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POWER-BOUNDED OPERATORS AND RELATED NORM ESTIMATES

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Abstract

It is considered whether $L = \limsup_{n\to\infty} n\|T^{n+1} - T^n\| < \infty$ implies that the operator T is power-bounded. It is shown that this is so if $L < 1/e$, but it does not necessarily hold if $L = 1/e$. As part of the methods, a result of Esterle is improved, showing that if $\sigma(T) = \{1\}$ and $T \neq I$, then $\liminf_{n\to\infty} n\|T^{n+1} - T^n\| \geqslant 1/e$. The constant $1/e$ is sharp. Finally, a way to create many generalizations of Esterle's result is described, and also many conditions are given on an operator which imply that its norm is equal to its spectral radius.

1. *Introduction*

Let T be a bounded linear operator on a complex Banach space X. One of the classical problems in operator theory is to determine the relation between the size of the resolvent $(T - \lambda I)^{-1}$ when λ is near the spectrum $\sigma(T)$, and the asymptotic properties of orbits $\{T^n x : n \geqslant 0\}$ for each $x \in X$. The inequality

$$\|(T - \lambda I)^{-1}\| \leqslant \frac{C}{\operatorname{dist}(\lambda, \sigma(T))}, \qquad \lambda \in \mathbb{C} \setminus \sigma(T),$$

has been extensively studied by, for example, Benamara and Nikolski [**2**] and also, very recently, by Borovykh, Drissi and Spijker [**8**], and El-Fallah and Ransford [**12**]; see also [**20**, **22**, **24**, **30**]. Such an inequality is extreme in the sense that the converse inequality (with $C = 1$) is always satisfied. In most cases the relationship to such an inequality and the properties of the orbits are very difficult to determine.

Thus it is interesting that one has a very clean equivalence for the resolvent condition introduced by Ritt [**27**], which says there is a constant $C > 0$ such that

$$\|(T - \lambda I)^{-1}\| \leqslant \frac{C}{|\lambda - 1|} \qquad (|\lambda| > 1).$$

Nagy and Zemánek [**22**], and independently Lyubich [**19**], proved the following result (see also [**23**, Theorem 4.5.4]).

Theorem 1.1. *Let T be an operator on a complex Banach space. Then T satisfies the Ritt resolvent condition if and only if*
(i) *T is power bounded;*
(ii) $\sup_n n\|T^{n+1} - T^n\| < \infty$.

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We recall a result of Esterle [**13**] saying that if $\sigma(T)=\{1\}$ and T is not the identity operator, then $\liminf_{n\to\infty} n\|T^{n+1}-T^n\| \geqslant 1/12$. (The citation given only has 1/96; this was improved by Berkani [**3**] to 1/12.) Moreover, it was noted in [**23**, Theorem 4.5.1] that if 1 is a limit point of $\sigma(T)$, then $\limsup_{n\to\infty} n\|T^{n+1}-T^n\| \geqslant 1/e$. Thus both the Ritt resolvent condition and condition (ii) are extremal, and it is natural to ask whether these two conditions are equivalent, at least in the case when $\sigma(T)=1$. Note that it was only recently that Lyubich [**20**] constructed operators satisfying the Ritt condition and $\sigma(T)=\{1\}$.

Another reason that such a question is interesting is because of the famous Esterle–Katznelson–Tzafriri theorem [**13**, **16**], which states that if T is power bounded, and its spectrum meets the unit circle only at the point 1, then $\|T^{n+1}-T^n\| \to 0$ as $n\to\infty$. Thus a positive answer to our question would provide a partial converse.

Towards this conjecture, it is known that if $\limsup_{n\to\infty} n\|T^{n+1}-T^n\| < 1/12$, then T is power bounded in a rather trivial manner, that is, it is the direct sum of an identity operator and an operator whose spectral radius is less than 1. This follows directly from the result of Esterle cited above.

In this paper, we improve these results. We answer a conjecture of Esterle [**13**] (see also [**3**]) and show that in his result 1/12 may be replaced by $1/e$. Furthermore an example shows that $1/e$ is sharp. As a corollary we show that if $\limsup_{n\to\infty} n\|T^{n+1}-T^n\| < 1/e$, then T is power bounded. Again we provide an example to show that $1/e$ is sharp. In particular, the condition $\sup_n n\|T^{n+1}-T^n\| < \infty$ does not necessarily imply that T is power bounded. We leave open the question as to whether it implies power boundedness in the case that $\sigma(T)=\{1\}$.

We create a general framework which shows how to easily create results in the same vein as Esterle's result. For example, one can give conditions concerning $\|T^n - T^m\|$ which imply that an operator with $\sigma(T)=\{1\}$ is the identity. We also give results similar to the special case of Sinclair's theorem [**28**] considered by Bonsall and Crabb [**7**], giving many different conditions on an operator that imply that its norm is equal to its spectral radius.

Finally, we note that the condition $\sup_n n\|T^{n+1}-T^n\| < \infty$ appears in the paper by Coulhon and Saloff-Coste [**11**], and also in the papers by Blunck [**5**, **6**], which give many applications of this condition to maximal regularity problems.

Throughout this paper, we will take the Fourier transform to be $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}\,dx$ and the inverse Fourier transform to be

$$\check{g}(x) = (1/2\pi)\int_{-\infty}^{\infty} g(\xi)e^{ix\xi}\,d\xi.$$

All Banach spaces will be complex in the remainder of the paper.

2. *Esterle's result*

To illustrate the ideas, let us first give a continuous time version. The methods used are similar to those in a paper by Bonsall and Crabb [**7**] in their proof of a special case of Sinclair's theorem [**28**]. After this present paper was finished, the authors learned of the papers by Berkani, Esterle and Mokhtari [**4**] and Esterle and Mokhtari [**14**] which use similar methods. The function W described below is often called the Lambert function (see [**10**]).

