Boundary Value Problems for Nonlinear Partial Differential Operators

ROBERT J. ELLIOTT AND NIGEL J. KALTON

Department of Mathematics, University of Toronto, Toronto, and
Department of Pure Mathematics, University College, Swansea

Submitted by J. P. LaSalle

1. Introduction

The present paper is a sequel to our paper [2] on differential games of survival and is also related to the work of Fleming [4] on Cauchy problems for degenerate parabolic equations. Our notation and terminology, again outlined in Section 2, is that of [2].

In our recent paper [3], using a related stochastic game, we proved that for fixed time differential games our value $U$ is the same as the upper value $V^+$ of Friedman [5] (see also [1, Sect. 4]). In Section 3 of this paper we extend this result to games of survival and prove that if the boundary $\partial F$ of the terminal set $F$ is regular then $U = V = Q^+$, where $Q^+$ is the upper value introduced in [2].

We next prove that if there are upper and lower solutions of the related Isaacs–Bellman equation then $U$ is Lipschitz continuous and so, almost everywhere, a solution of the equation.

We then consider a nonlinear equation of the form

$$Lv = \frac{\partial v}{\partial t} + G(t, x, \nabla v) = 0$$

together with the boundary condition $v |_{\partial F} = g$. Under certain hypotheses on $G$ we show by constructing a related differential game that if there are $C^1$ functions $\theta_1$ and $\theta_2$ such that $L\theta_1 \geq 0 \geq L\theta_2$ and $\theta_1 = \theta_2 = g$ on $\partial F$ then the value of the differential is a generalized solution of the above boundary value problem. Finally we prove that the solution thus obtained is independent of how a differential game is associated with the problem, that is, the "differential game solution" is unique.
2. GAMES OF SURVIVAL

We consider a differential game of survival with control sets $Y$ and $Z$ which are compact metric spaces. The dynamics of the game are given by

$$\frac{dx}{dt} = \dot{x}(t) = f(t, x, y, z)$$

where $x \in \mathbb{R}^m$ and $t \in [t_0, \infty)$. $f: \mathbb{R} \times \mathbb{R}^{m+1} \times Y \times Z \to \mathbb{R}^m$ is continuous and satisfies a constant Lipschitz condition in $x$:

$$\|f(t, x_1, y, z) - f(t, x_2, y, z)\| \leq K \|x_1 - x_2\|$$

We suppose the trajectory satisfies the initial condition

$$x(t_0) = x_0.$$ 

A closed terminal set $F \subset \mathbb{R} \times \mathbb{R}^m$ such that $F \subset [T, \infty) \times \mathbb{R}^m$ for some fixed $T$ is considered and the game ends the first time $t_F$, that the trajectory enters $F$. The payoff computed is then

$$P(y, z) = g(t_F, x(t_F)) + \int_{t_0}^{t_F} h(t, x(t), y(t), z(t)) \, dt$$

where $g: \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ and $h: \mathbb{R} \times \mathbb{R}^m \times Y \times Z$ are continuous. Player $J_1$ controlling $y \in Y$ is trying to maximize $P$ whilst player $J_2$ controlling $z \in Z$ is trying to minimize $P$. We denote by $G(t_0, x_0)$ the differential game with dynamics (1), initial condition (3) and payoff (4).

$M_1(t_0)$ denotes the space of measurable functions on $[t_0, \infty)$ with values in $Y$. (A function $y \in M_1(t_0)$ is measurable if for every continuous real valued function $\psi$ on $Y \psi \circ y$ is Lebesgue measurable, and functions equal almost everywhere are identified.) $M_2(t_0)$ denotes the space of measurable functions $[t_0, \infty) \to Z$. For $s \geq 0$ an $s$-delay strategy $\alpha$ for $J_2$ is a map $\alpha: M_2(t_0) \to M_1(t_0)$ such that wherever

$$z_1(t) = z_2(t) \quad \text{a.e.} \quad t_0 \leq t \leq t_1,$$

then

$$(\alpha z_1)(t) = (\alpha z_2)(t) \quad \text{a.e.} \quad t_0 \leq t \leq t_1 + s.$$ 

The set of $s$-delay strategies for $J_1$ is denoted by $\Gamma_{t_0}(s)$. $s$-delay strategies for $J_2$ are defined similarly and the set denoted by $A_{t_0}(s)$. For $\alpha \in \Gamma_{t_0}(0)$ its value is defined as

$$u(\alpha) = \inf \{P(\alpha z, z); z \in M_2(t_0)\}.$$
Then
\[ U(t_0, x_0) = \sup \{ u(\alpha) ; \alpha \in \Gamma_{t_0}(0) \}. \] (5)

