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Trace-Class Operators and Commutators

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We identify the linear span of commutators $AB-BA$, where A is a trace-class operator and B is any bounded operator on a separable Hilbert space, or where A, B are both Hilbert–Schmidt operators, thus answering questions raised by work of Anderson, Vaserstein, and Weiss.

1. INTRODUCTION

Let $\mathscr{H}$ be an infinite-dimensional separable Hilbert space and let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded operators on $\mathscr{H}$. If $\mathscr{I}$ and $\mathscr{J}$ are two-sided ideals in $\mathscr{B}(\mathscr{H})$ then we denote by $[\mathscr{I}, \mathscr{J}]$ the linear span of the commutators $[A, B]=AB-BA$, where $A \in \mathscr{I}$ and $B \in \mathscr{J}$. Let us denote by $\mathscr{C}_p$ the usual Schatten p-class. The question of identifying $[\mathscr{C}_p, \mathscr{B}(\mathscr{H})]$ has been investigated by several authors recently, motivated partly by considerations from K-theory (cf. [2]). For the cases $p<1$ and $p>1$ the solutions are known. For $p>1$, $[\mathscr{C}_p, \mathscr{B}(\mathscr{H})]=\mathscr{C}_p$; this result is implicit in the work of Pearcy and Topping [13] (see Anderson and Vaserstein [2, Proposition 1]). For $p \leqslant 1$ there is a continuous trace on $\mathscr{C}_p$ which we denote by tr. If $p<1$ then Anderson [1] shows that $[\mathscr{C}_p, \mathscr{B}(\mathscr{H})]$ coincides with $\mathscr{C}_p^{(0)}=\{T \in \mathscr{C}_p : \operatorname{tr} T=0\}$.

For $p=1$ the situation is more complicated. Pearcy and Topping [13] asked whether $[\mathscr{C}_2, \mathscr{C}_2]=\mathscr{C}_1^{(0)}$ and this question was resolved negatively by Weiss [15, 16]. As pointed out by Anderson and Vaserstein [2, Proposition 2] we have an automatic inclusion $[\mathscr{C}_2, \mathscr{C}_2] \supset [\mathscr{C}_1, \mathscr{B}(\mathscr{H})]$ so that Weiss' result also implies $[\mathscr{C}_1, \mathscr{B}(\mathscr{H})] \neq \mathscr{C}_1^{(0)}$. This result was also obtained by Figiel (unpublished). Weiss [15, 17] also considers a special type of diagonal operator $T=\operatorname{diag}(d_1, d_2, \ldots,)$, where $d_1<0$ and $d_2 \geqslant d_3 \geqslant \cdots \geqslant 0$. If $\operatorname{tr} T=0$, i.e., $\sum d_n=0$, then $T \in [\mathscr{C}_2, \mathscr{C}_2]$ if and only if $\sum d_n \log n<\infty$. Weiss' results were extended by Anderson and Vaserstein to diagonal operators satisfying more general conditions [2].

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In this paper we identify $[\mathscr{C}_2, \mathscr{C}_2]$ and $[\mathscr{C}_1, \mathscr{B}(\mathscr{H})]$. To describe the solution let us suppose that $T \in \mathscr{C}_1$; let (λ_n) be the sequence of nonzero eigenvalues, repeated according to algebraic multiplicity, of T arranged so that $|\lambda_n|$ is decreasing (i.e., $|\lambda_n| \geqslant |\lambda_{n+1}|$ for $n \geqslant 1$). If the set is finite, then the sequence (λ_n) is completed by adding zeros. We say $T \in \mathscr{C}h_1$ if

$$\sum_{n=1}^{\infty} \frac{|\lambda_1 + \cdots + \lambda_n|}{n} < \infty.$$

Clearly if $T \in \mathscr{C}h_1$ then tr $T = 0$ but the converse is false.

Somewhat surprisingly it turns out that $\mathscr{C}h_1$ is a linear space, and in fact a quasi-Banach space under a suitable quasi-norm (Theorem 6.9 below). Furthermore it is the solution of the commutator question described above (Theorem 7.4). Thus $\mathscr{C}h_1 = [\mathscr{C}_1, \mathscr{B}(\mathscr{H})] = [\mathscr{C}_p, \mathscr{C}_q]$ whenever $1/p + 1/q = 1$. It is shown that at most six commutators are needed to reach any $T \in \mathscr{C}h_1$. The space $\mathscr{C}h_1$ can also be identified with the linear span of the quasi-nilpotent operators in $\mathscr{C}_1$.

The techniques used in proving these results are an extension of ideas concerning nonlinear commutators arising in interpolation theory (see [6, 8]). We have, however, written this paper independently of the results of [8] and have refrained from exploring connections with interpolation theory for Schatten classes. We plan to pursue this direction in a future publication. Nevertheless the reader should be aware that the notions of centralizers and bicentralizers studied in [8] and in Sections 3–4 of this paper are motivated by interpolation theory.

There are also strong connections with the theory of twisted sums of Banach spaces (see, e.g., [7], [9], or [10]). The main results are proved without using any results from this theory, but in Section 8 we do use some properties of certain nonlocally convex twisted sums to obtain some additional results.

2. Notation and Prerequisites

Some notation has already been presented in the Introduction. In this section we list some further prerequisites we will need.

First we denote by $\mathscr{K}(\mathscr{H})$ the ideal of compact operators on $\mathscr{H}$. If $A \in \mathscr{K}(\mathscr{H})$ then the singular values or s-numbers of A (see Gohberg and Krein [4, p. 26]) are denoted by $s_n = s_n(A)$. Then A has a Schmidt expansion

$$Ax = \sum_{n=1}^{\infty} s_n(x, e_n) f_n,$$

where (e_n) and (f_n) are orthonormal sequences (see [4, p. 28]).

We will also need some facts concerning quasi-normed spaces and quasi-additive maps. If X is a (complex) vector space then a map $x \to \|x\|$, $(X \to \mathbf{R})$ is called a quasi norm if (a) $\|x\| > 0$ if $x \neq 0$, (b) $\|\alpha x\| = |\alpha| \, \|x\|$ for $\alpha \in \mathbf{C}$, $x \in X$, and (c) for some C we have $\|x+y\| \leqslant C(\|x\| + \|y\|)$ for all $x, y \in X$. A complete quasi-normed space is called a quasi-Banach space (see [10]).

Let X, Y be quasi-normed spaces. Let $F: X \to Y$ be any map. Then F is called quasi-additive if for some constant Δ we have

$$\| F(x_1 + x_2) - F(x_1) - F(x_2)\| \leqslant \Delta(\|x_1\| + \|x_2\|)$$

for all $x_1, x_2 \in X$. F is called quasi-linear if additionally we have $F(\alpha x) = \alpha F(x)$ for all $\alpha \in \mathbf{C}$, $x \in X$. Quasi-additive and quasi-linear maps are intimately connected with twisted sums of X and Y; we do not use much from this theory here but the reader may consult [7, 9, 10] for further details.

We also recall that a function $\Phi: \mathbf{C}^n \to \mathbf{R}$ is called plurisubharmonic if it is upper-semi-continuous and satisfies

$$\int_0^{2\pi} \Phi(\mathbf{v} + e^{i\theta}\mathbf{w}) \frac{d\theta}{2\pi} \geqslant \Phi(\mathbf{v})$$

whenever $\mathbf{v}$, $\mathbf{w} \in \mathbf{C}^n$. If Φ is plurisubharmonic and $f: \mathbf{C}^m \to \mathbf{C}^n$ is any analytic function then $\Phi \circ f$ is also plurisubharmonic; in particular if $m = 1$, then $\Phi \circ f$ is subharmonic. See, for example, [11].

Finally, we note that we will employ the convention that C is a constant independent of A, B, T, U, V, ϕ, ψ, etc., which may, however, depend on p, q, etc., and is allowed to vary from line to line.

3. The Commutative Case

Suppose $1 \leqslant p < \infty$. We shall say that a map $\Omega: l_p \to l_\infty$ is a (strong) *l_p-centralizer* if there is a constant δ so that if $s \in l_p$, $u \in l_\infty$, and $\|u\|_\infty \leqslant 1$ then $u\Omega(s) - \Omega(us) \in l_p$ and

$$\|u\Omega(s) - \Omega(us)\|_p \leqslant \delta \, \|s\|_p.$$

The least such constant δ will be denoted by $\delta_p(\Omega)$, which will be abbreviated to δ where no confusion can arise. We note first a simple proposition (cf. [8, Lemma 4.2]).

PROPOSITION 3.1. *If $\Omega: l_p \to l_\infty$ is an l_p-centralizer then for $s_1, s_2 \in l_p$ we have*

$$\|\Omega(s_1+s_2)-\Omega(s_1)-\Omega(s_2)\|_p \leqslant 3\delta(\|s_1\|_p+\|s_2\|_p),$$

where $\delta=\delta_p(\Omega)$.

Proof. Simply write $t=|s_1|+|s_2|$, $s_1=u_1 t$, $s_2=u_2 t$, $s_1+s_2=(u_1+u_2)t$, where $\|u_1\|_\infty$, $\|u_2\|_\infty$, $\|u_1+u_2\|_\infty \leqslant 1$, then use the definition of a centralizer.

We now introduce some specific centralizers. Let $\phi: \mathbf{R} \to \mathbf{R}$ be any Lipschitz map. We denote by $L=L(\phi)$ the Lipschitz constant of ϕ, so that

$$|\phi(x_1)-\phi(x_2)| \leqslant L\,|x_1-x_2|, \qquad x_1, x_2 \in \mathbf{R}.$$

If $s \in l_\infty$ define

$$\Omega_\phi(s)=(s_n \phi(\log|s_n|)),$$

where we set $\phi(\log 0)=0$. It is clear that Ω_ϕ maps l_∞ into itself.