THEOREM 2.1. *Let A be a bounded operator on a Banach space such that $\sigma(A)=\{0\}$. For each $t>0$ such that $\|Ae^{tA}\|\leqslant 1/et$, we have $\|A\|\leqslant 1/t$. In particular, if $\liminf_{t\to\infty} t\|Ae^{tA}\|<1/e$, then $A=0$.*

Proof. Let $f(z)=ze^z$. There is an analytic function W such that $W(f(z))=z$ in some neighborhood of 0. In particular, by the Riesz–Dunford functional calculus, $W(tAe^{tA})=tA$. Now

$$W(z)=\sum_{m=1}^{\infty} p_m z^m,$$

where, by Lagrange's inversion formula [**1**, Ch. 5, Ex. 33],

$$p_m=\frac{1}{m!}\frac{d^{m-1}}{dz^{m-1}}\left(\frac{z}{f(z)}\right)^m\bigg|_{z=0}=\frac{(-m)^{m-1}}{m!}.$$

The radius of convergence of W is $1/e$, and $\sum_{m=1}^{\infty}|p_m|e^{-m}=1$, since $f(-1)=-1/e$. Therefore $\|W(tAe^{tA})\|\leqslant 1$, and the result follows. □

THEOREM 2.2. *Let T be a bounded operator on a Banach space such that $\sigma(T)=\{1\}$. For each positive integer n such that $\|T^{n+1}-T^n\|\leqslant n^n/(n+1)^{n+1}$, we have $\|T-I\|\leqslant 1/(n+1)$. In particular, if $\liminf_{n\to\infty} n\|T^{n+1}-T^n\|<1/e$, then $T=I$.*

Proof. Let $f_n(z)=z(1+z/n)^n$. There is an analytic function W_n such that $W_n(f_n(z))=z$ in some neighborhood of 0. In particular, by the Riesz–Dunford functional calculus, $W_n(n(T^{n+1}-T^n))=n(T-I)$. Now

$$W_n(z)=\sum_{m=1}^{\infty} p_{nm} z^m$$

where

$$p_{nm}=\frac{1}{m!}\frac{d^{m-1}}{dz^{m-1}}\left(\frac{z}{f_n(z)}\right)^m\bigg|_{z=0}=\frac{(-1)^{m-1}}{n^{m-1}(nm+m-1)}\binom{nm+m-1}{m}.$$

The radius of convergence of W_n is $r_n=(n/(n+1))^{n+1}$, and $\sum_{m=1}^{\infty}|p_{nm}|r_n^m=n/(n+1)$, since $f_n(-n/(n+1))=-r_n$. Therefore $\|W_n(n(T^{n+1}-T^n))\|\leqslant n/(n+1)$ and the result follows. □

In Section 4, we will generalize this approach and give many extensions of these results.

Now let us turn our attention to whether the constant $1/e$ in Theorems 2.1 and 2.2 can be improved. By the results of Lyubich [**20**] combined with Theorem 1.1, we know that there must be some upper bound on the numbers $C>0$ such that $\sigma(T)=\{1\}$ and $\liminf_{n\to\infty} n\|T^{n+1}-T^n\|<C$ imply that $T=I$. In fact we will be able to modify the examples of Lyubich to show that $C=1/e$ is sharp.

We will consider the fractional Volterra operators, parameterized by $\alpha > 0$, on $L_p([0,1])$ for $1 \leqslant p \leqslant \infty$, given by the formula

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y)\, dy,$$

and also modified fractional Volterra operators

$$L^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} e^{y-x} f(y)\, dy.$$

It is well known (and easy to show) that $(J^\alpha)_{\alpha>0}$ is a C_0-semigroup. Similarly $(L^\alpha)_{\alpha>0}$ is also a C_0-semigroup. Thus it is easily seen that $\|(L^\alpha)^n\| = \|L^{\alpha n}\| \leqslant 1/\Gamma(\alpha n + 1)$, and hence the spectral radius of L^α is zero.

Let us also consider an extension of this operator $\tilde{L}^\alpha$ on $L_2(\mathbb{R})$ given by the formula

$$\tilde{L}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-y)^{\alpha-1} e^{y-x} f(y)\, dy.$$

This is a convolution operator. Therefore, $\widehat{\tilde{L}^\alpha f}(\xi) = m_\alpha(\xi) \hat{f}(\xi)$, where m_α is the Fourier transform of $x_+^{\alpha-1} e^{-x}/\Gamma(\alpha)$. Direct calculation shows that $m_\alpha(\xi) = (1+i\xi)^{-\alpha}$, where here we are taking the principal branch.

Next, let M denote the operator of multiplication by the indicator function of $[0,1]$, then it is not so hard to see that for any entire function f we have $f(L^\alpha) = M f(\tilde{L}^\alpha) M$, and so $\|f(L^\alpha)\| \leqslant \|f(\tilde{L}^\alpha)\|$.

Now we see that

$$\widehat{\tilde{L}^\alpha e^{-t\tilde{L}^\alpha} f}(\xi) = k(\xi) \hat{f}(\xi),$$

where $k(\xi) = m_\alpha(\xi) e^{-t m_\alpha(\xi)}$. If $0 < \alpha < 1$, then $\operatorname{Re}(m_\alpha(\xi)) > 0$, and $\lim_{\xi \to \pm\infty} \arg(m_\alpha(\xi)) = \alpha\pi/2$. Hence it is easy to see that

$$\limsup_{t\to\infty} t \left\| L^\alpha e^{-tL^\alpha} \right\| \leqslant \limsup_{t\to\infty} t \left\| \tilde{L}^\alpha e^{-t\tilde{L}^\alpha} \right\| \leqslant 1/e \cos(\alpha\pi/2).$$

This is enough to show that the constant $C = 1/e$ is sharp in Theorem 2.1. However, we can do a little better.

THEOREM 2.3. (i) *There exists an operator $A \neq 0$ on a Hilbert space, with $\sigma(A) = \{0\}$, and $\limsup_{t\to\infty} t\|Ae^{tA}\| \leqslant 1/e$.*

(ii) *There exists an operator $T \neq I$ on a Hilbert space, with $\sigma(T) = \{1\}$, and $\limsup_{n\to\infty} n\|T^{n+1} - T^n\| \leqslant 1/e$.*

Proof. Let us consider the operator on $L_2([0,1])$

$$A = -\int_0^{1/2} L^\alpha \, d\alpha,$$

where the integral converges in the strong operator topology. Lyubich [**20**] showed that the operator $B = \int_0^\infty J^\alpha \, d\alpha$ has spectral radius equal to 0 on $L_p([0,1])$ for all $1 \leqslant p \leqslant \infty$. Now both $-A$ and B are operators with positive kernels, and the kernel of $-A$ is bounded above by the kernel of B. It follows that on $L_p([0,1])$ for $p=1$ or $p=\infty$, $\|A^n\| \leqslant \|B^n\|$ for all positive integers n. Thus A has spectral radius equal to 0 on $L_p([0,1])$ for $p=1$ and $p=\infty$, and hence, by interpolation, for all $1 \leqslant p \leqslant \infty$.