For \( \beta \in \Delta_{t_0}(0) \) we define
\[ v(\beta) = \sup \{ P(y, \beta y) ; y \in \mathcal{M}_t(t_0) \} \]
and
\[ V(t_0, x_0) = \inf \{ v(\beta) ; B \in \Delta_{t_0}(0) \}. \] (6)

We also define upper and lower values
\[ V^+(t_0, x_0) = \inf \left[ v(\beta) ; \beta \in \bigcup_{s > 0} \Delta_{t_0}(s) \right], \] (7)
\[ V^-(t_0, x_0) = \sup \left[ u(\alpha) ; \alpha \in \bigcup_{s > 0} \Gamma_{t_0}(S) \right]. \] (8)

For \( \beta, \beta' \in \Delta_{t_0}(0) \) we say that \( \beta' \in O(\beta) \) if for any \( y \in \mathcal{M}_2(t_0) \)
\[ (\beta' y)(t) = (\beta y)(t) \quad \text{a.e.} \quad t_0 + s \leq t < \infty. \]

The \textit{s-delay value} of \( \beta \in \Delta_{t_0}(0) \) is defined as
\[ v_s(\beta) = \sup \{ v(\beta') ; \beta' \in \Delta^s_{t_0}(\beta) \}. \]

Write \( \mathcal{M}_2(t_0) \) for the space of measurable functions \( z : [t_0, t_0 + s] \to Z \) and also define \( \Delta_{t_0}[s \mid z] \) as the set of strategies \( \beta \in \Delta_{t_0}(s) \) such that \( \beta y(t) = z(t) \)
a.e. \( t_0 \leq t \leq t_0 + s \), where \( z \in \mathcal{M}_2[t_0] \) is fixed. We define the value
\[ Q_s^+(t_0, x_0) = \sup_{z \in \mathcal{M}_2[t_0]} \inf_{\beta \in \Delta_{t_0}[s \mid z]} v(\beta). \] (9)

Clearly we have the relation
\[ Q_s^+(t_0, x_0) \leq \inf_{\beta \in \Delta_{t_0}(s)} v_s(\beta). \]

Finally
\[ Q^+(t_0, x_0) = \lim_{s \to 0} Q_s^+(t_0, x_0) \]
\[ = \inf_{s > 0} Q_s^+(t_0, x_0). \] (10)

There are similar definitions (see [2]) for the \textit{s-delay value} \( u_s(\alpha) \) of \( \alpha \in \Gamma_{t_0}(0) \), \( Q_s^-(t_0, x_0) \) and \( Q^-(t_0, x_0) \).
From [1, 2, and 3] we collect the following results.

**Theorem 2.1.**

\[
Q^-(t_0, x_0) \leq V^-(t_0, x_0) \leq V(t_0, x_0), U(t_0, x_0) \\
\leq V^+(t_0, x_0) \leq Q^+(t_0, x_0).
\]  

(11)

For fixed time games (i.e., \(F = [T, \infty) \times R^n\))

\[
Q^-(t_0, x_0) = V^-(t_0, x_0) = V(t_0, x_0)
\]

(12)

and

\[
Q^+(t_0, x_0) = V^+(t_0, x_0) = U(t_0, x_0).
\]

(13)

Consider a map \(\tau: M(t_0) \times M(t_0) \rightarrow [t_0, \infty)\) which prescribes a stopping time corresponding to any pair of control functions. \(\tau\) is said to be nonanticipating if whenever

\[
y'(t) = y(t) \text{ a.e. } t_0 \leq t \leq \tau(y, z),
\]

\[
z'(t) = z(t) \text{ a.e. } t_0 \leq t \leq \tau(y, z),
\]

then \(\tau(y', z') = \tau(y, z)\). Consider a nonanticipating map \(\tau\) with \(\tau(y, z) \leq t_F(y, z)\) for all \(y, z\) and a real-valued function \(\theta\) defined on \(R \times R^n\).