We also define for $s \in c_0$

$$r_s(n)=\operatorname{card}\{k: |s_k|>|s_n| \text{ or } |s_k|=|s_n| \text{ and } k \leqslant n\}$$

as long as $s_n \neq 0$ and $r_s(n)=0$ otherwise. The sequence $r_s(n)$ assigns to each n the rank of $|s_n|$ in its decreasing rearrangement with a modification introduced to cover the cases when $|s_n|$ assumes some value more than once and also to cover the case when $|s_n|>0$ infinitely often but $|s_n|$ assumes the value zero. We then define

$$\Gamma_\phi(s)=(s_n \phi(\log(r_s(n)))).$$

Notice that Γ_ϕ may not map c_0 into l_∞. However, for $p<\infty$, Γ_ϕ does map l_p into l_∞ since we have $|s_n| \leqslant r_s(n)^{-1/p}\|s\|_p$ and $|\phi(\log r_s(n))| \leqslant |\phi(0)|| + L \log r_s(n)$ as long as $r_s(n)>0$.

LEMMA 3.2. *For each such Lipschitz function ϕ, and each p, $1 \leqslant p < \infty$, Ω_ϕ and Γ_ϕ are l_p-centralizers with*

$$\delta_p(\Omega_\phi) \leqslant CL(\phi)$$
$$\delta_p(\Gamma_\phi) \leqslant CL(\phi),$$

where C is a constant depending only on p.

Proof. In fact, if $\|u\|_\infty \leqslant 1$, $s \in l_p$,

$$|(\Omega_\phi(us) - u\Omega_\phi(s))_n| \leqslant |u_n s_n| \, |\phi(\log|s_n|) - \phi(\log|u_n s_n|)$$
$$\leqslant L(\phi)|s_n| \, |u_n \log |u_n||$$

and the first half of the lemma follows with $C = e^{-1}$.

The second part is more difficult. We may suppose for convenience that (s_n) is rearranged in an appropriate order with $|s_n|$ decreasing so that $r_s(n) = n$ or $r_s(n) = 0$; it clearly suffices to consider this case. Let $w = us$, where $\|u\|_\infty \leqslant 1$, and let $\rho = r_w$. Let w_n^* be the decreasing rearrangement of $|w_n|$ so that $|w_n| \leqslant w_{\rho_n}^*$ as long as $\rho_n > 0$ and $w_n^* \leqslant |s_n|$ for all n. Then

$$|\Gamma_{\phi,n}(us) - u_n \Gamma_{\phi,n}(s)| \leqslant L\,|w_n| \left|\log \frac{n}{\rho_n}\right|,$$

where $\Gamma_\phi(t) = (\Gamma_{\phi,n}(t))$, $L = L(\phi)$, and the right-hand side vanishes if $w_n = 0$ (and $\rho_n = 0$).

For $m \in \mathbf{Z}$ let

$$A_m = \{k \in \mathbf{N} : w_k \neq 0 \qquad \text{and} \qquad 2^m k \leqslant \rho_k < 2^{m+1} k\}.$$

If $m \geqslant 0$ and $k \in A_m$ then $\rho_k \geqslant 2^m k$ and so

$$\sum_{k \in A_m} |w_k|^p \leqslant \sum_{k=1}^{\infty} (w_{2^m k}^*)^p$$
$$\leqslant \sum_{k=1}^{\infty} |s_{2^m k}|^p$$
$$\leqslant 2^{-m} \|s\|_p^p.$$

If $m < 0$ and $k \in A_m$ then $\rho_k < 2^{m+1} k$ so that $k > 2^{-(m+1)} \rho_k$. Thus

$$\sum_{k \in A_m} |w_k|^p \leqslant \sum_{k \in A_m} |s_k|^p$$
$$\leqslant \sum_{k \in A_m} |s_{2^{-(m+1)}\rho_k}|^p$$
$$\leqslant \sum_{k=1}^{\infty} |s_{2^{-(m+1)}k}|^p$$
$$\leqslant 2^{m+1} \|s\|_p^p$$
$$= 2.2^{|m|} \|s\|_p^p.$$

Thus

$$\|\Gamma_\phi(us)-u\Gamma_\phi(s)\|_p^p \leqslant (L\log 2)^p \sum_{m\in Z}(|m|+1)^p \sum_{k\in A_m}|w_k|^p$$

$$\leqslant 2(L\log 2)^p\,\|s\|_p^p \sum_{m\in Z}(|m|+1)^p\,2^{-|m|}$$

$$\leqslant CL^p\,\|s\|_p^p.$$

The families $\{\Omega_\phi\}$ and $\{\Gamma_\phi\}$ are closely related. To see this we will need the following lemma.

LEMMA 3.3. *Let $s=(s_n)$ be a decreasing nonnegative sequence in l_p. For $0<\alpha<1/p<\beta$ there exists a constant $C=C(\alpha,\beta,p)$ so that if*

$$t_n=\max_{2^{k+1}>n} 2^{(k+1)\alpha}n^{-\alpha}s_{2^k}$$

$$v_n=\max_{2^k\leqslant n} 2^{(k+1)\beta}n^{-\beta}s_{2^k}$$

then we have $\|t\|_p\leqslant C\,\|s\|_p$, $\|v\|_p\leqslant C\,\|s\|_p$. Furthermore, we have t, $v\geqslant s$ and for $n>m$, $n^\alpha t_n\leqslant m^\alpha t_m$, $n^\beta v_n\geqslant m^\beta v_m$.

Proof. For $n\in\mathbf{N}$ let $l=l(n)\in\mathbf{N}$ be defined as the unique integer such that $2^{l-1}\leqslant n<2^l$. Then

$$t_n\geqslant 2^{l\alpha}n^{-\alpha}s_{2^{l-1}}\geqslant s_n$$

and similarly $v_n\geqslant s_n$.

Further we have

$$t_n=\max_{j\geqslant 0} 2^{l\alpha}n^{-\alpha}(2^{j\alpha}s_{2^{l-1+j}})$$

$$\leqslant 2^\alpha\sum_{j=0}^{\infty}2^{j\alpha}s_{2^{l-1+j}}.$$

Now

$$\sum_{n=1}^{\infty}s^p_{2^{l-1+j}}=\sum_{k=0}^{\infty}2^k s^p_{2^{k+j}}$$

$$\leqslant 2^{1-j}\sum_{k=0}^{\infty}2^{k-1}s^p_{2^k}$$

$$\leqslant 2^{1-j}\,\|s\|_p^p.$$

Thus by Minkowski's inequality in l_p we have

$$\|t\|_p \leqslant \left(\sum_{j=0}^{\infty} 2^{(j+1)\alpha+(1-j)(1/p)} \right) \|s\|_p \leqslant C \|s\|_p.$$

Similarly

$$\begin{aligned} v_n &= \max_{0 \leqslant j \leqslant l-1} 2^{l\beta} n^{-\beta} (2^{-j\beta} s_{2^{l-1-j}}) \\ &\leqslant 2^{\beta} \sum_{j=0}^{l-1} 2^{-j\beta} s_{2^{l-1-j}} \\ &= 2^{\beta} \sum_{j=0}^{\infty} 2^{-j\beta} \chi_{nj} s_{2^{l-1-j}}, \end{aligned}$$

where $\chi_{nj} = 1$ provided $n \geqslant 2^j$ and $\chi_{nj} = 0$ otherwise. Now

$$\begin{aligned} \sum_{n=1}^{\infty} \chi_{nj}^p s_{2^{l-1-j}}^p &= \sum_{n=2^j}^{\infty} s_{2^{l-1-j}}^p \\ &= \sum_{k=0}^{\infty} 2^{j+k} s_{2^k}^p \\ &\leqslant 2^{j+1} \|s\|_p^p \end{aligned}$$

so that

$$\|v\|_p \leqslant \left(\sum_{j=0}^{\infty} 2^{\beta(1-j)+(j+1)(1/p)} \right) \|s\|_p \leqslant C \|s\|_p.$$

The other properties are clear.

PROPOSITION 3.4. *For $1 \leqslant p < \infty$ there is a constant $C = C(p)$ so that if $s \in l_p$ and $\phi: \mathbf{R} \to \mathbf{R}$ is Lipschitz we can find $\psi_j: \mathbf{R} \to \mathbf{R}$ $(j = 1, 2)$, Lipschitz functions with*

$$L(\psi_j) \leqslant CL(\phi), \qquad j = 1, 2 \tag{1}$$

$$\|\Omega_\phi(w) - \Gamma_{\psi_1}(w)\|_p \leqslant CL(\phi) \|s\|_p \qquad \text{for} \quad |w| \leqslant |s| \tag{2}$$

$$\|\Gamma_\phi(w) - \Omega_{\psi_2}(w)\|_p \leqslant CL(\phi) \|s\|_p \qquad \text{for} \quad |w| \leqslant |s|. \tag{3}$$

Proof. Fix $0 < \alpha < p < \beta$. For given $s \in l_p$ we can use Lemma 3.3 to produce t, $v \in l_p$ with t, $v \geqslant |s|$ and such that their decreasing rearrangements satisfy

$$t_n^* = \max_{2^{k+1} > n} 2^{(k+1)\alpha} n^{-\alpha} s_{2^k}^*$$

$$v_n^* = \max_{2^k \leqslant n} 2^{(k+1)\beta} n^{-\beta} s_{2^k}^*.$$

Define $\psi_1: \mathbf{R} \to \mathbf{R}$ to be Lipschitz with constant $L(\psi_1) \leqslant CL(\phi)$ and satisfying

$$\psi_1(\log n) = \phi(\log |v_n^*|).$$

This is possible since for $n > m$,

$$|\phi(\log v_n^*) - \phi(\log v_m^*)| \leqslant L(\phi) \log \frac{v_m^*}{v_n^*} \leqslant \beta L(\phi) \log \frac{n}{m}.$$

Then if $|w| \leqslant |s|$ we have $w = uv$, where $\|u\|_\infty \leqslant 1$. Thus

$$\|\Omega_\phi(w) - u\Omega_\phi(v)\|_p \leqslant CL(\phi)\|s\|_p$$
$$\|\Gamma_{\psi_1}(w) - u\Gamma_{\psi_1}(v)\|_p \leqslant CL(\phi)\|s\|_p.$$

By choice, $\Omega_\phi(v) = \Gamma_{\psi_1}(v)$ so that we obtain (2). The proof of (3) is similar using t rather than v.