We also define the operator on $L_2(\mathbb{R})$

$$\tilde{A} = -\int_0^{1/2} \tilde{L}^\alpha \, d\alpha.$$

Following the above argument, we see that $\|Ae^{tA}\| \leqslant \|\tilde{A}e^{t\tilde{A}}\|$, and $\widehat{\tilde{A}e^{t\tilde{A}}f}(\xi) = k(\xi)\hat{f}(\xi)$, where

$$|k(\xi)| = |h(\xi)| \exp(-t\mathrm{Re}(h(\xi))),$$

and

$$h(\xi) = \int_0^{1/2} m_\alpha(\xi) \, d\alpha.$$

We see that $\arg(h(\xi)) \to 0$ as $\xi \to \infty$, and hence it is an easy matter to see that $\limsup_{t\to\infty} t\|Ae^{tA}\| \leqslant 1/e$.

The second example is given by $T = e^A$. Note that $T \neq I$, because otherwise $A = \log(T) = 0$. The estimate is easily obtained since $T^{n+1} - T^n = \int_n^{n+1} Ae^{tA}\, dt$. □

3. *Power boundedness*

THEOREM 3.1. *Let T be a bounded operator on a Banach space X such that $\limsup_{n\to\infty} n\|T^{n+1} - T^n\| < 1/e$. Then X decomposes as the direct sum of two closed T-invariant subspaces such that T is the identity on one of these subspaces, and the spectral radius of T on the other subspace is strictly less than 1. In particular, T^n converges to a projection.*

Proof. First note that $\sigma(T)$ must be contained in $\{1\} \cup \{z : |z| < \alpha\}$ for some $\alpha < 1$, otherwise it is easy to see that limit superior of the spectral radius of $T^{n+1} - T^n$ is at least $1/e$ (see, for example [**23**, Theorem 4.5.1]). Thus there is a projection P that commutes with T such that $\sigma(T|_{\mathrm{image}(P)}) = \{1\}$, and the spectral radius of $T|_{\ker(P)}$ is strictly less than 1. The result now follows by applying Theorem 2.2 to $T|_{\mathrm{image}(P)}$. □

A very similar proof works also for the following continuous time version. However, we were also able to produce a different proof of this same result.

THEOREM 3.2. *Let A be a bounded operator on a Banach space X such that $L = \limsup_{t\to\infty} t\|Ae^{tA}\| < 1/e$. Then X decomposes as the direct sum of two closed A-invariant subspaces such that A is the zero operator on one of these subspaces, and on the other subspace the supremum of the real part of the spectrum is strictly negative. In particular, e^{tA} converges to a projection.*

Proof. To illustrate the ideas, let us first prove that e^{tA} converges in the case that $L < 1/4$, that is, there are constants $c < 1/4$ and $t_0 > 0$ such that $\|Ae^{tA}\| \leqslant c/t$ for $t \geqslant t_0$. It follows that $\|A^2e^{2tA}\| \leqslant c^2/t^2$ for $t \geqslant t_0$, or $\|A^2e^{tA}\| \leqslant 4c^2/t^2$ for $t \geqslant 2t_0$. Then for $t \geqslant 2t_0$ we have

$$\|Ae^{tA}\| = \left\| \lim_{\tau\to\infty} \int_t^\tau A^2e^{sA}\, ds \right\| \leqslant \frac{4c^2}{t},$$

since $Ae^{\tau A} \to 0$ as $\tau \to \infty$. Iterating this process, we get $\|Ae^{tA}\| \leqslant (4c)^{2^k}/4t$ for $t \geqslant 2^k t_0$. To put this another way, $\|Ae^{tA}\| \leqslant (4c)^{t/2t_0}/4t$ for $t \geqslant t_0$. It follows that

$$e^{t_1 A} - e^{t_2 A} = \int_{t_2}^{t_1} Ae^{sA}\, ds$$

converges to zero as $t_1, t_2 \to \infty$, that is, e^{tA} is a Cauchy sequence. Hence it converges.

The case when $L < 1/e$ is only marginally more complicated. Again, there are constants $c < 1/e$ and $t_0 > 0$ such that $\|Ae^{tA}\| \leqslant c/t$ for $t \geqslant t_0$. For any integer $M \geqslant 2$ we have $\|A^M e^{tA}\| \leqslant (cM)^M/t^M$ for $t \geqslant Mt_0$. Integrating $(M-1)$ times we obtain

$$\|Ae^{tA}\| \leqslant \frac{(cM)^M}{t(M-1)!} \qquad \text{for } t \geqslant Mt_0.$$

A simple computation shows that

$$\frac{(cM)^M}{(M-1)!} \leqslant \frac{M}{e}(ce)^M,$$

and hence iterating we obtain that if $t > M^k t_0$ then

$$\|Ae^{tA}\| \leqslant \left(\frac{M}{e}\right)^{-1/(M-1)} \left(ce\left(\frac{M}{e}\right)^{1/(M-1)}\right)^{M^k} \frac{1}{t}.$$

By choosing M sufficiently large, we see that there exist constants $c_1, c_2 > 1$ such that $\|Ae^{tA}\| \leqslant c_1 c_2^{-t}/t$ for $t \geqslant t_0$, and hence $\|e^{tA}\|$ converges.

Now it is clear that $S = \lim_{t\to\infty} e^{tA}$ is a bounded projection (because $S^2 = S$) such that $Se^{tA} = e^{tA}S = S$. Let $X_1 = \mathrm{Im}(S)$, and $X_2 = \mathrm{Ker}(S)$, so $X = X_1 \oplus X_2$. These spaces are clearly invariant under e^{tA}, and hence invariant under $A = \lim_{t\to 0}(e^{tA} - I)/t$. Since $S|_{X_1} = I|_{X_1}$ we see immediately that $e^{tA}|_{X_1} = I|_{X_1}$, and so $A|_{X_1} = \lim_{t\to 0}(e^{tA}|_{X_1} - I|_{X_1})/t = 0$. Furthermore, we have $e^{tA}|_{X_2} \to 0$. Let t_0 be such that $\|e^{t_0 A}|_{X_2}\| \leqslant 1/2$. Then the spectral radius of $e^{t_0 A}|_{X_2}$ is bounded by $1/2$, and so $\sup \mathrm{Re}(A|_{X_2}) < -\log(2)/t_0$. □

We also point out that Theorem 3.1 could be proved in a similar manner, but the details can be quite complicated. It is also possible to deduce Theorem 3.1 from Theorem 3.2. Briefly, if $\|T^{n+1} - T^n\| \leqslant (1+\epsilon)L/(n+1)$ for large enough n, then by writing out the power series for $(T-I)e^{tT}$ about $t=0$ one obtains $\|(T-I)e^{tT}\| \leqslant (1+2\epsilon)Le^t/t$ for large enough t. The result now follows quickly by applying Theorem 3.2 to $A = T - I$, remembering that $\sigma(T) \subset \{1\} \cup \{z : |z| < 1\}$.

