BASES IN NON-CLOSED SUBSPACES OF ω

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A Schauder basis $\{x_n\}$ of a locally convex space E is called *equi-Schauder* if the projection maps P_n given by

$$P_n\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) = \sum_{i=1}^{n} \alpha_i x_i$$

are equicontinuous; recently, Cook [3] has shown that if E possesses a Schauder basis which is equi-Schauder for the weak topology on E, then E is isomorphic to a subspace of ω, the space of all scalar (real or complex) sequences, with the topology of co-ordinatewise convergence. In this paper I shall characterize subspaces of ω in which every Schauder basis is equi-Schauder, or in which every Schauder basis is unconditional; this will be achieved by establishing dual results for locally convex spaces of countable algebraic dimension.

In general if $\{x_n\}$ is a Schauder basis of E, the dual sequence in E' will be denoted by $\{f_n\}$ so that for $x \in E$, $x = \sum_{n=1}^{\infty} f_n(x)\, x_n$. The Schauder basis $\{x_n\}$ is *shrinking* if $\{f_n\}$ is a basis for E' in the strong topology; it is *boundedly-complete* if, whenever $\{\sum_{n=1}^{k} a_n x_n;\ k = 1, 2, \ldots\}$ is bounded, $\sum_{n=1}^{\infty} a_n x_n$ converges.

1. Spaces of countable dimension

PROPOSITION 1.1. *If E is a locally convex space of countable dimension and F is an infinite-dimensional subspace of E, then E possesses a Hamel Schauder basis $\{x_n\}$ such that $x_n \in F$ infinitely often.*

Proof. As E has countable dimension, there exists an increasing sequence E_n of subspaces such that $\dim E_n = n$ and $\bigcup E_n = E$. Choose $x_1 \in F$; then one may choose an increasing sequence $\{m_n\}$ of integers, and sequences $\{x_n\}$ in E and $\{f_n\}$ in E' such that

(i) $\{x_1, x_2, \ldots, x_{m_n}\}$ is a Hamel basis of E_{m_n},

(ii) $f_i(x_j) = \delta_{ij}$ ($\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ otherwise),

(iii) $x_{m_n+1} \in F$ for all n.

Suppose that $\{m_n\}_{n=1}^{k}$, $\{x_n\}_{n=1}^{m_k}$ and $\{f_n\}_{n=1}^{m_k}$ have been determined; then as F is of infinite dimension, there exists $x_{m_k+1} \in F$ such that $f_i(x_{m_k+1}) = 0$ for $i = 1, 2, \ldots, m_k$. Then there exists m_{k+1} such that $x_{m_k+1} \in E_{m_{k+1}}$; extend the sequence $x_1, \ldots, x_{m_k+1}$ to a basis $x_1, \ldots, x_{m_{k+1}}$ of $E_{m_{k+1}}$ so that $f_i(x_j) = 0$ for $i \leqslant m_k$ and $j > m_k$. By using the Hahn–Banach theorem one may determine $f_{m_k+1}, \ldots, f_{m_{k+1}}$ in E' such that $f_i(x_j) = \delta_{ij}$ for $1 \leqslant j \leqslant m_{k+1}$, and $m_k+1 \leqslant i \leqslant m_{k+1}$.

It is clear that (x_n) is a Hamel basis of E, while if $x \in E$, and $x = \sum_{i=1}^{\infty} \alpha_i x_i$, then $f_i(x) = \alpha_i$, so that (x_n) is also a Schauder basis of E.

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It is clear that there exist locally convex spaces of countable dimension with Schauder bases which are not Hamel bases (e.g. the subspace of the Banach space l_1 spanned by (e_n) and $e = \sum(1/n^2)\, e_n$ where $\{e_n\}$ is the natural basis of l). The following two lemmas concern the construction of Schauder bases in a space of countable dimension, and are preparatory for Theorem 1.4.

LEMMA 1.2. *Let $\{x_n\}$ be a Hamel Schauder basis of E, and let $\{m_n\}$ be an increasing sequence of integers; then, if $z_{m_k} = \sum_{i=1}^{k} x_{m_i}$ and $z_n = x_n$ for $n \neq m_k$. the sequence $\{z_n\}$ is a Hamel Schauder basis of E.*

Proof. Clearly $\{z_n\}$ is a Hamel basis of E; if further $g_n = f_n$ for $n \neq m_k$ and $g_{m_k} = f_{m_k} - f_{m_{k+1}}$, then $g_i(z_j) = \delta_{ij}$. It follows that $\{z_n\}$ is a Schauder basis of E.