Remark. If we additionally suppose ϕ to be bounded then we may also choose ψ_1 and ψ_2 to be bounded.

Let us now focus on l_1. We define the trace for $s \in l_1$ by

$$\operatorname{tr}(s) = \sum_{n=1}^{\infty} s_n.$$

Let $\mathscr{L}$ be the class of Lipschitz functions $\phi: \mathbf{R} \to \mathbf{R}$ so that $L(\phi) \leqslant 1$ and ϕ is bounded. For each $\phi \in \mathscr{L}$, the map $\Omega_\phi: l_1 \to l_1$ is continuous. For $s \in l_1$ with $\operatorname{tr}(s) = 0$ we define

$$\|s\|_h = \|s\|_1 + \sup_{\phi \in \mathscr{L}} |\operatorname{tr}(\Omega_\phi(s)|.$$

Then $\| \ \|_h$ is lower-semi-continuous. If $\lambda \in \mathbf{C}$,

$$\operatorname{tr} \Omega_\phi(\lambda s) = \lambda \operatorname{tr} \Omega_{\phi'}(s),$$

where $\phi'(x) = \phi(x + \log|\lambda|)$ so that if $\phi \in \mathscr{L}$ then $\phi' \in \mathscr{L}$. Thus $\| \ \|_h$ is homogeneous.

Also we have

$$|\operatorname{tr} \Omega_\phi(s+t)| \leqslant |\operatorname{tr} \Omega_\phi(s)| + |\operatorname{tr} \Omega_\phi(t)| + C(\|s\|_1 + \|t\|_1)$$

by Proposition 3.1, so that

$$\|s+t\|_h \leqslant (C+1)(\|s\|_h + \|t\|_h).$$

It follows that the set $h_1^{\text{sym}} = \{s : \|s\|_h < \infty\}$ is a linear space and further under the quasi norm $\| \ \|_h$ it is a rearrangement-invariant quasi-Banach space; here, completeness follows from the lower-semi-continuity of the quasi norm on l_1.

THEOREM 3.5. *Suppose* $s \in l_1$ *satisfies* $\operatorname{tr}(s) = 0$ *and* $|s_n|$ *is decreasing. Then the following are equivalent.*

(i) $s \in h_1^{\text{sym}}$.

(ii) $\sum_{n=1}^{\infty} (|s_1 + \cdots + s_n|/n) < \infty$.

(iii) $\sup_{\phi \in \mathscr{L}} |\operatorname{tr} \Gamma_\phi(s)| < \infty$.

Proof. First, it is clear that if $\phi : \mathbf{R} \to \mathbf{R}$ is any bounded Lipschitz function,

$$|\operatorname{tr} \Omega_\phi(s)| \leqslant L(\phi) \| s \|_h$$

and so Proposition 3.4 yields (i) $\Rightarrow$ (iii). Similarly (iii) $\Rightarrow$ (i).

For (ii) $\Rightarrow$ (iii), note that if $\phi \in \mathscr{L}$,

$$\sum_{n=1}^{N} s_n \phi(\log n) = \sum_{n=1}^{N} (s_1 + \cdots + s_n)(\phi(\log n) - \phi(\log(n+1))) + \phi(\log(N+1)) \sum_{n=1}^{N} s_n$$

so that

$$\left| \sum_{n=1}^{N} s_n \phi(\log n) \right| \leqslant \sum_{n=1}^{N} \frac{|s_1 + \cdots + s_n|}{n} + B \left| \sum_{n=1}^{N} s_n \right|,$$

where $B = \sup |\phi(x)|$. Letting $N \to \infty$ we obtain (iii).

For (iii) $\Rightarrow$ (ii) note that

$$\sum_{n=1}^{\infty} \frac{|\Re(s_1 + \cdots + s_n)|}{n} \leqslant \sup_{N} \sup_{\varepsilon_k = \pm 1} \left| \sum_{k=1}^{N} \varepsilon_k \frac{s_1 + \cdots + s_k}{k} \right|$$

so that, using a similar inequality for imaginary parts,

$$\sum_{n=1}^{\infty} \frac{|s_1 + \cdots + s_n|}{n} \leqslant 2 \sup_{N} \sup_{\varepsilon_k = \pm 1} \left| \sum_{k=1}^{N} \varepsilon_k \frac{s_1 + \cdots + s_k}{k} \right|$$

$$= 2 \sup_{N, \varepsilon_1, \ldots, \varepsilon_N} \left| \sum_{k=1}^{N} s_k \left(\sum_{n=k}^{N} \frac{\varepsilon_n}{n} \right) \right|.$$

Now we can choose, by linear interpolation, a bounded Lipschitz function $\phi: \mathbf{R} \to \mathbf{R}$ with

$$\phi(\log k) = \sum_{n=k}^{N} \frac{\varepsilon_n}{n}, \qquad k = 1, 2, ..., N$$

and $L(\phi) \leqslant (\log 2)^{-1}$.

Thus

$$\sum_{n=1}^{\infty} \frac{|s_1 + \cdots + s_n|}{n} \leqslant \frac{2}{\log 2} \sup_{\phi \in \mathscr{L}} |\operatorname{tr} \Gamma_\phi(s)|.$$

Remark. An equivalent quasi norm on h_1^{sym} can be given by

$$\sum_{n=1}^{\infty} |s_n| + \sum_{n=1}^{\infty} \frac{|\tilde{s}_1 + \cdots + \tilde{s}_n|}{n},$$

where $\tilde{s}$ is any rearrangement of s so that $|\tilde{s}_k|$ is decreasing. Here we define rearrangement in the loose sense that if $\alpha \in \mathbf{C} \backslash \{0\}$ then the sets $\{k \in \mathbf{N}: s_k = \alpha\}$ and $\{k \in \mathbf{N}: \tilde{s}_k = \alpha\}$ have equal cardinalities. The space h_1^{sym} is the discrete analogue of the space $H_{1,0}^{\text{sym}}$ introduced in [8].

4. The Noncommutative Case: Basic Results

Let $\mathscr{H}$ be an infinite-dimensional Hilbert space. A map $\Omega: \mathscr{C}_p \to \mathscr{B}(\mathscr{H})$ is called a *(strong) $\mathscr{C}_p$-centralizer* if there is a constant δ so that

$$\| \Omega(VA) - V\Omega(A) \|_p \leqslant \delta \, \| A \|_p$$

whenever $A \in \mathscr{C}_p$ and $V \in \mathscr{B}(\mathscr{H})$ with $\| V \| \leqslant 1$. The least such constant is denoted by $\delta_p(\Omega)$.

Proposition 4.1. *If $\Omega: \mathscr{C}_p \to \mathscr{B}(\mathscr{H})$ is a $\mathscr{C}_p$-centralizer then for $A_1, A_2 \in \mathscr{C}_p$,*

$$\| \Omega(A_1 + A_2) - \Omega(A_1) - \Omega(A_2) \|_p \leqslant 8\delta(\| A_1 \|_p + \| A_2 \|_p).$$

Proof. Let $\mathscr{H}_0$, $\mathscr{H}_1$ be two orthogonally complement closed subspaces of $\mathscr{H}$ such that $\dim \mathscr{H} = \dim \mathscr{H}_0 = \dim \mathscr{H}_1$. Let $V: \mathscr{H} \to \mathscr{H}_0$ and $W: \mathscr{H} \to \mathscr{H}_1$ be isometries, so that $V^*V = W^*W = I$, VV^* is the orthogonal projection on $\mathscr{H}_0$, WW^* is the orthogonal projection on $\mathscr{H}_1$, and $V^*W = W^*V = 0$.

Then

$$\begin{aligned} A_1 &= V^*(VA_1 + WA_2) \\ A_2 &= W^*(VA_1 + WA_2) \\ A_1 + A_2 &= (V^* + W^*)(VA_1 + WA_2) \\ &= \tfrac{1}{2}(V^* + W^*)(2VA_1 + 2WA_2). \end{aligned}$$

Thus

$$\| \Omega(A_1 + A_2) - \tfrac{1}{2}(V^* + W^*)\, \Omega(2VA_1 + 2WA_2)\|_p \leqslant 4\delta(\| A_1 \|_p + \| A_2 \|_p)$$
$$\| \Omega(A_1) - \tfrac{1}{2}V^*\Omega(2VA_1 + 2WA_2)\|_p \leqslant 2\delta(\| A_1 \|_p + \| A_2 \|_p),$$

etc., so that

$$\| \Omega(A_1 + A_2) - \Omega(A_1) - \Omega(A_2)\|_p \leqslant 8\delta(\| A_1 \|_p + \| A_2 \|_p).$$

We shall say that Ω is a *$\mathscr{C}_p$-bicentralizer* if for some constant $\Delta = \Delta_p(\Omega)$ we have

$$\| \Omega(VAW) - V\Omega(A)\, W\|_p \leqslant \Delta \, \| A \|_p$$

for $A \in \mathscr{C}_p$, $V, W \in \mathscr{B}(\mathscr{H})$, with $\| V\|$, $\| W\| \leqslant 1$. Note here that if Ω is a $\mathscr{C}_p$-centralizer with $\Omega(A^*) = \Omega(A)^*$ for every $A \in \mathscr{C}_p$ then Ω is also a bicentralizer with $\Delta_p(\Omega) \leqslant 2\delta_p(\Omega)$.

Now, for any Lipschitz function $\phi : \mathbf{R} \to \mathbf{R}$, we define maps $\Omega_\phi, \Gamma_\phi : \to \mathscr{B}(\mathscr{H})$ using the corresponding l_p-centralizers introduced in the previous section.