Now we give some counterexamples to show that in general the condition $\sup_n n\|T^{n+1} - T^n\| < \infty$ does not necessarily imply power boundedness.

THEOREM 3.3. *There exists a bounded operator T on $L_1(\mathbb{R})$ such that $\sup_n n\|T^{n+1} - T^n\| < \infty$, and $\|T^n\| \approx \log n$.*

Proof. The example is a multiplier on $L_1(\mathbb{R})$ given by $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$. It is well known that such an operator is bounded if the inverse Fourier transform $\check{m}$ is a measure of bounded variation, and indeed that the norm is equal to the variation of $\check{m}$.

Let us consider the case

$$m(\xi) = \begin{cases} 1 & \text{if } |\xi| \leqslant 1 \\ \exp(1-|\xi|) & \text{if } |\xi| > 1. \end{cases}$$

An explicit computation shows that the inverse Fourier transform of m^n is

$$\frac{nx\cos(x) + n^2\sin(x)}{\pi x(x^2+n^2)}$$

and that the inverse Fourier transform of $m^{n+1} - m^n$ is

$$\frac{(x^2 - n(n+1))\cos(x) + (2nx + x)\sin(x)}{\pi(x^2+n^2)(x^2+(n+1)^2)},$$

and it is now easy to verify the claims. □

We now show that for any infinite-dimensional Banach space we can find an operator $T: X \longrightarrow X$ with $\limsup_{n\to\infty} n\|T^n - T^{n+1}\| = 1/e$ but such that $\lim_{n\to\infty}\|T^n\| = \infty$. To do this we will need to construct a special bi-orthogonal system in an arbitrary Banach space. We recall that a family $(e_j, e_j^*)_{j\in J}$ where $e_j \in X$, $e_j^* \in X^*$ for $j \in J$ is called a bi-orthogonal system if $e_j^*(e_j) = 1$ for $j \in J$ and $e_j^*(e_k) = 0$ whenever $j \neq k$. We refer to [**18**, **25**, **26**] for known results on the construction of bi-orthogonal systems in a separable Banach space.

The following proposition is the key to the construction. We will give a short proof valid in a Hilbert space and then prove a lemma which allows us to remove this restriction in an arbitrary Banach space; the reader whose main interest is in construction of an operator on a Hilbert space may simply omit this lemma.

Proposition 3.4. *Let X be an infinite-dimensional Banach space and suppose that $(c_n)_{n=1}^\infty$ is a sequence such that $\lim_{n\to\infty} c_n = \infty$ and $\lim_{n\to\infty} c_n n^{-1/2} = 0$. Then X contains a bi-orthogonal system $(e_n, e_n^*)_{n=1}^\infty$ such that the following hold.*

(a) *If $P_n x = \sum_{k=1}^n e_k^*(x)e_k$ then $\|P_n\| \geqslant c_n$.*

(b) $\lim_{n\to\infty}\|e_n^*\|\|e_n\| = 1$.

Proof. Let us suppose that X is a Hilbert space. We pick an orthonormal sequence $(f_n)_{n=0}^\infty$ and a decreasing sequence of positive reals $(\tau_m)_{m=1}^\infty$ such that $\lim_{m\to\infty}\tau_m = 0$ and $\tau_m \geqslant 2c_n n^{-1/2}$ whenever $2^{m-1} \leqslant n < 2^m$. Note that this implies that $\lim_{m\to\infty} 2^{m/2}\tau_m = \infty$ since $\lim_{n\to\infty} c_n = \infty$. Denote by $(f_n^*)_{n=0}^\infty$ the sequence bi-orthogonal to (f_n) with $\|f_n^*\| = 1$ (that is $f_n^*(x) = (x, f_n)$).

Define $e_n = f_n + \tau_m f_0$ for $n \geqslant 1$ and $2^m \leqslant n < 2^{m+1}$. Let $e_n^* = f_n^*$. Then $(e_n, e_n^*)_{n=1}^\infty$ is a bi-orthogonal system with $\lim_{n\to\infty}\|e_n\|\|e_n^*\| = 1$. Note that $\|P_1\| \geqslant \tau_1 \geqslant c_1$. Now suppose that $2^m \leqslant n < 2^{m+1}$ where $m \geqslant 1$. Then

$$\left\| \sum_{k=2^{m-1}}^{2^m-1} e_k \right\| \geqslant \tau_m 2^{m-1}.$$

On the other hand, for any $r > m+1$

$$\left\| \sum_{k=2^{m-1}}^{2^m-1} e_k - \tau_m\tau_r^{-1}2^{m-r}\sum_{k=2^{r-1}}^{2^r-1} e_k \right\| \leqslant 2^{(m-1)/2} + \tau_m\tau_r^{-1}2^{m-(1/2)(r+1)}.$$

The second term on the right tends to zero as $r\to\infty$. We deduce that $\|P_n\|\geqslant \tau_m 2^{(m-1)/2}\geqslant \frac{1}{2}\tau_{m+1}\sqrt{n}\geqslant c_n$. □

Now let us indicate how to extend this to an arbitrary Banach space. In fact it is clear that the argument goes through with minor modifications if we have the following lemma.

LEMMA 3.5. *If X is an infinite-dimensional Banach space then X contains a bi-orthogonal system $(f_n,f_n^*)_{n=0}^\infty$ such that $\|f_n\|=1$ for $n\geqslant 0$, $\|f_0^*\|=1$,*

$$\lim_{n\to\infty}\|f_n\|\|f_n^*\|=1$$

and for each $m=1,2,\ldots$ and scalars $(a_n)_{n=2^{m-1}}^{2^m-1}$,

$$\Big\|\sum_{k=2^{m-1}}^{2^m-1}a_kf_k\Big\|\leqslant 2\Big(\sum_{k=2^{m-1}}^{2^m-1}|a_k|^2\Big)^{1/2}.$$

Proof. We will need two basic facts from Banach space theory, which we review for the convenience of the reader.