We define a game \(G_\epsilon(t_0, x_0; \theta)\) as the game with initial condition (3), dynamics (1) and payoff:

\[
P_{\tau, \theta}(y, z) = \theta(\tau(y, z), x(\tau(y, z))) + \int_{t_0}^{\tau(y, z)} h(t, x(t), y(t), z(t)) \, dt.
\]

(14)

The various upper and lower values for \(G_\epsilon(t_0, x_0; \theta)\) are defined in analogous ways in [2, Section 3] but the particularly important cases are when \(\theta\) is itself a value of the game \(G(\tau(y, z), x(\tau(y, z)))\). The following dynamic programming results are then obtained in [2].

**Theorem 2.2.** Suppose \(\tau\) is non-anticipating and \(\tau \leq t_F\). Then

\[
U(t_0, x_0) = U(t_0, x_0; U),
\]

(15)

\[
V^+(t_0, x_0) \geq V^+(t_0, x_0; V^+),
\]

(16)

and

\[
Q_s^+(t_0, x_0) \leq Q_s^+(t_0, x_0; Q_s^+),
\]

(17)

\[
Q_s^-(t_0, x_0) \geq Q_s^-(t_0, x_0; U).
\]

(18)
Similar results hold for the various lower values. Further, if
\[ \lim_{s \to 0} Q_{s}^{+}(t, x) = Q^{+}(t, x) \]
uniformly on compacta then
\[ Q^{+}(t_0, x_0) \leq Q^{+}(t_0, x_0; Q^{+}). \]  

3. Equalities for Value Functions

In this section we extend the identity (13) of Theorem 2.1 to games of survival. For each \((t, x) \in \mathbb{R}^{m+1} - \text{int} F\) define
\[ \bar{Q}^{+}(t, x) = \limsup_{(\tau, \dot{y}) \to (t, x)} Q^{+}(t, x). \]

DEFINITION 3.1. A point \((t, x)\) of \(\partial F\) is \(Q^{+}\)-regular if it satisfies
\[ \lim_{s \to 0} \bar{Q}^{+}(t, x) = g(t, x) \]
and \(U\) is continuous at \((t, x)\). \(\partial F\) is \(Q^{+}\)-regular if every point of \(\partial F\) is \(Q^{+}\)-regular.

We can now prove the following result.

THEOREM 3.2. Suppose \(f\) satisfies a Lipschitz condition in both \(t\) and \(x\)
\[ \|f(t_1, x_1, y, z) - f(t_2, x_2, y, z)\| \leq K(|t_1 - t_2| + \|x_1 - x_2\|). \]  
and \(\partial F\) is \(Q^{+}\)-regular. Then
\[ U(t_0, x_0) = V^{+}(t_0, x_0) = Q^{+}(t_0, x_0) \quad \text{for all } (t_0, x_0) \in \mathbb{R}^{m+1} - \text{int} F. \]  

Proof. Theorem 6.4 of [2] states that under these hypotheses
\[ V^{-}(t_0, x_0) = Q^{+}(t_0, x_0) \]
so only the identity involving \(U\) remains to be proved; the present proof, by contradiction, is similar. As quoted in Theorem 2.1, for fixed time games we know
\[ U(t_0, x_0) = V^{+}(t_0, x_0) = Q^{+}(t_0, x_0). \]
Suppose \((t_0, x_0) \in \partial F\). Theorem 6.3 of [2] states that if \(\partial F\) is \(Q^+\)-regular then \(Q^+\) is continuous on \(R^{m+1} - \text{int} F\) and \(\lim_{\epsilon \to 0} Q_+^+(t, x) = Q^+(t, x)\) uniformly on compacta. Therefore, from the definitions, for \((t_0, x_0) \in \partial F\)

\[
U(t_0, x_0) = V^+(t_0, x_0) = Q^+(t_0, x_0).
\]

Suppose now \((t_0, x_0) \in R^{m+1} - F\) is a point where

\[
Q^+(t_0, x_0) - U(t_0, x_0) = \epsilon > 0.
\]

Then every trajectory with initial point \((t_0, x_0)\) is contained in a sphere

\[
B_R = \{(t, x) : |t| + \|x\| < R\}
\]

for some \(R > 0\). Let

\[
\sup_{B_R \times Y \times Z} ||f(t, x, y, z)|| = M.
\]

Determine inductively a sequence \((t_n, x_n)\) in \(B_R\) for \(n \geq 1\) with \(\rho(t_n, x_n) > 0\) for all \(n\). Here \(\rho(t, x)\) denotes the distance from \((t, x)\) to \(F\). Suppose \((t_k, x_k)\) has been determined. Write

\[
\tau_k = (2(M + 1)^{-1} \rho(t_k, x_k) + t_k
\]

and consider \(G_{\tau_k}(t_k, x_k; Q^+)\).