For $A \in \mathscr{K}(\mathscr{H})$ we denote by $s_n = s_n(A)$ the singular values of A. Then A has a Schmidt representation in the form

$$Ax = \sum_{n=1}^{\infty} s_n(x, e_n) f_n$$

or

$$A = \sum_{n=1}^{\infty} s_n e_n \otimes f_n,$$

where (e_n), (f_n) are orthonormal sequences. For each such A, we fix such a representation which henceforward will be called the prescribed representation for A.

Now define

$$\Omega_\phi(A) = \sum_{n=1}^{\infty} s_n \phi(\log |s_n|)\, e_n \otimes f_n$$

$$\Gamma_\phi(A) = \sum_{n=1}^{\infty} s_n \phi(\log n)\, e_n \otimes f_n.$$

Note that Ω_ϕ is independent of the choice of prescribed representation, but that Γ_ϕ does depend on this choice if the singular values of A are not distinct.

THEOREM 4.2. *Suppose $1 < p < \infty$. Then there is a constant $C = C(p)$ so that if $\phi: \mathbf{R} \to \mathbf{R}$ is Lipschitz then Γ_ϕ is a $\mathscr{C}_p$-bicentralizer with $\Delta_p(\Gamma_\phi) \leqslant CL(\phi)$.*

Proof. First we fix $\alpha > 0$ so that $\alpha < \min(p^{-1}, q^{-1})$, where $p^{-1} + q^{-1} = 1$. Next, suppose that $A \in \mathscr{C}_p$, $B \in \mathscr{C}_q$, and $V_1, V_2 \in \mathscr{B}(\mathscr{H})$. Let the prescribed representations of A, B be given by

$$A = \sum_{n=1}^{\infty} s_n(A)\, e_n \otimes f_n$$

$$B = \sum_{n=1}^{\infty} s_n(B)\, g_n \otimes h_n.$$

As in Lemma 3.3, we introduce

$$t_n(A) = \max_{2^{k+1} > n} 2^{(k+1)\alpha} n^{-\alpha} s_{2^k}(A)$$

$$t_n(B) = \max_{2^{k+1} > n} 2^{(k+1)\alpha} n^{-\alpha} s_{2^k}(B).$$

Note that for some constant C depending on p, q, α we have

$$\left(\sum |t_n(A)|^p\right)^{1/p} \leqslant C\left(\sum s_n(A)^p\right)^{1/p} = C\, \|A\|_p$$

and

$$\left(\sum |t_n(B)|^q\right)^{1/q} \leqslant C\, \|B\|_q.$$

For $N \in \mathbf{N}$ and $z \in \mathbf{C}$ let us define

$$F_A^N(z) = \sum_{n=1}^{N} s_n(A) \exp(z\phi(\log n))\, e_n \otimes f_n$$

$$F_B^N(z) = \sum_{n=1}^{N} s_n(B) \exp(z\phi(\log n))\, g_n \otimes h_n.$$

For any $z \in \mathbf{C}$, the operator $F_A^N(z)$ has rank N at most and its nonzero singular values are the decreasing rearrangement of the finite sequence $\{s_k(A)\exp(x\phi(\log k))\}_{k=1}^N$, where $x = \Re z$. Now $t_n(A)$ satisfies, for $n > m$,

$$t_n(A) \leqslant m^\alpha n^{-\alpha} t_m(A)$$

so that since

$$\exp(x\phi(\log n)) \leqslant (n/m)^{L|x|} \exp(x\phi(\log m))$$

the sequence $\{\exp(x\phi(\log n))\, t_n(A)\}$ is decreasing provided $L|x| \leqslant \alpha$. Hence for $|x| \leqslant L^{-1}\alpha$ we have, for $1 \leqslant k \leqslant N$,

$$s_k(F_A^N(z)) \leqslant t_k(A) \exp(x\phi(\log k))$$

and similarly,

$$s_k(F_B^N(z)) \leqslant t_k(B) \exp(x\phi(\log k)).$$

Thus

$$s_k(V_1 F_A^N(z)) \leqslant \| V_1 \|\; t_k(A) \exp(x\phi(\log k))$$

$$s_k(V_2 F_B^N(-z)) \leqslant \| V_2 \|\; t_k(B) \exp(-x\phi(\log k)).$$

Now, by [4, p. 63], we have, for $|\Re z| \leqslant L^{-1}\alpha$, and $\| V_1 \|, \| V_2 \| \leqslant 1$,

$$\begin{aligned}\sum_{k=1}^N s_k(V_1 F_A^N(z)\, V_2 F_B^N(-z)) &\leqslant \sum_{k=1}^N t_k(A)\, t_k(B) \\ &\leqslant \left(\sum_{k=1} t_k(A)^p\right)^{1/p} \left(\sum_{k=1}^N t_k(B)^q\right)^{1/q} \\ &\leqslant C \|A\|_p \|B\|_q.\end{aligned}$$

We conclude that

$$|\operatorname{tr}(V_1 F_A^N(z)\, V_2 F_B^N(-z))| \leqslant C \|A\|_p \|B\|_q$$

for $|\Re z| \leqslant L^{-1}\alpha$. Now we use Cauchy formulae to estimate the derivative of the analytic function $\varphi(z) = \operatorname{tr}(V_1 F_A^N(z)\, V_2 F_B^N(-z))$ at the origin. We see that

$$|\varphi'(0)| \leqslant CL \|A\|_p \|B\|_q,$$

where C depends only on p, q, α. On evaluation we have

$$\varphi'(0) = \operatorname{tr}(V_1 \Gamma_\phi^N(A)\, V_2 B_N - V_1 A_N V_2 \Gamma_\phi^N(B)),$$

where

$$A_N = \sum_{k=1}^{N} s_k(A)\, e_k \otimes f_k$$

$$B_N = \sum_{k=1}^{N} s_k(B)\, g_k \otimes h_k$$

$$\Gamma_\phi^N(A) = \sum_{k=1}^{N} s_k(A)\, \phi(\log k)\, e_k \otimes f_k$$

$$\Gamma_\phi^N(B) = \sum_{k=1}^{N} s_k(B)\, \phi(\log k)\, g_k \otimes h_k.$$

Hence, we have the estimate

$$|\operatorname{tr}(V_1 \Gamma_\phi^N(A)\, V_2 B_N - V_1 A_N V_2 \Gamma_\phi^N(B))| \leqslant CL\, \|A\|_p\, \|B\|_q.$$

Now suppose B is of finite rank, so that $B_N = B$ and $\Gamma_\phi^N(B) = \Gamma_\phi(B)$ for large enough N. Then, in operator norm, the sequence A_N converges to A and $\Gamma_\phi^N(A)$ converges to $\Gamma_\phi(A)$. Thus in trace-class norm the sequence $V_1 \Gamma_\phi^N(A) V_2 B_N - V_1 A_N V_2 \Gamma_\phi^N(B)$ converges to $V_1 \Gamma_\phi(A) V_2 B - V_1 A V_2 \Gamma_\phi(B)$. Thus we have the estimate

$$|\operatorname{tr}(V_1 \Gamma_\phi(A)\, V_2 B - V_1 A V_2 \Gamma_\phi(B))| \leqslant CL\, \|A\|_p\, \|B\|_q$$

as long as B is finite rank. This implies a similar estimate, replacing A by $V_1 A V_2$,

$$|\operatorname{tr}(\Gamma_\phi(V_1 A V_2)\, B - V_1 A V_2 \Gamma_\phi(B))| \leqslant CL\, \|A\|_p\, \|B\|_q$$

and hence

$$|\operatorname{tr}(V_1 \Gamma_\phi(A)\, V_2 B - \Gamma_\phi(V_1 A V_2)\, B)| \leqslant CL\, \|A\|_p\, \|B\|_q.$$

Since $B \in \mathscr{C}_q$ is arbitrary, subject to being finite rank, we have

$$\|V_1 \Gamma_\phi(A)\, V_2 - \Gamma_\phi(V_1 A V_2)\|_p \leqslant CL\, \|A\|_p$$

and the theorem is proved.

THEOREM 4.3. *Suppose $1 < p < \infty$. Then there is a constant $C = C(p)$ so that if $\phi: \mathbf{R} \to \mathbf{R}$ is Lipschitz then Ω_ϕ is a $\mathscr{C}_p$-bicentralizer with $\Delta_p(\Omega_\phi) \leqslant CL(\phi)$.*

Proof. We use Proposition 3.4. Suppose $A \in \mathscr{C}_p$ and let $s = (s_n(A))$. We can find ψ with $L(\psi) \leqslant CL(\phi)$ so that

$$\|\Omega_\phi(w) - \Gamma_\psi(w)\|_p \leqslant CL(\phi)\|s\|_p$$

for $|w| \leqslant |s|$.

Now if $\|V_1\|, \|V_2\| \leqslant 1$ then $s_n(V_1AV_2) \leqslant s_n(A)$ and so

$$\|\Omega_\phi(V_1AV_2) - \Gamma_\psi(V_1AV_2)\|_p \leqslant CL\,\|A\|_p.$$

However, by Theorem 4.2,

$$\|\Gamma_\psi(V_1AV_2) - V_1\Gamma_\psi(A)\,V_2\|_p \leqslant CL\,\|A\|_p$$

and we have

$$\|\Gamma_\psi(A) - \Omega_\phi(A)\|_p \leqslant CL\,\|A\|_p$$

so that

$$\|\Omega_\phi(V_1AV_2) - V_1\Omega_\phi(A)\,V_2\|_p \leqslant CL\,\|A\|_p.$$

Remark. We only need the above results for $p=2$. In fact, we only require the following conclusion from Proposition 4.1 and Theorem 4.3:

$$\|\Omega_\phi(A_1+A_2) - \Omega_\phi(A_1) - \Omega_\phi(A_2)\|_2 \leqslant CL(\phi)(\|A_1\|_2 + \|A_2\|_2).$$

Theorem 4.2 is modelled after [8, Theorem 5.6]. In our special circumstances, since the bicentralizer Γ_ϕ is not used again in the paper, it is possible to prove Theorem 4.3 directly by somewhat similar arguments. However, we chose our particular route to Theorem 4.3 because Theorem 4.2 can be easily modified to a general interpolation theorem; this point will be pursued in a later publication.