(1) *Dvoretzky's theorem* [**21**]: If $\epsilon>0, m\in\mathbb{N}$ there exists $N=N(m,\epsilon)$ so that if X is an N-dimensional (real or complex) Banach space then X contains a subspace E of dimension m whose Banach–Mazur distance to ℓ_2^m is at most $1+\epsilon$.

(2) *Lemma of Krein, Krasnoselskii and Milman* [**17**] (see also [**29**, p. 269]): If E and F are two finite-dimensional subspaces of a Banach space X and $\dim F>\dim E$ then there exists $f\in F$ with $d(f,E)=\min_{e\in E}\|f-e\|=\|f\|$.

Let (σ_n) be a descending sequence with $\sigma_1<2$ and $\lim\sigma_n=1$. We will construct $(f_n,f_n^*)_{n=0}^\infty$ inductively to satisfy the conditions of the lemma and $\|f_n^*\|\leqslant\sigma_m^2$ for $2^{m-1}\leqslant n<2^m$. We start by picking f_0,f_0^* so that $\|f_0\|=\|f_0^*\|=f_0^*(f_0)$. Now suppose that $(f_n,f_n^*)_{n=0}^{2^{m-1}-1}$ have been chosen (where $m\geqslant 1$).

Let F be the linear span $[f_n]_{n=0}^{2^{m-1}-1}$. Let $X_0=\{x\in X:\ f_n^*(x)=0,\ 1\leqslant n\leqslant 2^{m-1}-1\}$. By using Dvoretzky's theorem twice we may find a subspace V of X_0 of dimension 2^m so that there are Hilbertian norms $|\cdot|_0$ and $|\cdot|_1$ on V with the properties that

$$\|x\|\leqslant|x|_0\leqslant\sigma_m\|x\|,\qquad x\in V$$

and

$$\sigma_m^{-1}d(x,F)\leqslant|x|_1\leqslant d(x,F),\qquad x\in V.$$

Let $(v_j)_{j=1}^{2^m}$ be an orthonormal basis of $(V,|\cdot|_0)$ which is also orthogonal in $(V,|\cdot|_1)$. We may assume that $|v_j|_1$ decreases in j; note that $|v_j|_1\leqslant 1$ for all j. Then for $x\in[v_j]_{j=2^{m-1}}^{2^m}$ we have $|x|_1\leqslant|v_{2^{m-1}}|_1|x|_0$ and hence $d(x,F)\leqslant\sigma_m^2|v_{2^m}|_1\|x\|$. Since $2^m+1>\dim F=2^m$ it follows from the result of Krein, Krasnoselskii and Milman cited above that $|v_{2^m}|_1\geqslant\sigma_m^{-2}$. Let $V_0=[v_j]_{j=1}^{2^m}$; then for $x\in V_0$ we have $|x|_0\leqslant\sigma_m^2|x|_1$ and hence $\|x\|\leqslant\sigma_m^2 d(x,F)$. We then define $f_{2^{m-1}+k-1}=v_k/\|v_k\|$ for $1\leqslant k\leqslant 2^m$; note that $\sigma_m^{-1}\leqslant\|v_k\|\leqslant 1$. Suppose that $a_1,\ldots,a_{2^m-1}$ are scalars and

$2^m \leqslant k \leqslant 2^{m+1}-1$. Then

$$|a_k| \leqslant \left|\sum_{j=2^m-1}^{2^m-1} a_j f_j\right|_0 \leqslant \sigma_m^2 \left|\sum_{j=2^m-1}^{2^m-1} a_j f_j\right|_1$$
$$\leqslant \sigma_m^2 d\left(\sum_{j=2^m-1}^{2^m-1} a_j f_j, F\right) \leqslant \sigma_m^2 \left\|\sum_{j=1}^{2^m-1} a_j f_j\right\|.$$

Hence by the Hahn–Banach theorem we can define bi-orthogonal functionals f_k^* for $2^{m-1} \leqslant k \leqslant 2^m - 1$ so that $\|f_k^*\| \leqslant \sigma_m^2$. To complete the inductive step we need only observe that

$$\left\|\sum_{k=2^m-1}^{2^m-1} a_k f_k\right\| \leqslant \left|\sum_{k=2^m-1}^{2^m-1} a_k \|v_k\|^{-1} v_k\right|_0 \leqslant \sigma_m \left(\sum_{k=2^m-1}^{2^m-1} |a_k|^2\right)^{1/2}. \qquad \square$$

THEOREM 3.6. *Suppose that* $0<a<\frac{1}{2}$. *On any infinite-dimensional Banach space* X*, there exists a bounded operator* $T: X \longrightarrow X$ *such that* $\limsup_{n\to\infty} n\|T^{n+1}-T^n\| = 1/e$ *and for some* $c>0$ *we have* $\|T^n\| \geqslant c(\log n)^a$ *for all* $n \geqslant 2$.

Proof. Suppose that $a < b < \frac{1}{2}$. By Proposition 3.4 we may pick a bi-orthogonal sequence $(e_n, e_n^*)_{n=1}^\infty$ in X so that $\lim_{n\to\infty} \|e_n\|\|e_n^*\| = 1$ and the operators P_n satisfy $\|P_n\| \geqslant n^b$. Let $M = \max_{n\geqslant 1} \|e_n\|\|e_n^*\|$.

Define $T: X \longrightarrow X$ by

$$Tx = x + \sum_{k=1}^\infty (\lambda_k - 1) e_k^*(x) e_k,$$

where $\lambda_k = \exp(-1/(2k)!)$. Since $|\lambda_k - 1| \leqslant 1/(2k)!$ it follows that T is bounded and $\|T\| \leqslant Me + 1$.

Consider

$$(T^n - T^{n+1})x = \sum_{k=1}^\infty (\lambda_k^n - \lambda_k^{n+1}) e_k^*(x) e_k.$$

Hence

$$n\|T^n - T^{n+1}\| \leqslant \sum_{k=1}^\infty \frac{ne^{-n/(2k)!}}{(2k)!}\|e_k\|\|e_k^*\|.$$

To estimate this sum suppose that $(2m-1)! \leqslant n < (2m+1)!$. Then

$$n\|T^n - T^{n+1}\| \leqslant M\left(\sum_{k\neq m} \frac{n}{(2k)!}e^{-n/(2k)!}\right) + \frac{n}{(2m)!}e^{-n/(2m)!}\|e_n\|\|e_n^*\|.$$

Simple estimates show that the first term converges to 0 as $n \to \infty$. We also note that $te^{-t} \leqslant e^{-1}$ for $t>0$. Hence $\limsup_n n\|T^n - T^{n+1}\| \leqslant 1/e$.