LEMMA 1.3. *Let $\{x_n\}$ be a Hamel Schauder basis of E and let $\{m_n\}$ be an increasing sequence of integers such that $x_{m_n} \to 0$; then, if $z_{m_k} = x_{m_k} - x_{m_{k+1}}$, and $z_n = x_n$ for $n \neq m_k$, $\{z_n\}$ is a Schauder basis of E.*

Proof. Let $g_{m_k} = \sum_{i=1}^{k} f_{m_i}$ and let $g_n = f_n$ for $n \neq m_k$.

For given n, suppose $m_k \leqslant n < m_{k+1}$; for $x \in E$

$$\sum_{i=1}^{n} f_i(x)\, x_i - \sum_{i=1}^{n} g_i(x)\, z_i = \sum_{i=1}^{k} f_{m_i}(x)\, x_{m_i} - \sum_{i=1}^{k} g_{m_i}(x)\, z_{m_i}$$

$$= g_{m_k}(x)\, x_{m_{k+1}}.$$

However

$$\sup_k |g_{m_k}(x)| = \sup_k \left| \sum_{i=1}^{k} f_{m_i}(x) \right|$$

$$< \infty$$

as $\{x_n\}$ is a Hamel basis of E; also $x_{m_{k+1}} \to 0$. Thus

$$x = \sum_{i=1}^{\infty} f_i(x)\, x_i = \sum_{i=1}^{\infty} g_i(x)\, z_i.$$

As $g_i(z_j) = \delta_{ij}$, it follows that $\{z_i\}$ is a Schauder basis of E.

THEOREM 1.4. *Let E be a locally convex space of countable dimension; the following conditions on E are equivalent.*

(i) *Every bounded set in E is contained in a subspace of finite dimension.*

(ii) *E is sequentially complete.*

(iii) *E is semi-reflexive.*

(iv) *Every Hamel Schauder basis of E is boundedly-complete.*

(v) *Every Schauder basis of E is a Hamel basis.*

(vi) *Every Schauder basis of E is unconditional.*

(vii) *No subsequence of any Hamel Schauder basis of E converges to zero.*

Proof. The theorem will be proved according to the logical scheme

(i) ⇒ (iii) ⇒ (ii) ⇒ (iv) ⇒ (vii) ⇒ (i), and (i) ⇒ (v) ⇒ (vi) ⇒ (vii).

(i) ⇒ (iii): If (i) holds, then clearly the strong topology on E' coincides with the weak topology.

(iii) ⇒ (ii): Immediate.

(ii) ⇒ (iv): Suppose that $\{x_n\}$ is a Hamel Schauder basis of E and $\{\sum_{i=1}^{n} a_i x_i;\ n = 1, 2, \ldots\}$ is bounded; then $(a_n x_n)$ is bounded and hence, if E is sequentially complete, $\sum_{n=1}^{\infty} (1/n^2)\, a_n x_n$ converges. As $\{x_n\}$ is a Hamel basis, $(1/n^2)\, a_n = 0$ eventually so that $a_n = 0$ eventually; hence $\sum_{i=1}^{\infty} a_i x_i$ converges.

(iv) ⇒ (vii): Suppose that $\{x_n\}$ is a Hamel Schauder basis of E and $x_{m_n} \to 0$; then $(\sum_{n=1}^{k} (1/n^2)\, x_{m_n};\ k = 1, 2, \ldots)$ is bounded and hence converges. This is a contradiction as $\{x_n\}$ is a Hamel basis.

(vii) ⇒ (i): Suppose that B is a bounded absolutely convex subset of E not contained in any subspace of finite dimension. Then $F = \bigcup_{n=1}^{\infty} nB$ is an infinite-dimensional subspace of E; hence E possesses a Hamel Schauder basis $\{x_n\}$ with a subsequence $x_{m_n} \in F$. As $F = \bigcup_{n=1}^{\infty} nB$, there exist $\alpha_n \neq 0$ such that $\alpha_n x_{m_n} \in B$, and so there exists a Hamel Schauder basis $\{z_n\}$ with $z_{m_n} = (1/n)\, \alpha_n x_{m_n}$ and $z_k = x_k$ for $k \neq m_n$ such that $z_{m_n} \to 0$.

(i) ⇒ (v): If $\{x_n\}$ is a Schauder basis of E and if $\sum_{n=1}^{\infty} a_n x_n$ converges, $(a_n x_n)$ is bounded, and hence $a_n = 0$ eventually.

(v) ⇒ (vi): A Hamel basis is unconditional.

(vi) ⇒ (vii): Let $\{x_n\}$ be a Hamel Schauder basis of E with a bounded subsequence $\{x_{m_n}\}$, and suppose that every Schauder basis of E is unconditional.