It should be noted that special cases of Theorems 4.2 and 4.3 corresponding to the choice $\phi(x) = x$ are implied by results of Section 4.6 of [6].

5. Some Finite-Dimensional Results

We first prove a proposition which is probably well known.

PROPOSITION 5.1. *Let $\Phi\colon \mathbf{C}^n \to \mathbf{R}$ be a plurisubharmonic function which is symmetric, i.e., for every permutation π of $\{1, 2, ..., n\}$ we have*

$$\Phi(z_1, ..., z_n) = \Phi(z_{\pi(1)}, ..., z_{\pi(n)}).$$

Define $\Psi\colon \mathbf{C}^n \to \mathbf{R}$ by

$$\Psi(z_1, ..., z_n) = \Phi(\alpha_1, ..., \alpha_n),$$

where $(\alpha_1, ..., \alpha_n)$ are the roots of the polynomial

$$\lambda^n + \sum_{k=1}^{n} z_k \lambda^{k-1} = 0$$

repeated according to multiplicity. Then Ψ is also plurisubharmonic.

Proof. We omit the simple proof that Ψ is upper-semi-continuous. Let us fix $\mathbf{w}=(w_1, ..., w_n)$ and $\mathbf{u}=(u_1, ..., u_n)$. We shall prove that for some $\delta>0$ and for all $0<r<\delta$,

$$\int_0^{2\pi} \Psi(\mathbf{w}+re^{i\theta}\,\mathbf{u})\,\frac{d\theta}{2\pi} \geqslant \Psi(\mathbf{w}).$$

Let $m=n!$ and note that it suffices to prove that for some $\delta>0$ and all $0<r<\delta$,

$$\int_0^{2\pi} \Psi(\mathbf{w}+r^m e^{im\,\theta}\,\mathbf{u})\,\frac{d\theta}{2\pi} \geqslant \Psi(\mathbf{w}).$$

Let $f(\lambda)=\lambda^n+\sum w_k\lambda^{k-1}$ and let $g(\lambda)=\sum u_k\lambda^{k-1}$, and then set $P(z, \lambda)=f(\lambda)+z^m g(\lambda)$. We now claim:

Claim. There exist $\delta>0$ and analytic functions $\zeta_1, ..., \zeta_n$ defined on the disk $\{z: |z|<\delta\}$ such that for each z, $(\zeta_j(z))$ is an enumeration of the roots of $P(z, \lambda)$.

Let us first complete the proof, assuming the chaim. We simply write

$$\begin{aligned}\int_0^{2\pi} \Psi(\mathbf{w}+r^m e^{im\theta}\mathbf{u})\,\frac{d\theta}{2\pi} &= \int_0^{2\pi} \Phi(\zeta_1(re^{i\theta}), ..., \zeta_n(re^{i\theta}))\,\frac{d\theta}{2\pi}\\ &\geqslant \Phi(\zeta_1(0), ..., \zeta_n(0))\\ &= \Psi(\mathbf{w})\end{aligned}$$

for $|z|<\delta$.

It therefore remains to establish the claim. To do this, it suffices to consider a single r-fold root, α, of f and show the existence of r analytic functions $\zeta_1, ..., \zeta_r$ defined on a neighborhood of zero such that the set $(\zeta_1(z), ..., \zeta_r(z))$ is contained in the roots of $P(z, \lambda)$ counting multiplicities.

Let us suppose α is a k-fold root of g. If $k \geqslant r$ then we simply put $\zeta_j(z)=\alpha$ for $1 \leqslant j \leqslant r$ and we are done. If $k<r$, we put $p=r-k$, $\nu=m/p$. Let $f(\lambda)=(\lambda-\alpha)^r f_0(\lambda)$, $g(\lambda)=(\lambda-\alpha)^k g_0(\lambda)$, and consider the equation

$$\mu^p f_0(\alpha+z^\nu\mu)+g_0(\alpha+z^\nu\mu)=0.$$

It follows from the Implicit Function Theorem that we can find p distinct analytic functions $\mu_1, ..., \mu_p$ defined on a neighborhood of zero satisfying this equation, corresponding to the p distinct roots of $\mu^p f_0(\alpha)+g_0(\alpha)=0$. Then, if we set $\zeta_j(z)=\alpha+z^\nu\mu_j(z)$ $(1 \leqslant j \leqslant p)$, $\zeta_j(z)=\alpha$ $(p+1 \leqslant j \leqslant r)$ we have our claimed analytic solutions and the lemma is proved.

Now let us suppose $\mathscr{H}_0$ is an n-dimensional Hilbert space.

PROPOSITION 5.2. *Let $\Phi: \mathbf{C}^n \to \mathbf{R}$ be a symmetric plurisubharmonic function and define $\hat{\Phi}: \mathscr{B}(\mathscr{H}_0) \to \mathbf{R}$ by $\hat{\Phi}(A) = \Phi(\lambda_1, ..., \lambda_n)$, where $(\lambda_1, ..., \lambda_n)$ are the eigenvalues of A repeated according to algebraic multiplicity. Then $\hat{\Phi}$ is also plurisubharmonic.*

Proof. The map which assigns to each $A \in \mathscr{B}(\mathscr{H}_0)$ the coefficients of its characteristic polynomial is holomorphic and thus the proposition is immediate from Proposition 5.1.

For $A \in \mathscr{B}(\mathscr{H}_0)$, $\phi \in \mathscr{L}$, we set

$$\sigma(A) = \sum_{k=1}^{n} |\lambda_k|$$

$$\tau_\phi(A) = \sum_{k=1}^{n} \lambda_k \phi(\log |\lambda_k|),$$

where, as usual, the eigenvalues of A, repeated according to algebraic multiplicity, are denoted $(\lambda_1, ..., \lambda_n)$ and the summand is taken to vanish if $\lambda_k = 0$.

PROPOSITION 5.3. *Let $F: \mathbf{C} \to \mathscr{B}(\mathscr{H}_0)$ be a a polynomial map, i.e., $F(z) = A_0 + A_1 z + \cdots + z^m A_m$. Then for any $\phi \in \mathscr{L}$,*

$$\left| \int_0^{2\pi} (\tau_\phi(F(e^{i\theta})) - \tau_\phi(F(0))) \frac{d\theta}{2\pi} \right| \leqslant 6 \int_0^{2\pi} \sigma(F(e^{i\theta})) \frac{d\theta}{2\pi}.$$

Proof. We start by smoothing ϕ as follows: set

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x+u)\, e^{-u^2/2}\, du.$$

Then $\psi \in \mathscr{L}$ and

$$|\psi(x) - \phi(x)| \leqslant \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} |u|\, e^{-u^2/2}\, du < 1$$

for $-\infty < x < \infty$. Also

$$\begin{aligned}
\psi''(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(u) \frac{\partial^2}{\partial x^2} (e^{-(1/2)(x-u)^2})\, du \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (u^2 - 1)\, \phi(x+u)\, e^{-u^2/2}\, du \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (u^2 - 1)(\phi(x+u) - \phi(x))\, e^{-u^2/2}\, du
\end{aligned}$$

so that

$$|\psi''(x)| \leqslant \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |u|\,|u^2-1|\, e^{-u^2/2}\, du < 3.$$

Now define $h: \mathbf{C} \to \mathbf{R}$ by

$$h(z) = 5\,|z| - (\Re z)\, \psi(\log |z|), \qquad z \neq 0$$
$$h(0) = 0.$$

h is then continuous on $\mathbf{C}$. We claim further that h is subharmonic. In fact

$$\frac{1}{2\pi} \int_0^{2\pi} h(re^{i\theta})\, d\theta = 5r \geqslant h(0)$$

for all $r > 0$, so that h verifies the mean-value property at the origin. For all other $z = re^{i\theta}$

$$\begin{aligned} \nabla^2 h &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) h \\ &= \frac{1}{r}\,(5 - \cos\theta(2\psi'(\log r) + \psi''(\log r)) \\ &\geqslant 0. \end{aligned}$$

Then $\Phi(z_1, ..., z_n) = \sum h(z_k)$ is plurisubharmonic on $\mathbf{C}^n$ and $\hat{\Phi}$, defined in Proposition 5.2, is plurisubharmonic on $\mathscr{B}(\mathscr{H}_0)$.

Hence

$$\hat{\Phi}(F(0)) \leqslant \int_0^{2\pi} \hat{\Phi}(F(e^{i\theta}))\, \frac{d\theta}{2\pi}.$$

Now, for any $A \in \mathscr{B}(\mathscr{H}_0)$,

$$|\tau_\phi(A) - \tau_\psi(A)| \leqslant \sigma(A)$$

since $|\phi(x) - \psi(x)| \leqslant 1$, and

$$\hat{\Phi}(A) = 5\sigma(A) - \Re\tau_\psi(A).$$

Thus

$$\Re \int_0^{2\pi} (\tau_\psi(F(e^{i\theta})) - \tau_\psi(F(0)))\, \frac{d\theta}{2\pi} \leqslant 5 \int_0^{2\pi} (\sigma(F(e^{i\theta})) - \sigma(F(0)))\, \frac{d\theta}{2\pi}$$

and

$$\Re \int_0^{2\pi} (\tau_\phi(F(e^{i\theta})) - \tau_\phi(F(0))) \frac{d\theta}{2\pi} \leqslant \int_0^{2\pi} 6\sigma(F(e^{i\theta})) - 4\sigma(F(0)) \frac{d\theta}{2\pi}$$

$$\leqslant 6 \int_0^{2\pi} \sigma(F(e^{i\theta})) \frac{d\theta}{2\pi}.$$

Replacing F by αF where $|\alpha| = 1$ we obain the result, since $\tau_\phi(\alpha A) = \alpha(\tau_\phi(A))$.