Next we estimate $\|T^n\|$. If $(2m-1)! \leqslant n \leqslant (2m+1)!$ then

$$(P_m + T^n)x = x + \sum_{k=1}^m \lambda_k^n e_k^*(x) e_k + \sum_{k=m+1}^\infty (\lambda_k^n - 1) e_k^*(x) e_k.$$

Hence

$$\|P_m + T^n\| \leqslant 1 + M\left(e^{-n/(2m)!} + \sum_{k=1}^{m-1} e^{-n/(2k)!} + \sum_{k=m+1}^{\infty} \frac{n}{(2k)!}\right).$$

Again it is simple to see that

$$\|P_m + T^n\| \leqslant M_1$$

for some suitable constant M_1 independent of n. Thus $\|T^n\| \geqslant \|P_m\| - M_1 \geqslant m^b - M_1$. Since $\log n \leqslant (2m+1)\log(2m+1)$ we have $(\log n)^a \leqslant C_1 m^b$ for a suitable constant C_1 and the result follows. □

Remark 3.7. It would be interesting to know if one can do better than the growth rate for $\|T^n\|$ of $(\log n)^{1/2-\epsilon}$ in this theorem in the case of a Hilbert space. If $X = \ell_p$, when $p > 2$ one can use the canonical basis in the construction and get an example where $\|T^n\| \geqslant c(\log n)^{1-1/p-\epsilon}$, and by duality if $p < 2$ one has an example with $\|T^n\| \geqslant c(\log n)^{1/p-\epsilon}$.

4. *A general approach*

In this section we will discuss how to extend Theorems 2.1 and 2.2 by a more general approach. We first isolate the argument used.

To do this, let us introduce a class of analytic functions. Let f be an analytic function defined on a disk $\{z : |z| < R\}$ (we allow the case when f is entire and $R = \infty$).

We will say that $f \in \mathcal{P}$ if the following hold.

(1) $f(0) = 0$.

(2) $f'(0) \neq 0$.

(3) $f(x) \in \mathbb{R}$ if $-R < x < R$.

(4) The local inverse function $\varphi = f^{-1}$ of f at the origin, which is defined in a neighborhood of 0 with $\varphi(0) = 0$, satisfies the conditions $\varphi^{(n)}(0) \geqslant 0$ for all $n \geqslant 1$.

We remark that in [**7**] the key idea is that $f(z) = \sin z$ is in class $\mathcal{P}$. In §2, we essentially used the fact that the functions ze^{-z} and $z(1 - z/n)^n$ are in class $\mathcal{P}$. Before proceeding let us include another simple example which illustrates the basic ideas. During the late 1960s a series of papers investigated conditions on the sequence of norms $\|I - T^n\|$ which imply that $T = I$. A typical result is that of Chernoff [**9**], which says that if $\sup_{n\geqslant 0} \|I - T^{2^n}\| < 1$ then $T = I$. Later Gorin [**15**] considered similar results for sequences $(q_n)_{n=0}^{\infty}$ replacing (2^n); he showed that the result is also true for sequences $q_n = 3^n, 4^n, 5^n$ but not 6^n. More generally the conclusion is true if $q_0 = 1$ and $q_{n+1}/q_n \leqslant 5$. Let us prove the following simple result.

Theorem 4.1. *Suppose that T is a bounded operator on a Banach space X. Suppose that $\lambda = 1$ is the only complex solution of the system of inequalities*

$$|1 - \lambda^n| \leqslant \|I - T^n\|, \qquad n = 1, 2, \ldots.$$

Then $T = I$.

Proof. It is clear that $\sigma(T)=\{1\}$. Assume that $0 < a < 1$. Then there exists $n \in \mathbb{N}$ so that $\|I - T^n\| < 1 - a^n$. Consider the function $f(z)=1-(1-z)^n$. This is in class $\mathcal{P}$ and φ is given by $\varphi(z)=1-(1-z)^{1/n}$ for $|z| < 1$. Let $A=I-T$ so that $A, f(A)$ are quasi-nilpotent. By the Riesz–Dunford functional calculus,

$$A=\varphi(f(A))=\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}f(A)^k.$$

In particular, $\|A\| \leqslant \varphi(\|f(A)\|) < 1-a$. It follows that $A=0$ and $T=I$. □

We now derive a corollary which is a slightly stronger form of the results of Gorin cited above. Note that if $c < 5$ we have $2\sin(\pi/(c+1)) > 1$.

Corollary 4.2. *Suppose that T is an operator on a Banach space such that $\liminf_{n\to\infty}\|I-T^n\| < 1$. Suppose that for some $c>1$ there is a sequence of positive integers $(q_n)_{n=0}^{\infty}$ with $q_0=1$ and $q_{n+1} \leqslant cq_n$ if $n \geqslant 0$ such that $\|I - T^{q_n}\| < 2\sin(\pi/(c+1))$ for $n \geqslant 0$. Then $T=I$.*

Proof. This follows very simply from Theorem 4.1. Indeed if $|1-\lambda^n| \leqslant \|I-T^n\|$ for all n then the fact that $\liminf_{n\to\infty}\|I-T^n\| < 1$ is enough to imply that $|\lambda|=1$. Now if $\lambda=e^{i\theta}$ where $|\theta| \leqslant \pi$ we have $|\theta| < 2\pi/(c+1)$. If $\theta \neq 0$, let N be the least integer such that $q_{N+1}|\theta| \geqslant 2\pi/(c+1)$. Then $q_{N+1}|\theta| \leqslant cq_N|\theta| \leqslant 2c\pi/(c+1)$ so that $|1-\lambda^{q_{N+1}}| \geqslant 2\sin(\pi/(c+1))$. This yields a contradiction and so $\lambda=1$. □

Our next lemma gives us a recipe for constructing examples of functions in class $\mathcal{P}$, when explicit calculation of the inverse function φ may be difficult.