Then by (19)

\[
Q^+(t_k, x_k) \leq Q^+_\tau_k(t_k, x_k; Q^+).
\]

\(G_{\tau_k}\) is a fixed duration game so \(Q^+(t_k, x_k) \leq U_{\tau_k}(t_k, x_k; Q^+)\). But by (15)

\[
U(t_k, x_k) = U_{\tau_k}(t_k, x_k; U)
\]

so

\[
Q^+(t_k, x_k) - U(t_k, x_k) \leq U_{\tau_k}(t_k, x_k; Q^+) - U_{\tau_k}(t_k, x_k; U).
\]

Therefore, there is a trajectory \(\xi(t)\) in \(G_{\tau_k}(t_k, x_k)\) such that

\[
Q^+(t_{k+1}, x_{k+1}) - U(t_{k+1}, x_{k+1}) \geq Q^+(t_k, x_k) - U(t_k, x_k) - \epsilon/2k^2,
\]

where

\[
t_{k+1} = \tau_k \quad \text{and} \quad \xi(t_{k+1}) = x_{k+1}.
\]

Also

\[
\rho(t_{k+1}, x_{k+1}) \geq \rho(t_k, x_k) - (M + 1)(t_{k+1} - t_k) > 0
\]
and certainly \((t_n, x_n)\) is a sequence in \(B_R\) because each \((t_n, x_n)\) is reached from \((t_0, x_0)\) by a trajectory. By construction

\[
Q^+(t_n, x_n) - U(t_n, x_n) \geq \epsilon \left(1 - \sum_{k=1}^{n} 2^{-k+1}\right) \geq \epsilon/2.
\]

For each \(n\) \((t_n, x_n) \not\in F\) so \(t_n \leq T\) for all \(n\). Thus

\[
\lim_{n \to \infty} t_n = t_0 + \sum_{k=1}^{\infty} (t_{k+1} - t_k) = \bar{t}
\]

exists and we have

\[
\lim_{k \to \infty} (t_{k+1} - t_k) = 0.
\]

As \(\|x_{k+1} - x_k\| \leq M(t_{k+1} - t_k)\) the sequence \(x_k\) converges to \(\bar{x}\) say and by (23) \(\lim_{k \to \infty} \rho(t_k, x_k) = 0\). Therefore \((t, \bar{x}) \in F\) so by the continuity of \(Q^+\) and \(U\):

\[
\lim_{k \to \infty} (Q^+(t_k, x_k) - U(t_k, x_k)) = 0.
\]

However, this contradicts (24) so we must have

\[
Q^+(t_0, x_0) = U(t_0, x_0) = \nu^+(t_0, x_0).
\]

4. LIPSCHITZ CONTINUITY

In this section we prove under certain hypotheses that the value functions are Lipschitz continuous and so almost everywhere they are solutions of the Isaac's-Bellman equation. A consequence of Section 3 is that we can confine our attention to the value \(U\), which because it satisfies, for example, identity (15), has several advantages.

**Theorem 4.1.** Suppose \(f\) satisfies the Lipschitz condition (21) in \((t, x)\), suppose \(h\) is Lipschitz in \((t, x)\) and suppose \(U\) is uniformly Lipschitz in \((t, x)\) on \(\partial F\) in any bounded region of \(R^{m+1}\). Then \(U\) is uniformly Lipschitz in any region of the form

\[
(R^{m+1} - \text{int} \ F) \cap ((T_0 \leq t \leq T) \times R^{m}).
\]

**Proof.** The initial part of the proof is similar to that of Theorem 6.3 of [2]. Consider the set \(C = B_{r} \cap (R^{m+1} - \text{int} \ F)\) for some \(r > 0\). Then any trajectory of (1) with initial point in \(C\) is contained for \(r \leq T\) in some set \(B_{R}\).