6. THE CLASS $\mathscr{C}h_1$

Let $\mathscr{H}$ be an infinite-dimensional Hilbert space and suppose $A \in \mathscr{K}(\mathscr{H})$. Then the spectrum of A consists of zero and a finite or countably infinite subset of eigenvalues each with finite algebraic multiplicity. Thus we can associate to A a sequence $\lambda_n(A) = \lambda_n$ of eigenvalues so that if $\alpha \neq 0$ then $\operatorname{card}\{n : \lambda_n = \alpha\}$ is equal to the multiplicity of the eigenvalue α. We refer to $(\lambda_n)_{n=1}^{\infty}$ as the eigenvalues of A, disregarding the possible ambiguity about the eigenvalue zero. Note that if $A \in \mathscr{C}_1$ then $\operatorname{tr} A = \sum \lambda_n$ by a result of Lidskii [4, p. 101].

We define $\mathscr{C}h_1$ to be the set of $A \in \mathscr{C}_1$ so that $(\lambda_n) \in h_1^{\text{sym}}$. Note that this is independent of the order or choice of (λ_n) since h_1^{sym} is rearrangement-invariant. If we choose (λ_n) so that $|\lambda_n|$ is decreasing then we see from Section 3 that $A \in \mathscr{C}h_1$ if and only if

$$\sum_{n=1}^{\infty} \frac{|\lambda_1 + \cdots + \lambda_n|}{n} < \infty.$$

In particular, if $A \in \mathscr{C}h_1$ then $\sum \lambda_n = \operatorname{tr} A = 0$. Thus $\mathscr{C}h_1 \subset \mathscr{C}_1^{(0)}$ and the inclusion is clearly strict. Note further that $A \in \mathscr{C}h_1$ if and only if $A^* \in \mathscr{C}h_1$.

Now, for $\phi \in \mathscr{L}$, define, as in the previous section,

$$\tau_\phi(A) = \sum_{n=1}^{\infty} \lambda_n \phi(\log |\lambda_n|),$$

where the summand vanishes whenever $\lambda_n = 0$. Since ϕ is bounded we have

$$|\tau_\phi(A)| \leqslant k_\phi \sum_{n=1}^{\infty} |\lambda_n| \leqslant k_\phi \, \|A\|_1,$$

where $k_\phi = \sup |\phi(t)|$. Here we use Corollary 3.1, p. 41 of [4]. We also observe that if $|\alpha| = 1$ then $\tau_\phi(\alpha A) = \alpha \tau_\phi(A)$.

For $A \in \mathscr{C}_1$ set

$$\|A\|_h = \sup_{\phi \in \mathscr{L}} |\tau_\phi(A)| + \|A\|_1.$$

From the results of Section 3, $\mathscr{C}h_1 = \{A \in \mathscr{C}_1 : \|A\|_h < \infty\}$. Exactly as in the commutative case we establish that for $\alpha \in \mathbf{C}$ we have $\|\alpha A\|_h = |\alpha| \, \|A\|_h$. However, it should be stressed that we do not know as yet that $\mathscr{C}h_1$ is a linear space or that $\| \ \|_h$ is a quasi norm.

LEMMA 6.1. *Suppose $\phi \in \mathscr{L}$ and $\lim_{t \to -\infty} \phi(t)$ exists. Then τ_ϕ is continuous on $\mathscr{C}_1$.*

Proof. We shall prove this with the assumption that $\lim_{c \to -\infty} \phi(t) = 0$. Indeed, if not we can replace ϕ by $\phi - c$ for suitable c and then obtain the conclusion.

Suppose $A_n \to A$ in $\mathscr{C}_1$ and that $\mathscr{M} = \sup \|A_n\|_1$. Pick $\delta_0 > 0$ so that $|\phi(t)| < \varepsilon/(4M)$ for $t < \log \delta_0$ and A has no eigenvalue of absolute value δ_0. Suppose $\lambda_1, ..., \lambda_m$ are the eigenvalues of A with $|\lambda_k| > \delta_0$, and let $r_1, ..., r_m$ be their associated multiplicities. Let $r = \sum r_k$. For each k, $1 \leqslant k \leqslant m$, let $\delta_k > 0$ be chosen so that

$$|z\phi(\log |z|) - \lambda_k \phi(\log |\lambda_k)| < \varepsilon/2r$$

for $|z - \lambda_k| < \delta_k$ and also so that if $1 \leqslant j \neq k \leqslant m$ then $|\lambda_j - \lambda_k| > \delta_j + \delta_k$.

Let $\gamma_0 = \{z : |z| < \delta_0\}$ and $\gamma_k = \{z : |z - \lambda_k| < \delta_k\}$ for $1 \leqslant k \leqslant m$. Then [4, p. 18] there exists n_0 so that if $n \geqslant n_0$ the spectrum of A_n is entirely contained in $\gamma_0 \cup \gamma_1 \cup \cdots \cup \gamma_m$ and the total number of eigenvalues, counting multiplicities, in each γ_k $(1 \leqslant k \leqslant m)$ is exactly r_k.

For $n \geqslant n_0$ let (μ_j) be the nonzero eigenvalues of A_n counted according to multiplicity. Then for $1 \leqslant k \leqslant m$,

$$\left| \sum_{\mu_j \in \gamma_k} \mu_j \phi(\log |\mu_j|) - r_k \lambda_k \phi(\log |\lambda_k|) \right| \leqslant \frac{r_j \varepsilon}{2r}.$$

Also

$$\left| \sum_{\mu_j \in \gamma_0} \mu_j \phi(\log |\mu_j|) \right| \leqslant \frac{\varepsilon}{4M} \sum |\mu_j| \leqslant \frac{\varepsilon}{4}.$$

Similarly

$$\left| \tau_\phi(A) - \sum_{k=1}^{m} r_k \lambda_k \phi(\log |\lambda_k|) \right| \leqslant \frac{\varepsilon}{4}.$$

Hence for $n \geqslant n_0$,

$$|\tau_\phi(A) - \tau_\phi(A_n)| \leqslant \varepsilon.$$

It is immediate from Lemma 6.1 that:

LEMMA 6.2. *If $\phi \in \mathscr{L}$, there is a sequence $\phi_n \in \mathscr{L}$ so that τ_{ϕ_n} is continuous and for each $A \in \mathscr{C}_1$*

$$\tau_\phi(A) = \lim_{n \to \infty} \tau_{\phi_n}(A)$$

$$\sup |\tau_{\phi_n}(A)| \leqslant k_\phi \, \|A\|_1.$$

Proof. Simply suppose each ϕ_n satisfies the conditions of Lemma 6.1 together with $\phi_n(t) \to \phi(t)$, $|\phi_n(t)| \leqslant k_\phi$ for all n, t.

LEMMA 6.3. *$\| \ \|_h$ is lower-semi-continuous on $\mathscr{C}_1$ as a map into* $[0, \infty]$.

Proof. This is immediate from Lemma 6.2.

LEMMA 6.4. *Suppose $\phi \in \mathscr{L}$ and that $F: \mathbf{C} \to \mathscr{C}_1$ is a polynomial. Then*

$$\left| \int_0^{2\pi} (\tau_\phi(F(e^{i\theta})) - \tau_\phi(F(0))) \frac{d\theta}{2\pi} \right| \leqslant 6 \int_0^{2\pi} \|F(e^{i\theta})\|_1 \frac{d\theta}{2\pi}.$$

Proof. It follows from Lemma 6.2 that it is sufficient to prove the lemma under the additional hypothesis that τ_ϕ is continuous on $\mathscr{C}_1$. In this case, however, it is sufficient to prove the lemma under the hypothesis that $F(z) = A_0 + A_1 z + \cdots + A_m z^m$ with $\mathscr{R}(A_k) < \infty$ for $1 \leqslant k \leqslant m$; this follows from an easy approximation argument. Now we may restrict to the finite-dimensional space $\mathscr{H}_0 = \mathscr{R}(A_0) + \mathscr{R}(A_1) + \cdots + \mathscr{R}(A_m)$ and apply Proposition 5.3. If we note that

$$\sigma(F(e^{i\theta})|_{\mathscr{H}_0}) \leqslant \| F(e^{i\theta})\|_1$$

then the result will follow.

The next result is the central theorem of the section

THEOREM 6.5. *There is a constant C so that if A, $B \in \mathscr{C}_2$, $\phi \in \mathscr{L}$ then*

$$|\tau_\phi(AB) - \operatorname{tr}(A\Omega_\psi(B))| \leqslant C(\|A\|_2 + \|B\|_2)^2,$$

where $\psi(t) = \phi(2t)$.

Proof. We consider the polynomial $F(z) = (A + zB^*)(B + zA^*)$. Then $F(0) = AB$.

For $0 \leqslant \theta < 2\pi$, let

$$P_\theta = (B + e^{i\theta}A^*)^* \, (B + e^{i\theta}A^*)$$

so that P_θ is a positive operator, and $F(e^{i\theta}) = e^{i\theta}P_\theta$. Thus

$$\begin{aligned}\int_0^{2\pi} \| F(e^{i\theta})\|_1 \frac{d\theta}{2\pi} &= \int_0^{2\pi} \operatorname{tr}(P_\theta) \frac{d\theta}{2\pi} \\ &= \operatorname{tr}(AA^* + B^*B) \\ &= \| A \|_2^2 + \| B \|_2^2.\end{aligned}$$

For fixed θ suppose

$$B + e^{i\theta}A^* = \sum_{n=1}^{\infty} s_n e_n \otimes f_n$$

is the prescribed representation. Then

$$P_\theta = \sum_{n=1}^{\infty} s_n^2 e_n \otimes e_n$$

and

$$\begin{aligned}\tau_\phi(P_\theta) &= \sum_{n=1}^{\infty} s_n^2 \phi(2 \log s_n) \\ &= \sum_{n=1}^{\infty} s_n^2 \psi(\log s_n) \\ &= \operatorname{tr}((B + e^{i\theta}A^*)^* \, \Omega_\psi(B + e^{i\theta}A^*)).\end{aligned}$$

Now by the remark at the end of Section 4,

$$\| \Omega_\psi(B + e^{i\theta}A^*) - \Omega_\psi(B) - \Omega_\psi(e^{i\theta}A^*) \|_2 \leqslant C(\| A \|_2 + \| B \|_2),$$

where C is independent of ϕ, A, B since $L(\psi) \leqslant 2$. We note also that $\Omega_\psi(e^{i\theta}A^*) = e^{i\theta}\Omega_\psi(A^*)$. Thus

$$|\tau_\phi(F(e^{i\theta})) - \operatorname{tr}((A + e^{i\theta}B^*)(\Omega_\psi(B) + e^{i\theta}\Omega_\psi(A^*)))| \leqslant C(\| A \|_2 + \| B \|_2)^2.$$

Integrating over $[0, 2\pi]$ we obtain

$$\left| \int_0^{2\pi} \tau_\phi(F(e^{i\theta})) \frac{d\theta}{2\pi} - \operatorname{tr}(A\Omega_\psi(B)) \right| \leqslant C(\| A \|_2 + \| B \|_2)^2.$$

The result now follows from Lemma 6.4.