Suppose that $\{\theta_n\}$ is a sequence of scalars with $\theta_n \neq 0$ and $\theta_n \to 0$. Then the sequence $\{z_n\}$ given by $z_{m_k} = \theta_k x_{m_k} - \theta_{k+1} x_{m_{k+1}}$, $z_n = x_n$ for $n \neq m_k$, is a Schauder basis of E by Lemma 1.3. Hence $\{z_n\}$ is unconditional; thus, since

$$\theta_1 x_{m_1} = \sum_{k=1}^{\infty} z_{m_k},$$

for any $f \in E'$

$$\sum_{k=1}^{\infty} |f(z_{m_k})| < \infty,$$

i.e.

$$\sum_{k=1}^{\infty} |\theta_k f(x_{m_k}) - \theta_{k+1} f(x_{m_{k+1}})| < \infty.$$

For fixed f, the signs of θ_k may be chosen so that

$$|\theta_k f(x_{m_k}) - \theta_{k+1} f(x_{m_{k+1}})| = |\theta_k|\, |f(x_{m_k})| + |\theta_{k+1}|\, |f(x_{m_{k+1}})|.$$

Hence

$$\sum_{k=1}^{\infty} |\theta_k|\, |f(x_{m_k})| < \infty$$

whenever $\theta_k \to 0$.

Therefore

$$\sum_{k=1}^{\infty} |f(x_{m_k})| < \infty.$$

In particular the sequence $\{\sum_{i=1}^{k} x_{m_i}\}_{k=1}^{\infty}$ is bounded; by Lemma 1.2, the sequence w_n, given by

$$w_{m_k} = \sum_{i=1}^{k} x_{m_i} \qquad k = 1, 2, \ldots$$

and
$$w_n = x_n \qquad n \neq m_k,$$

is a Hamel Schauder basis of E, with $\{w_{m_k}\}$ a bounded subsequence; hence, using the argument above for $f \in E'$,

$$\sum_{k=1}^{\infty} |f(w_{m_k})| < \infty.$$

However $f_1(w_{m_k}) = 1$ for all k, and so this is a contradiction; thus no subsequence of $\{x_n\}$ is bounded, and in particular no subsequence converges to zero.

2. Subspaces of ω

The results of §1 can be dualized to give results about subspaces of the space ω. All the results of this section apply trivially to finite-dimensional subspaces of ω, and so are proved for infinite dimensional subspaces. Since any subspace of ω has a weak topology, the next proposition is an immediate consequence of the theorem of Cook [3].

PROPOSITION 2.1. *Let E be a subspace of ω; then a Schauder basis $\{x_n\}$ of E is an equi-Schauder basis if and only if $\{f_n\}$ is a Hamel basis of E'.*

Using Proposition 1.1 on the space $\{E', \sigma(E', E)\}$ which is of countable dimension one obtains:

PROPOSITION 2.2. *Every subspace of ω possesses an equi-Schauder basis.*

PROPOSITION 2.3. *If E is a subspace of ω, then a Schauder basis $\{x_n\}$ of E is shrinking if and only if it is equi-Schauder.*

Proof. If (x_n) is equi-Schauder then (f_n) is a Hamel basis of E' and hence a basis in the strong topology on E'. Conversely† as E is metrizable, the strong dual of E is complete (see [1; Théorème 3, corollaire]) and so by Theorem 1.4, if (x_n) is shrinking, (f_n) is a Hamel basis of E'; therefore (x_n) is equi-Schauder.

Theorem 1.4 may be dualized in the following form:

THEOREM 2.4. *If E is a subspace of ω, then the following conditions on E are equivalent.*

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† I am grateful to the referee for pointing out a simplification in the proof of this proposition.

(i) *E is barrelled.*

(ii) *Every Schauder basis of E is shrinking.*

(iii) *Every Schauder basis of E is equi-Schauder.*

(iv) *Every Schauder basis of E is unconditional.*

Proof. Consider the space E' with the weak topology $\sigma(E', E)$. Then, as the topology on E is metrizable and hence equal to the Mackey topology $\tau(E, E')$, E' satisfies condition (iii) of Theorem 1.4 if and only if E satisfies condition (i). By Proposition 2.1, (iii) is similarly equivalent to condition (v) of Theorem 1.4; and (iv) is obviously equivalent to condition (vi) of Theorem 1.4. Finally conditions (ii) and (iii) were shown to be equivalent in Proposition 2.3.

Not every barrelled subspace of ω is closed (an example is constructed by Webb [5]). A theorem of Dynin and Mitiagin [4] states every Schauder basis of a nuclear Fréchet space is unconditional; Theorem 2.4 raises then the following problem.

Problem 1. Suppose E is nuclear and metrizable and every Schauder basis of E is unconditional; is E barrelled?

The other main problem which the theorem raises is

Problem 2. Suppose E is a metrizable locally convex space which has a Schauder basis; if every Schauder basis is equi-Schauder, is E barrelled?