Suppose \((t_1, x_1)\) and \((t_0, x_0) \in C\) and write:

\[
\sigma = t_2 - t_1
\]

\[
\delta = |t_2 - t_1| + \|x_2 - x_1\|.
\]
For \( y = y(t) \in \mathcal{M}_1(t_1) \) and \( z = z(t) \in \mathcal{M}_2(t_1) \) we associate controls \( y^* = y^*(t) \in \mathcal{M}_1(t_2) \) and \( z^* = z^*(t) \in \mathcal{M}_2(t_2) \) by
\[
y^*(t) = y(t - \sigma), \quad z^*(t) = z(t - \sigma).
\]
The map \((y, z) \rightarrow (y^*, z^*)\) is a nonanticipating bijection and it induces a map between the spaces of strategies in \( G(t_1, x_1) \) and \( G(t_2, x_2) \). Suppose \( \xi_1(t) \) and \( \xi_2(t) \) are trajectories corresponding to \((y, z)\) and \((y^*, z^*)\), respectively, in \( G(t_1, x_1) \) and \( G(t_2, x_2) \). Then
\[
\frac{d}{dt} (\xi_1(t) - \xi_2(t + \sigma)) = f(t, \xi_1(t), y(t), z(t)) - f(t + \sigma, \xi_2(t + \sigma), y(t), z(t))
\]
from which we deduce
\[
\| \xi_1(t) - \xi_2(t + \sigma) \| \leq \delta e^{\mathcal{K}(t_1 - t)} - |\sigma| \leq \delta e^{\mathcal{K}(T - t)} - |\sigma| \quad \text{for} \quad t_1 \leq t \leq T.
\]
A stopping time \( \tau: \mathcal{M}_1(t_1) \times \mathcal{M}_2(t_1) \rightarrow R \) is defined by \( \tau = \tau(y, z) \) where
\[
\min\{\rho(\tau, \xi_1(\tau)); \rho(\tau + \sigma, \xi_2(\tau + \sigma))\} = 0
\]
but
\[
\min\{\rho(t, \xi_1(t)); \rho(t + \sigma, \xi_2(t + \sigma))\} > 0 \quad \text{for} \quad t_1 \leq t < \tau.
\]
Then \( \tau \) is non-anticipating and \( \tau \leq T \).

Suppose, without loss of generality, that \( (\tau, \xi_1(\tau)) \in \mathcal{F} \). Then by (27)
\[
\rho(\tau + \sigma, \xi_2(\tau + \sigma)) \leq \delta e^{\mathcal{K}(T + \sigma)} - |\sigma|.
\]

By hypothesis there is a constant \( K' \) such that
\[
\| h(t_1, x_1, y, z) - h(t_2, x_2, y, z) \| \leq K'(|t_1 - t_2| + \|x_1 - x_2\|)
\]
and a constant \( L \) such that
\[
\| U(\tau, \xi_1(\tau)) - U(\tau + \sigma, \xi_2(\tau + \sigma)) \| \leq L(\|\sigma\| + \|\xi_1(\tau) - \xi_2(\tau + \sigma)\|),
\]
that is:
\[
| g(\tau, \xi_1(\tau)) = U(\tau + \sigma, \xi_2(\tau + \sigma)) | \leq L \delta e^{\mathcal{K}(T + \sigma)}.
\]

Therefore,
\[
\left| \int_{t_1}^{\tau} h(t, \xi_1(t), y(t), z(t)) - \int_{t_2}^{\tau} h(t, \xi_2(t), y^*(t), z^*(t)) \, dt \right|
\leq K' \int_{t_1}^{\tau} (\|\sigma\| + \delta e^{\mathcal{K}(T + \sigma)} - |\sigma|) \, dt
\leq K'(\tau + T) \delta e^{\mathcal{K}(T + \sigma)}
\]
So,

$$| P_{r,n}(t_1, x_1; y, z) - P_{r+o,n}(t_2, x_2; y^*, z^*)| \leq e^{k(T+r)}(L + K'(r + T)).$$

(30)

The same inequality is true if $(r + \sigma, \xi_+(\theta + \sigma)) \in \partial F$ so by the remarks above on strategies in $G(t_1, x_1)$ and $G(t_2, x_2)$:

$$| U_{r,n}(t_1, x_1) - U_{r+o,n}(t_2, x_2)| \leq M\delta$$

(31)

where

$$M = e^{k(T+r)}(L + K'(r + T)).$$

By (15)

$$| U(t_1, x_1) - U(t_2, x_2)| \leq M\delta$$

and so $U$ is uniformly Lipschitz continuous in regions of the form

$$(R^{m+1} - \text{int } F) \cap ([T_0 \leq t \leq T] \times \mathbb{R}^m).$$

**Theorem 4.2.** Suppose $f$ and $h$ are Lipschitz in $(t, x)$ and suppose that there exist functions $\theta_1$, $\theta_2$, both $C^1$ on $R^{m+1} - \text{int } F$, such that $\theta_1 = \theta_2 = g$ on $\partial F$ and $L^+\theta_1 \leq 0 \leq L^+\theta_2$ on $R^{m+1} - \text{int } F$, where

$$L^+\theta = \frac{\partial \theta}{\partial t} + \min_y \max_z (\nabla \theta f + h).$$

Then the value $U$ of the game is almost everywhere a solution of the equation $L^+U = 0$ satisfying the boundary condition $U|_{\partial F} = g$.