THEOREM 6.6. *There is a constant C so that if $A \in \mathscr{C}_1$ and $\phi \in \mathscr{L}$ then*

$$|\tau_\phi(A) - \operatorname{tr} \Omega_\phi(A)| \leqslant C \, \|A\|_1.$$

Proof. Suppose

$$A = \sum_{n=1}^{\infty} s_n e_n \otimes f_n$$

is the prescribed representation. Let

$$A_1 = \sum_{n=1}^{\infty} \sqrt{s_n}\, e_n \otimes f_n$$

$$A_2 = \sum_{n=1}^{\infty} \sqrt{s_n} f_n \otimes f_n.$$

Then $\Omega_\phi(A) = A_1 \Omega_\psi(A_2)$ and the result follows from Theorem 6.5 since $\|A_1\|_2 = \|A_2\|_2 = \sqrt{\|A_1\|_1}$.

THEOREM 6.7. *There is a constant C so that if $\phi \in \mathscr{L}$, then Ω_ϕ is a $\mathscr{C}_1$-bicentralizer with $\Delta(\Omega_\phi) \leqslant C$.*

Proof. Suppose $A \in \mathscr{C}_1$, U, $V \in \mathscr{B}(\mathscr{H})$ with $\|U\|$, $\|V\| \leqslant 1$. Let $A = A_1 A_2$ be the factorization used in the preceding theorem. Then $U\Omega_\phi(A) = UA_1\Omega_\psi(A_2)$ and so

$$|\operatorname{tr}(U\Omega_\phi(A)) - \tau_\phi(UA)| \leqslant C \, \|A\|_1.$$

By Theorem 6.6,

$$|\operatorname{tr}(\Omega_\phi(UA)) - \tau_\phi(UA)| \leqslant C \, \|A\|_1$$

and hence

$$|\operatorname{tr}(U\Omega_\phi(A) - \Omega_\phi(UA))| \leqslant C \, \|A\|_1.$$

It follows that

$$|\operatorname{tr}(VU\Omega_\phi(A) - \Omega_\phi(VUA))| \leqslant C \, \|A\|_1$$

and

$$|\operatorname{tr}(V\Omega_\phi(UA) - \Omega_\phi(VUA))| \leqslant C \, \|A\|_1$$

so that

$$|\operatorname{tr}(V(U\Omega_\phi(A) - \Omega_\phi(UA)))| \leqslant C \, \|A\|_1.$$

Thus

$$\|U\Omega_\phi(A) - \Omega_\phi(UA)\|_1 \leqslant C \, \|A\|_1.$$

To complete the argument note that from the definition we have $\Omega_\phi(A^*) = (\Omega_\phi(A))^*$ and so we also have, on taking adjoints,

$$\|\Omega_\phi(A)\,U - \Omega_\phi(AU)\|_1 \leqslant C\,\|A\|_1$$

and the theorem follows.

THEOREM 6.8. *There is a constant C so that if $\phi \in \mathscr{L}$, $A, B \in \mathscr{C}_1$ then*

$$|\tau_\phi(A+B) - \tau_\phi(A) - \tau_\phi(B)| \leqslant C(\|A\|_1 + \|B\|_1).$$

Proof. By Proposition 4.1 and Theorem 6.7,

$$\|\Omega_\phi(A+B) - \Omega_\phi(A) - \Omega_\phi(B)\|_1 \leqslant C(\|A\|_1 + \|B\|_1).$$

It remains only to apply Theorem 6.6.

THEOREM 6.9. *$\mathscr{C}h_1$ is a linear space and $\|\ \|_h$ is a quasi norm such that $\mathscr{C}h_1$ is a quasi-Banach space.*

Proof. Theorem 6.8 immediately yields

$$\|A+B\|_h \leqslant (C+1)(\|A\|_h + \|B\|_h).$$

Thus $\mathscr{C}h_1$ is a linear space equipped with a quasi norm. Since this quasi norm is lower-semi-continuous on $\mathscr{C}_1$ (Lemma 6.3) it is routine to establish completeness.

7. THE MAIN RESULTS

We recall that if $\mathscr{I}$, $\mathscr{J}$ are ideals in $\mathscr{B}(\mathscr{H})$ then $T \in \mathscr{B}(\mathscr{H})$ is called an $(\mathscr{I}, \mathscr{J})$-commutator if $T = [A, B] = AB - BA$ for $A \in \mathscr{I}$, $B \in \mathscr{B}$. The linear span of $(\mathscr{I}, \mathscr{J})$-commutators is denoted by $[\mathscr{I}, \mathscr{J}]$.

The proof of the next proposition uses ideas similar to those of Anderson and Vaserstein [2].

PROPOSITION 7.1. *Suppose $T \in \mathscr{C}h_1$ is self-adjoint. Then T is the sum of at most three $(\mathscr{C}_1, \mathscr{K}(\mathscr{H}))$-commutators. Similarly, if $1 < p < \infty$ and $1/p + 1/q = 1$ then T is the sum of at most three $(\mathscr{C}_p, \mathscr{C}_q)$-commutators.*

Proof. We may assume that rank $T = \infty$. Let (λ_n) be the nonzero eigenvalues of T counted according to multiplicity and arranged so that $(|\lambda_n|)$ is decreasing. For $n \in \mathbf{N}$ let

$$\beta_n = \sum_{k=1}^{2^n - 1} \lambda_k$$

and

$$\theta_n = |\lambda_{2^n - 1}|.$$

For $n \geqslant 2$ we have

$$\sum_{2^{n-2}+1}^{2^{n-1}} |\lambda_k| \geqslant 2^{n-2}\theta_n$$

so that

$$\sum_{n=1}^{\infty} 2^{n-2}\theta_n \leqslant \|T\|_1 < \infty.$$

We also observe that if $2^{n-1} \leqslant r < 2^n - 1$,

$$|\beta_n| \leqslant \left| \sum_{k=1}^{r} \lambda_k \right| + \sum_{k=2^{n-1}}^{2^n - 1} |\lambda_k|$$

so that

$$\begin{aligned} \sum_{n=1}^{\infty} |\beta_n| &\leqslant \sum_{n=1}^{\infty} 2^{-(n-1)} \sum_{r=2^{n-1}}^{2^n-1} \left| \sum_{k=1}^{r} \lambda_k \right| + \|T\|_1 \\ &\leqslant 2 \sum_{r=1}^{\infty} \frac{1}{r} \left| \sum_{k=1}^{r} \lambda_k \right| + \|T\|_1 \\ &< \infty \end{aligned}$$

since $(\lambda_n) \in h_1^{\text{sym}}$.

Now let

$$\alpha_n = 2^{-(n-1)} \sum_{k=2^{n-1}}^{2^n-1} \lambda_k$$

for $n \in \mathbf{N}$. Thus $|\alpha_n| \leqslant \theta_n$ and $|\lambda_k - \alpha_n| \leqslant 2\theta_n$ for $2^{n-1} \leqslant k < 2^n$. Observe also that

$$\sum_{k=2^{n-1}}^{2^n-1} (\lambda_k - \alpha_n) = 0$$

and recall that since T is self-adjoint the λ_k are real. It is thus possible to rearrange the set $\{\lambda_k : 2^{n-1} \leqslant k < 2^n\}$ to produce $\{\mu_k : 2^{n-1} \leqslant k < 2^n\}$, where for $2^{n-1} \leqslant r < 2^n$,

$$\left| \sum_{k=2^{n-1}}^{r} (\mu_k - \alpha_n) \right| \leqslant 2\theta_n.$$

Pick an orthonormal sequence (e_k) so that $Te_k = \mu_k e_k$. Let $\mathscr{H}_0$ be the orthogonal complement of $[e_k]$ and let S be the operator defined by $S(\mathscr{H}_0) = 0$ and $Se_k = \alpha_n e_k$ for $2^{n-1} \leqslant k < 2^n$, $n \in \mathbf{N}$. We will prove that S is the sum of at most two $(\mathscr{C}_1, \mathscr{K}(\mathscr{H}))$ or $(\mathscr{C}_p, \mathscr{C}_q)$-commutators, using a method similar to that of [2], and then show that $T - S$ is itself a commutator of the required type.

