi[V^+(c + t\chi_E)] &= V^+(\Phi \circ (c + t\chi_E)) \\ &= V^+(\Phi(c) + (\Phi(c + t) - \Phi(c))\, \chi_E). \end{aligned}$$

Therefore

$$\Phi(c + t\delta) = \Phi(c) + \delta(\Phi(c + t) - \Phi(c))$$

or

$$\Phi(c + t\delta) - \Phi(c) = \delta(\Phi(c + t) - \Phi(c)).$$

By induction,

$$\Phi(c + t\delta^n) - \Phi(c) = \delta^n(\Phi(c + t) - \Phi(c))$$

and therefore, taking limits as $n \to \infty$,

$$t\Phi'(c) = \Phi(c + t) - \Phi(c).$$

It follows that Φ is linear on $[c, \infty)$ with slope $\Phi'(c)$. However, c can be any of a dense subset of points of R, and hence Φ is linear contrary to assumption. Thus $\delta = 0$ or $\delta = 1$.

(ii) $\Rightarrow$ (i). First we observe that in general,

$$V(f) \leqslant \min_{E \in \mathscr{E}} \max_{s \in E} f(s).$$

Let

$$\min_{E \in \mathscr{E}} \max_{s \in E} f(s) = \alpha \qquad \text{and} \qquad V(f) = \beta$$

and suppose $\beta < \alpha$. Let $\gamma = \frac{1}{2}(\alpha + \beta)$, and χ be the characteristic function of the set $E_\gamma = \{s : f(s) \geqslant \gamma\}$.

Then

$$\min f + (\gamma - \min f)\,\chi \leqslant f$$

and therefore

$$\min f + (\gamma - \min f)\,I_V(E_\gamma) \leqslant \beta.$$

Hence $I_V(E_\gamma) < 1$ and so $I_V(E_\gamma) = 0$.

Now let $F_\gamma = \{s: f(s) \leqslant \gamma\}$; then for any $h \in C(S)$,

$$h \leqslant \max_{s\in F_\gamma} h(s) + (\max_{s\in S} h(s) - \max_{s\in F_\gamma} h(s))\,\chi$$

and therefore

$$V(h) \leqslant \max_{s\in F_\gamma} h(s).$$

Hence F_γ contains a set $E_0 \in \mathscr{E}$ and

$$\max_{s\in E_0} f(s) \leqslant \gamma < \alpha,$$

contrary to assumption. Hence $V(f) = \min_{E\in\mathscr{E}} \max_{s\in E} f(s)$ as required.

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