**Remark.** It is proved in Theorem 7.1 of [2] that the existence of functions $\theta_1$ and $\theta_2$ as above implies $\partial F$ is $Q^+$-regular. Therefore, by Theorem 3.2 of the present paper $U = V^+ = Q^+$, so that $V^+$ and $Q^+$ also satisfy the equation and boundary condition.

**Proof of 4.2.** By Theorems 5.3 and 5.4 of [2], for $(t, x) \in R^{m+1} - \text{int } F$:

$$\theta_2(t, x) \leq U(t, x) \leq \theta_1(t, x).$$

Certainly

$$\theta_2(\tau, \xi) = U(\tau, \xi) = \theta_1(\tau, \xi) = g(\tau, \xi) \quad \text{for } (\tau, \xi) \in \partial F.$$ 

and so

$$\theta_2(t, x) - \theta_2(\tau, \xi) \leq U(t, x) - U(\tau, \xi) \leq \theta_1(t, x) - \theta_1(\tau, \xi).$$
By the mean value theorem for \( \theta_1 \) and \( \theta_2 \) there is a constant \( L \) such that
\[
| U(t, x) - U(\tau, \xi) | \leq L( | t - \tau | + \| x - \xi \| ) \quad \text{for } (\tau, \xi) \in \partial F.
\]
\( L \) is just a bound for the derivative of \( \theta_1 \) and \( \theta_2 \), and so is bounded in compact sets of \( \mathbb{R}^{m+1} \). \( U \) is, therefore, uniformly Lipschitz in \((t, x)\) on \( \partial F \) in any bounded region of \( \mathbb{R}^{m+1} \) and so by Theorem 4.1, \( U \) is uniformly Lipschitz in bounded regions of \( \mathbb{R}^{m+1} - \text{int } F \).

By Rademacher's Theorem (see Section 4.1 of [5]) \( U(t, x) \) is thus differentiable at almost all points \((t, x)\) and so by Theorem 4.4 of [2] \( L^+U = 0 \) almost everywhere.

We state without proof the analogous theorem for lower values:

**Theorem 4.3.** Suppose \( f \) and \( h \) are Lipschitz in \((t, x)\) and suppose there exist functions \( \theta_1, \theta_2 \) both \( C^1 \) on \( \mathbb{R}^{m+1} - \text{int } F \), such that \( \theta_1 = \theta_2 = g \) on \( \partial F \) and \( L^{-}\theta_1 \geq 0 \geq L^{-}\theta_2 \) on \( \mathbb{R}^{m+1} - \text{int } F \) where
\[
L^{-}\theta = \partial / \partial t + \max_{y} \min_{z} (\nabla \theta \cdot f + h).
\]

Then the value \( V \) of the game is almost everywhere a solution of the equation \( L^{-}V = 0 \) satisfying the boundary condition \( V |_{\partial F} = g \).

5. **Nonlinear Partial Differential Equations**

We now apply Theorem 4.3 to obtain results for boundary value problems for certain non-linear partial differential equations. This extends work of Fleming [4] on the Cauchy problem for such equations. We construct a generalized solution, that is a function differentiable almost everywhere which at points of differentiability satisfies the equation.