Lemma 4.3. *Let f, h be analytic functions on the disk $\{z : |z| < R\}$. Suppose that $f \in \mathcal{P}$ and that h satisfies $h(0)>0$, $h^{(n)}(0) \geqslant 0$ for all $n \geqslant 1$ and h is nonvanishing. Then if $F(z)=f(z)/h(z)$ we have $F \in \mathcal{P}$.*

Proof. The first three conditions are obvious. For the last condition, let φ be the local inverse of f at the origin defined on some disk centered at the origin. Let $0 < \rho < \frac{1}{2}$ be chosen so that ρ is smaller than the radius of convergence of the power series expansions of h and φ around the origin and let $M \geqslant 1$ be an upper bound for $|h|, |h'|, |\varphi|$ and $|\varphi'|$ on the disk $\{z : |z| \leqslant \rho\}$. For fixed w consider the map $\Phi_w(z)=\varphi(wh(z))$ for $|z| \leqslant \rho$. Then if $M|w| < \rho$, we have $|\Phi_w(z)| \leqslant M|w||h(z)| \leqslant M^2|w|$. Thus if $|w| < M^{-2}\rho$ then Φ_w maps $\{z : |z| \leqslant \rho\}$ to itself. We also have $|\Phi_w'(z)| \leqslant M^2|w| < \rho$. We conclude that if $|w| < M^{-2}\rho$ then Φ_w maps the disk $\{z : |z| \leqslant \rho\}$ to itself and satisfies $|\Phi_w'(z)| \leqslant \frac{1}{2}$ for $|z| \leqslant \rho$. By the Banach contraction mapping principle, if $|w| < M^{-2}\rho$ we can define $g_n(w)$ by $g_n(0)=0$ and then $g_n(w)=\Phi_w(g_{n-1}(w))$ and $g_n(w)$ converges to the unique fixed point $\psi(w)$ of Φ_w. The convergence is uniform on the disk $\{w : |w| < M^{-2}\rho\}$. By induction each g_n is analytic and has non-negative coefficients in its Taylor series expansion about the origin. It follows that ψ has the same properties, and ψ is clearly the inverse function of F. □

Let us say $f \in \mathcal{P}$ is *admissible* if there exists $0 < x < R$ such that $f'(x) = 0$. If f is admissible let ξ be the least positive solution of $f'(x) = 0$ and suppose that δ is the radius of convergence of the power series expansion of φ.

LEMMA 4.4. *If f is admissible then $\delta = f(\xi)$ and*

$$\xi = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} f(\xi)^k.$$

Proof. Clearly, we have $\varphi(x) < \xi$ if $0 < x < \delta$. Let $\eta = \lim_{x \to \delta} \varphi(x)$ so that $\eta \leqslant \xi$. If $\eta = \xi$ we are done. Assume that $\eta < \xi$. Then it is clear that φ' is bounded above, for $|z| < \delta$, by $L = f'(\eta)^{-1}$. Let $U = \{\varphi(z) : |z| < \delta\})$. Let $U_n = \{z : d(z, U) < 1/n\}$. Then U is contained in the disk $\{z : |z| < \eta\}$ and so for large enough n, U_n is contained in the domain of f. Then f cannot be univalent on any U_n, for, if it were, φ could be extended to an analytic function on a disk of radius greater than δ. Pick $z_n, w_n \in U_n$ so that $w_n \neq z_n$ and $f(w_n) = f(z_n)$. We can find $w, z \in \overline{U}$ so that (w, z) is an accumulation point of (w_n, z_n). If $w = z$ then $f'(w) = 0$ and this implies that φ' cannot be bounded above, yielding a contradiction. If $w \neq z$, then we choose u_n, v_n with $|u_n| < \delta$, $|v_n| < \delta$ and $\varphi(u_n) \to w$, $\varphi(v_n) \to z$. Then $u_n, v_n \to f(w) = f(z)$ but

$$|w - z| \leqslant \limsup_{n \to \infty} L|u_n - v_n| = 0.$$

This also yields a contradiction and the proof is complete. □

THEOREM 4.5. *Let A be a quasi-nilpotent operator on a Banach space X. Suppose that f is an admissible analytic function defined on a disk $\{z : |z| < R\}$ and suppose that ξ is the smallest positive solution of $f'(x) = 0$. Then if $\|f(A)\| \leqslant f(\xi)$ we have $\|A\| \leqslant \xi$.*

Proof. Let φ be the local inverse at the origin. Then we have

$$A = \varphi(f(A)) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} (f(A))^n.$$

Hence by Lemma 4.4

$$\|A\| \leqslant \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} f(\xi)^n = \xi. \qquad \square$$

Let us note at this point that we can recapture Theorems 2.1 and 2.2 (without computing derivatives explicitly). Indeed, z belongs to $\mathcal{P}$ and hence $f(z) = ze^{-z}$ is admissible with $\xi = 1$ and $f(\xi) = 1/e$. Similarly $f(z) = (1-z)^n - (1-z)^{n+1} = z(1-z)^n$ is admissible with $\xi = 1/(n+1)$ and $f(\xi) = n^n(n+1)^{-n-1}$.

Let us now extend these results slightly. The first theorem below is a trivial application of the same ideas.

THEOREM 4.6. *Suppose that A is a quasi-nilpotent operator and for some positive integer m, $\|Ae^{-A^m}\| \leqslant (me)^{-1/m}$. Then $\|A\| \leqslant m^{-1/m}$. Hence if $\liminf_{t \to \infty} \|tAe^{-t^m A^m}\| < (me)^{-1/m}$ then $A = 0$.*

THEOREM 4.7. *Suppose that T is a bounded operator with $\sigma(T)=\{1\}$ and for some $m,n\in\mathbb{N}$ with $m>n$ we have*

$$\|T^m - T^n\| \leqslant \left(1-\frac{n}{m}\right)\left(\frac{n}{m}\right)^{n/(m-n)}.$$

Then $\|T-I\| \leqslant 1-(n/m)^{1/(m-n)}$.

Proof. We show that $f(z)=(1-z)^n-(1-z)^m$ is admissible. This follows from Lemma 4.3 since $f(z)=(1-z)^n(1-(1-z)^{m-n})$ and the function $1-(1-z)^{m-n}$ is in $\mathcal{P}$ since its local inverse at the origin is given by $1-(1-z)^{1/(m-n)}$. Now apply Theorem 4.5 to $I-T$. □

It is possible to derive other formulas of the type of Theorem 2.2 from Theorem 4.7. For example we have the following corollaries.