I conclude with a few remarks about Schauder bases in subspaces of ω. Every closed infinite dimensional subspace of ω is isomorphic to ω.

PROPOSITION 2.5. *Let E be a subspace of ω in which every Schauder basis is boundedly-complete; then E is closed.*

Proof. By Propositions 2.2 and 2.3, E possesses a shrinking Schauder basis; by a theorem of Cook [2], E is therefore semi-reflexive and hence quasi-complete. As ω is metrizable, E is closed in ω.

Two bases $\{x_n\}$ and $\{y_n\}$ are said to be *equivalent* if $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum_{n=i}^{\infty} a_n y_n$ converges. It is easy to show that any two Schauder bases of ω are equivalent.

PROPOSITION 2.6. *If E is a subspace of ω then all Schauder bases of E are equivalent if and only if E is closed.*

Proof. Suppose that all Schauder bases of E are equivalent. Let $\{x_n\}$ be an equi-Schauder basis of E (using Proposition 2.2); then for any sequence of scalars ε_n such that $|\varepsilon_n| = 1$ for all n, $(\varepsilon_n x_n)$ is a Schauder basis of E. Thus if $\sum_{n=1}^{\infty} a_n x_n$ converges, then $\sum_{n=1}^{\infty} \varepsilon_n a_n x_n$ converges. Consequently $\{x_n\}$ is an unconditional basis of E and by Theorem 2.4, E is barrelled.

If $\{x_n\}$ is a Hamel basis of E, then so is every Schauder basis and, by Theorem 1.4, E is sequentially complete and hence closed; however no closed subspace of ω is of countably infinite dimension. Hence it may be assumed that there exists a sequence $\{a_n\}$, with $a_1 \neq 0$ and a_n not eventually zero, such that $\sum_{n=1}^{\infty} a_n x_n$ converges.

Now $\{f_n\}$ is a Hamel Schauder basis of E', and by Lemma 1.2, so is $g_n = \sum_{i=1}^{n} f_i$.

Let $\{z_n\}$ be the dual sequence to $\{g_n\}$; then $\{z_n\}$ is a Schauder basis of E. As $\sum_{n=1}^{\infty} a_n x_n$ converges, so does $\sum_{n=1}^{\infty} |a_n| x_n$, and hence, if $x = \sum_{n=1}^{\infty} |a_n| x_n$, then

$$x = \sum_{n=1}^{\infty} g_n(x) z_n = \sum_{n=1}^{\infty} \left[\sum_{i=1}^{n} |a_i| \right] z_n.$$

Thus

$$\sum_{n=1}^{\infty} \left[\sum_{i=1}^{n} |a_i| \right] x_n$$

converges, i.e. there exists a sequence, $b_i \neq 0$ for all i with $\sum_{i=1}^{\infty} b_i x_i$ convergent. If (e_n) is any scalar sequence then $c_n = d_n + e_n$ with $d_n \neq 0$ for all n, and $e_n \neq 0$ for all n. As $(b_n^{-1} d_n x_n)$ and $(b_n^{-1} e_n x_n)$ are Schauder bases of E, $\sum_{n=1}^{\infty} d_n x_n$ and $\sum_{n=1}^{\infty} e_n x_n$ converge, so that $\sum_{n=1}^{\infty} c_n x_n$ converges.

Thus $\sum_{n=1}^{\infty} c_n x_n$ converges for all scalar sequences (c_n); hence any Schauder basis of E is boundedly-complete, and so by Proposition 2.5, E is closed.

If E is closed, any Schauder basis $\{x_n\}$ of E is equi-Schauder (since E is barrelled); thus $\{f_n\}$ is a Hamel basis of E', and so, for any scalar sequence c_n, $\{\sum_{n=1}^{k} c_n x_n\}_{k=1}^{\infty}$ is a Cauchy sequence in E. As E is complete $\sum_{n=1}^{\infty} c_n x_n$ converges, and so all Schauder bases are equivalent; clearly E is isomorphic to ω.

References

1. N. Bourbaki, " Sur certains espaces vectoriels topologiques ", *Ann. de l'Inst. Fourier*, II (1950), 5–16.
2. T. A. Cook, " Schauder decompositions and semi-reflexive spaces ", *Math. Ann.*, 82 (1969), 232–235.
3. T. A. Cook, " Weakly equicontinuous Schauder bases ", *Proc. Amer. Math. Soc.*, 23 (1969), 536–537.
4. A. Dynin and B. S. Mitiagin, " Criterion for nuclearity in terms of approximate dimension ", *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.*, 8 (1960), 535–540.
5. J. H. Webb, " Sequential convergence in locally convex spaces ", *Proc. Cambridge Philos. Soc.* 64 (1968), 341–364.

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