As in Section 1 \( F \subset \mathbb{R} \times \mathbb{R}^n \) denotes a closed set such that \( F \supset [T, \infty) \times \mathbb{R}^n \) for some fixed \( T \). Also \( t \in [t_0, \infty) \) and \( x \times \mathbb{R}^n \). Consider the nonlinear partial differential operator
\[
L\psi = \partial / \partial t + G(t, x, \nabla \psi).
\]
and \( G(t, x, \rho) \) is uniformly Lipschitz in all variables. Further, we suppose there is a constant \( B_1 \) such that
\[
| G(t, x, 0) | \leq B_1.
\]

For suitable bounds \( \alpha, \beta \) we introduce the control sets
\[
Y = \{ y \in \mathbb{R}^m : | y | \leq \alpha \}
\]
\[
Z = \{ z \in \mathbb{R}^m : | z | \leq \beta \}
\]
238  ELLIOTT AND KALTON

and the functions

\[ f(t, x, y, z) = \frac{G(t, x, y) y}{1 + |y|^2} + z \]  \hspace{1cm} (36)

\[ h(t, x, y, z) = \frac{G(t, x, y)}{1 + |y|^2} - y \cdot x. \]  \hspace{1cm} (37)

Writing

\[ G_0(y) = G(t, x, y)/1 + |y|^2 \]

and, for \( i = 1, \ldots, m, \)

\[ G_i(y) = G(t, x, y) y_i/1 + |y|^2 \]

it is proved in Fleming [4, p. 998] that there is a constant \( B_2 \) such that

\[ \left| G_0(y) - G_0(p) + \sum_{i=1}^{m} (G_i(y) - G_i(p)) p_i \right| \leq B_2 |y - p|. \]  \hspace{1cm} (38)

From this Fleming deduces that if \( |p| \leq \alpha \) and \( B_2 = \beta \) then

\[ \begin{align*}
\max_{\bar{y}} \min_{\bar{z}} (p \cdot f + h) \\
= \max_{\bar{y}} \min_{\bar{z}} (G(t, x, y)(1 + p \cdot y)(1 + |y|^2)^{-1} + \beta (p - y)) \\
= G(t, x, p).
\end{align*} \]  \hspace{1cm} (39)

**Theorem 5.1.** Consider the nonlinear partial differential equation

\[ Lv = \partial v/\partial t + G(t, x, \nabla v) = 0 \]  \hspace{1cm} (40)

together with the boundary condition \( v |_{\partial F} = g. \)

Here \( G \) is uniformly Lipschitz in \((t, x, p)\) and satisfies inequality (33). Suppose there are functions \( \theta_1 \) and \( \theta_2 \) both \( C^1 \) on \( R^{m+1} - \text{int} \ F \) such that the derivatives of \( \theta_1 \) and \( \theta_2 \) are uniformly bounded on \( R^{m+1} - \text{int} \ F \), \( \theta_1 = \theta_2 = g \) on \( \partial F \) and

\[ L(\theta_1) \geq 0 > L(\theta_2) \quad \text{on} \quad R^{m+1} - \text{int} \ F. \]

Then there is a generalized solution \( V(t, x) \) of the boundary value problem (40) in any region of the form

\[ (R^m - \text{int} F) \cap ((T_0 \leq t \leq T) \times R^m). \]

**Proof.** Consider the differential game with dynamics

\[ \dot{x}(t) = f(t, x, y, z), \]  \hspace{1cm} (40)
initial condition \( x(\tau) = \xi \) and payoff

\[
P(y, z) = g(t_F, x(T)) + \int_\tau^t h(t, x(t), y(t), z(t)) \, dt
\]

where \( f \) and \( h \) are the functions defined in (36) and (37). The control sets \( Y \) and \( Z \) are subsets of Euclidean space as in (34) and (35). The bound \( \beta \) is taken to be the constant \( B_2 \) given by inequality (38). Because the derivatives of \( \theta_1 \) and \( \theta_2 \) are bounded on \( \mathbb{R}^{m+1} - \text{int } F \) by \( L' \) say, we see as in Theorem 4.2 that the value \( V(t, x) \) of this game is uniformly Lipschitz continuous on \( \partial F \) and so on all \( \mathbb{R}^{m+1} - \text{int } F \). This Lipschitz constant \( L' \) for \( V(t, x) \) is, therefore, a bound for the derivatives of \( V \) at points of differentiability. Taking \( \alpha = \max(L', \beta) \) we see from (39) that

\[
\max_Z \min_Y (\nabla \theta_i \cdot f + h) = G(t, x, \nabla \theta_i), \quad i = 1, 2
\]

and at points of differentiability:

\[
\max_Z \min_Y (\nabla V \cdot f + h) = G(t, x, \nabla V).
\]

Therefore, for the differential game (40), (41) we have that

\[
L^{-\theta_1} = \partial \theta_1 / \partial t + G(t, x, \nabla \theta_1)
= L \theta_1 \geq 0,
\]

\[
L^{-\theta_2} = L \theta_2 \leq 0
\]

and so by Theorem 4.3 the value \( V \) of this game is a generalized solution of \( L^{-} V = LV = 0 \) satisfying \( V |_{\partial F} = g \).