COROLLARY 4.8. *Suppose that T is a bounded operator with $\sigma(T)=\{1\}$. If*

$$\liminf_{m/n\to\infty} \|T^m - T^n\| < 1,$$

then $T=I$.

More precisely, if

$$\limsup_{m/n\to\infty} \frac{m}{n\log(m/n)}(1-\|T^m-T^n\|) > 1,$$

then $T=I$.

COROLLARY 4.9. *Suppose that T is a bounded operator with $\sigma(T)=\{1\}$. If*

$$\liminf_{p/n\to 0} \frac{n}{p}\|T^{n+p}-T^n\| < \frac{1}{e},$$

then $T=I$.

COROLLARY 4.10. *Suppose that T is a bounded operator with $\sigma(T)=\{1\}$. Suppose that $0<s<1$. If*

$$\liminf_{\substack{m/n\to s\\ m,n\to\infty}} \|T^m - T^n\| < (1-s)s^{s/(1-s)},$$

then $T=I$.

The next theorem is a generalization of the argument used by Bonsall and Crabb [**7**] to prove a special case of Sinclair's theorem [**28**], namely that the norm of a hermitian element A of a Banach algebra coincides with its spectral radius $r(A)$.

THEOREM 4.11. *Suppose that f is an admissible entire function. Suppose that for every $-\pi<\theta\leqslant\pi$, we have one of the following.*

(i) $\sup_{t>0}|f(te^{i\theta})| > f(\xi)$.

(ii) $|f(te^{i\theta})| < f(\xi)$ *for* $0<t<\xi$.

Let A be any operator satisfying

$$\sup_{t>0}\|f(tA)\| \leqslant f(\xi).$$

Then $r(A)=\|A\|$. In particular, if A is quasi-nilpotent then $A=0$. Furthermore, if

$$\sup_{t>0}\|f(tA)\| < f(\xi)$$

then $A=0$.

Proof. We start by observing that if $\lambda\in\sigma(A)$ then $\sup_{t>0}|f(t\lambda)|\leqslant f(\xi)$. Let $r=r(A)$. If $tr<\xi$ then by (i) and (ii) we have $|f(t\lambda)|<f(\xi)$ for every $\lambda\in\sigma(A)$. Thus applying the Riesz–Dunford functional calculus to tA we have $tA=\varphi(f(tA))$ and so

$$t\|A\| < \sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}f(\xi)^n=\xi.$$

Hence $\|A\|<\xi/t$ and it follows that $\|A\|\leqslant r(A)$.

For the last part of the theorem, assume that $\sigma(A)\neq\{0\}$. Then there exists $-\pi<\theta\leqslant\pi$ with $\sup_{t>0}|f(te^{i\theta})|<f(\xi)$. It is easy to see that this implies that φ is unbounded on the disk $\{z:|z|<f(\xi)\}$ which contradicts Lemma 4.4. Hence A is quasi-nilpotent and the conclusion follows. □

In the Bonsall–Crabb argument for Sinclair's theorem one takes $f(z)=\sin z$ and shows that it verifies the hypotheses and hence $\|\sin tT\|\leqslant 1$ for all $t>0$ implies that the norm and spectral radius of T coincide. Other functions are permissible however, and lead to more general results of this type.

THEOREM 4.12. *Let A be an operator on a Banach space X. Then each of the following conditions implies that $r(A)=\|A\|$.*

(i) $\sup_{t>0}t\|Ae^{-tA}\|\leqslant e^{-1}$.

(ii) $\sup_{t>0}t\|Ae^{-tA^m}\|\leqslant (me)^{-1/m}$ *for $m>1$ an integer.*

(iii) $\sup_{t>0}\|e^{-tA}-e^{-stA}\|\leqslant (s-1)s^{-s/(s-1)}$ *for some $s>1$.*

(iv) $\sup_{t>0}\|e^{-(s+i)tA}-e^{-(s-i)tA)}\|\leqslant 2e^{-s\arctan(1/s)}/\sqrt{1+s^2}$ *for some $s\geqslant 0$.*

In each case a strict inequality implies that $A=0$.

Proof. The first two are immediate deductions from Theorem 4.11. We then must show for the remaining cases that $e^{-z}-e^{-sz}$ for $s>1$ and $e^{-sz}\sin z$ for $s>0$ satisfy the conditions of Theorem 4.11 (the case $s=0$ is Sinclair's theorem).

Note first that $f(z)=e^{-z}(1-e^{-(s-1)z})$ is admissible by Lemma 4.3, since $1-e^{(s-1)z}\in\mathcal{P}$. In this case $\xi=(s-1)^{-1}\log s$ and $f(\xi)<1$. Let us assume that $-\pi<\theta<\pi$ and $\theta\neq 0$. If $|\theta|>\pi/2$ then $f(te^{i\theta})$ is unbounded; if $|\theta|=\pi/2$ then $\sup_{t>0}|f(te^{i\theta})|=2>1$. If $|\theta|<\pi/2$ then we observe that

$$|f(te^{i\theta})|=e^{-t\cos\theta}\big|1-e^{-(s-1)te^{i\theta}}\big|.$$

Assume that $\sup_{t>0}|f(te^{i\theta})|\leqslant f(\xi)$. Pick t_0 so that $(s-1)t_0|\sin\theta|=\pi/2$. Then

$$e^{-\xi}>f(\xi)\geqslant|f(t_0e^{i\theta})|\geqslant e^{-t_0\cos\theta}.$$

Hence $t_0\cos\theta>\xi$. Choose $t_1<t_0$ so that $t_1\cos\theta=\xi$. Then $|f(t_1e^{i\theta})|\leqslant f(\xi)$ implies that $(s-1)t_1|\sin\theta|$ is a multiple of 2π. Since $t_1<t_0$ this is impossible.

Next consider $f(z)=e^{-sz}\sin z$ where $0<\theta<\pi/2$. In this case $\xi=\arctan s^{-1}$. We can again use Lemma 4.3 to see that f is admissible. Clearly, if $|\theta|\geqslant\pi/2$ then

$f(te^{i\theta})$ is unbounded on $\{t>0\}$. If $0<|\theta|<\pi/2$ we use the fact that if $z=x+iy$ then

$$|f(z)| \geqslant e^{-sx}\cosh y|\sin x|.$$

Hence $|f(te^{i\theta})|>|f(t\cos\theta)|$ and so $\sup_{t>0}|f(te^{i\theta})|>f(\xi)$. □

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