Remark 5.2. If the complement \( \Omega \) of the terminal set \( F \) is bounded then we need only require that \( G(t, x, 0) \) and the derivatives of \( \theta_1 \) and \( \theta_2 \) are bounded in \( \Omega \).

In Equations (36) and (37) we have, following Fleming, given one way of constructing a differential game associated with a nonlinear equation. However, the solution obtained is independent of how the differential game is constructed.

**Theorem 5.3.** The differential game solution of the boundary value problem of Theorem 5.1 is unique.

**Proof.** We first prove that for a nonlinear equation of the form (40) with a fixed final time Cauchy condition

\[
v(T, x) = g(x)
\]
the differential game solution is unique. Suppose $A_1$ and $A_2$ are two differential games of fixed duration associated with this equation. That is, if $A_i$ has dynamics $\dot{x}_i(t) = f(t, x_i, y_i, z_i)$, initial condition $x_i(\tau) = \xi \in \mathbb{R}^m$ and payoff $\int_0^T h_i(t, x_i, y_i, z_i) \, dt$ then for $p$,

$$x \in \mathbb{R}^m \max_{y_i, z_i} \min_{p} (p \cdot f + h) = G(t, x, p)$$

for $i = 1$ and 2.

The value functions $V_1(\tau, \xi)$, $V_2(\tau, \xi)$ obtained from $A_1$ and $A_2$ respectively are then generalized solutions of the Equation (40) satisfying $V_i(T, x) = g(x)$, $i = 1, 2$. However, as proved in [3] or [4], each $V_i$ is equal almost everywhere to $\lim_{\epsilon \to 0} \phi_{\epsilon}$ where $\phi_{\epsilon}$ is the unique solution of the non-linear parabolic equation $\epsilon \nabla^2 v + L_v = 0$ with boundary condition (42). By continuity $V_1 = V_2$ everywhere.

Consider now the situation of Theorem 5.1. Suppose again that $A_1$ and $A_2$ are differential games associated with (40), with value functions $V_1(t, x)$, $V_2(t, x)$, respectively, both satisfying the boundary value problem. Suppose there is some point $(t_0, x_0) \in \mathbb{R}^{m+1} - F$ where

$$V_1(t_0, x_0) - V_2(t_0, x_0) = \epsilon > 0.$$

Now suppose $\tau$ is a constant such that $\tau \leq t_F$ for any trajectory from $(t_0, x_0)$ in both $A_1$ and $A_2$. Consider fixed time games with the same dynamics and integral payoff as $A_1$ and $A_2$, starting at $(t_0, x_0)$ and ending at time $\tau$. By the identity for $V$ corresponding to (15) we have:

$$V_1(t_0, x_0) = V_{1, \tau}(t_0, x_0; V_1),$$

$$V_2(t_0, x_0) = V_{2, \tau}(t_0, x_0; V_2).$$

However, we could consider $A_1$ with a terminal payoff $V_1 \tau$ at time $\tau$, so by uniqueness for fixed time games we have:

$$V_2(t_0, x_0) = V_{1, \tau}(t_0, x_0; V_2).$$

Therefore

$$V_1(t_0, x_0) - V_2(t_0, x_0) = V_{1, \tau}(t_0, x_0; V_1) - V_{1, \tau}(t_0, x_0; V_2)$$

and we can apply the inductive construction of Theorem 3.2 to deduce a contradiction. Therefore, $V_1(t_0, x_0) = V_2(t_0, x_0)$ for all $(t_0, x_0) \in \mathbb{R}^{m+1} - \text{int}F$ and the differential game solution is unique.

**Theorem 5.4.** Under the hypotheses of Theorem 5.1 a unique differential game solution of the boundary value problem (40) exists in $\mathbb{R}^{m+1} - \text{int}F$. 


Proof. We proved that a differential game solution exists in
\[(R^{m+1} - \text{int } F) \cap ((T_0 \leq t \leq T) \times R^m).\]
If \(T_0' \leq T_0\) a differential game solution also exists in
\[(R^{m+1} - \text{int } F) \times ((T_0' \leq t \leq T) \times R^m).\]
By Theorem 5.3 the two solutions are equal on the intersection of their domains.

REFERENCES