In order to prove the first assertion, note that since $\sum |\beta_n| < \infty$ we can factor $\beta_n = -2^n u_n v_n$, where $\lim v_n = 0$ and $\sum 2^n |u_n| < \infty$. For the case of $(\mathscr{C}_p, \mathscr{C}_q)$-commutators we instead require $\sum 2^n |u_n|^p < \infty$ and $\sum 2^n |v_n|^q < \infty$. Next we define four operators U_1, U_2, V_1, V_2 by setting each to zero on $\mathscr{H}_0$ and then

$$
\begin{aligned}
U_1 e_k &= u_n e_{2k}, && 2^{n-1} \leqslant k < 2^n \\
U_2 e_k &= u_n e_{2k+1}, && 2^{n-1} \leqslant k < 2^n \\
V_1 e_{2k} &= v_n e_k, && 2^{n-1} \leqslant k < 2^n \\
V_1 e_{2k+1} &= 0, && k \in \mathbf{N} \\
V_2 e_{2k+1} &= v_n e_k, && 2^{n-1} \leqslant k \leqslant 2^n \\
V_2 e_1 &= 0, && \\
V_2 e_{2k} &= 0, && k \in \mathbf{N}.
\end{aligned}
$$

Then for $2^{n-1} \leqslant k < 2^n$,

$$
\begin{aligned}
[U_1, V_1]\, e_k &= (u_{n-1} v_{n-1} - u_n v_n)\, e_k, && k \text{ even} \\
&= -u_n v_n e_k, && k \text{ odd} \\
[U_2, V_2]\, e_k &= (u_{n-1} v_{n-1} - u_n v_n)\, e_k, && k > 1 \text{ odd} \\
&= -u_1 v_1 e_1, && k = 1 \\
&= -u_n v_n e_k, && k \text{ even.}
\end{aligned}
$$

Thus

$$([U_1, V_1] + [U_2, V_2])\, e_1 = -2u_1 v_1 e_1 = \beta_1 e_1 = \alpha_1 e_1$$

while for $2^{n-1} \leqslant k < 2^n$,

$$
\begin{aligned}
([U_1, V_1] + [U_2, V_2])\, e_k &= (u_{n-1} v_{n-1} - 2u_n v_n)\, e_k \\
&= 2^{-(n-1)}(\beta_n - \beta_{n-1})\, e_k \\
&= \alpha_n e_k.
\end{aligned}
$$

Thus $[U_1, V_1]+[U_2, V_2]=S$. It remains to observe that, for the first case of the theorem,

$$\|U_1\|_1=\|U_2\|_1=\sum 2^{n-1}|u_n|<\infty$$

while V_1, V_2 are each compact since $\lim v_n=0$. In the other case, we similarly have $U_1, U_2\in\mathscr{C}_p$, $V_1, V_2\in\mathscr{C}_q$.

Now we consider $T-S$. Let σ be the permutation of $\mathbf{N}$ defined by $\sigma(k)=k+1$ unless $k+1$ is a power of 2, in which case $\sigma(k)=\frac{1}{2}(k+1)$. For $2^{n-1}\leqslant r<2^n$ set

$$\gamma_r=\sum_{k=2^{n-1}}^{r}(\mu_k-\alpha_n)$$

so that we have $|\gamma_r|\leqslant 2\theta_n$, and $\gamma_{2^n-1}=0$. then

$$\sum_{r=1}^{\infty}|\gamma_r|\leqslant\sum_{n=1}^{\infty}2^n\theta_n<\infty.$$

Again we factorize $\gamma_r=-x_r y_r$, where $\sum|x_r|<\infty$ and $\lim y_r=0$. For the $(\mathscr{C}_p, \mathscr{C}_q)$ cases we take $\sum|x_r|^p<\infty$, $\sum|y_r|^q<\infty$.

Now define U_3, V_3 to be zero on $\mathscr{H}_0$ and such that

$$U_3 e_k=x_k e_{\sigma(k)}$$
$$V_3 e_k=y_{\sigma^{-1}(k)}e_{\sigma^{-1}(k)}.$$

Then

$$[U_3, V_3]\,e_k=(x_{\sigma^{-1}(k)}\,y_{\sigma^{-1}(k)}-x_k\,y_k)\,e_k,$$

Hence if $2^{n-1}<k<2^n$,

$$\begin{aligned}[U_3, V_3]\,e_k&=(x_{k-1}\,y_{k-1}-x_k\,y_k)\,e_k\\&=(\gamma_k-\gamma_{k-1})\,e_k\\&=(\mu_k-\alpha_n)\,e_k,\end{aligned}$$

while if $k=2^{n-1}$,

$$\begin{aligned}[U_3, V_3]\,e_k&=(u_{2^n-1}v_{2^n-1}-u_k v_k)\,e_k\\&=(\gamma_k-\gamma_{2^n-1})\,e_k\\&=\gamma_k e_k\\&=(\mu_k-\alpha_n)\,e_k.\end{aligned}$$

Hence $[U_3, V_3] = T - S$ and as before we have $U_3 \in \mathscr{C}_1$, $V_3 \in \mathscr{K}(\mathscr{H})$ or $U_3 \in \mathscr{C}_p$, $V_3 \in \mathscr{C}_q$. Since $T = [U_1, V_1] + [U_2, V_2] + [U_3, V_3]$ the proposition is proved.

Remarks. The preceding proposition is very similar to the tree constructions in the paper of Anderson and Vaserstein [2]; it is perhaps also worth remarking that the result of Anderson [1] that if $p < 1$ then $[\mathscr{C}_p, \mathscr{B}(\mathscr{H})] = \mathscr{C}_p^{(0)}$ follows by exactly the same method once one observes that

$$\sum_{n=1}^{\infty} \left| \sum_{k=1}^{2^n - 1} \lambda_k \right|^p < \infty$$

when $T \in \mathscr{C}_p$. Indeed

$$\begin{aligned} \left| \sum_{k=1}^{2^n - 1} \lambda_k \right| &= \left| \sum_{k=2^n}^{\infty} \lambda_k \right| \\ &\leqslant |\lambda_{2^n}|^{(1-p)} \, \| T \|_p^p \\ &\leqslant 2^{n(1/p - 1)} \, \| T \|_p \end{aligned}$$

so that

$$\sum_{n=1}^{\infty} \left| \sum_{k=1}^{2^n - 1} \lambda_k \right|^p \leqslant \| T \|_p^p \sum_{n=1}^{\infty} 2^{-n(1-p)} < \infty.$$

Similarly if $1/r + 1/s = 1/p$ then $[\mathscr{C}_r, \mathscr{C}_s] = \mathscr{C}_p^{(0)}$.

PROPOSITION 7.2. *Suppose $1 < p < \infty$. Then any $(\mathscr{C}_p, \mathscr{C}_q)$-commutator is the sum of at most four quasi-nilpotent trace class operators.*

Proof. Suppose $A \in \mathscr{C}_p$, $B \in \mathscr{C}_q$ and that $A = H + iK$ with H, K self-adjoint. Then $[A, B] = [H, B] + i[K, B]$ so that it suffices to show that $[H, B]$ is the sum of at most two quasi-nilpotent trace-class operators.

By the spectral theorem we may pick an orthonormal basis (e_n) of $\mathscr{H}$ so that $He_n = \lambda_n e_n$. By the boundedness of the triangular projection in $\mathscr{C}_q$ (cf. [3, 5, 12]) we may define operators B_1, $B_2 \in \mathscr{C}_q$ so that

$$\begin{aligned} (B_1 e_j, e_k) &= (Be_j, e_k), && j > k \\ &= 0, && j \leqslant k \end{aligned}$$

$$\begin{aligned} (B_2 e_j, e_k) &= (Be_j, e_k), && j < k \\ &= 0, && j \geqslant k. \end{aligned}$$

Then for $j \leqslant k$

$$([H, B_1] e_j, e_k) = (\lambda_k - \lambda_j)(B_1 e_j, e_k) = 0$$

so that $[H, B_1]$ is quasi-nilpotent; similarly $[H, B_2]$ is quasi-nilpotent. Clearly, $[H, B] = [H, B_1] + [H, B_2]$.

PROPOSITION 7.3. *Let $A, B \in \mathscr{B}(\mathscr{H})$ be such that AB and BA are both trace-class. Then $[A, B] \in \mathscr{C}h_1$.*

Proof. Note that AB and BA have the same nonzero eigenvalues with the same multiplicities. Thus for $\phi \in \mathscr{L}$ we have $\tau_\phi(AB) = \tau_\phi(BA)$. Now, by Theorem 6.8, we have $\sup |\tau_\phi([A, B])| < \infty$ or $[A, B] \in \mathscr{C}h_1$ as required.

THEOREM 7.4. *Suppose $A \in \mathscr{C}_1$ and that $1 < p, q < \infty$ with $1/p + 1/q = 1$. Then the following conditions on A are equivalent:*

(i) $A \in \mathscr{C}h_1$.

(ii) $A \in [\mathscr{C}_p, \mathscr{C}_q]$.

(iii) $A \in [\mathscr{C}_1, \mathscr{B}(\mathscr{H})]$.

(iv) $A \in [\mathscr{C}_1, \mathscr{K}(\mathscr{H})]$.

(v) *A is in the linear span of the trace-class quasi-nilpotent operators.*

(vi) *A is the sum of at most six $(\mathscr{C}_p, \mathscr{C}_q)$-commutators.*

(vii) *A is the sum of at most six $(\mathscr{C}_1, \mathscr{K}(\mathscr{H}))$-commutators.*

(viii) *A is the sum of at most 24 quasi-nilpotent trace-class operators.*

Proof. (i) $\Rightarrow$ (vi). If $A \in \mathscr{C}h_1$ then $A^* \in \mathscr{C}h_1$. Thus by Theorem 6.9, $\frac{1}{2}(A + A^*)$ and $(1/2i)(A - A^*)$ are in $\mathscr{C}h_1$. Now apply Proposition 7.1.

(vi) $\Rightarrow$ (viii). Proposition 7.2.

(viii) $\Rightarrow$ (v). Trivial.

(v) $\Rightarrow$ (i). Theorem 6.9 and the definition of $\mathscr{C}h_1$.

Thus (i), (v), (vi), and (viii) are equivalent.

(i) $\Rightarrow$ (vii). Proposition 7.1.

(vii) $\Rightarrow$ (iv) $\Rightarrow$ (iii) and (vi) $\Rightarrow$ (ii). Trivial.

(ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (i). Proposition 7.3 and Theorem 6.9.

This completes the proof.

Remarks. The fact that $[\mathscr{C}_2, C_2] \neq \mathscr{C}_1^{(0)}$ and hence that $[\mathscr{C}_1, \mathscr{B}(\mathscr{H})] \neq \mathscr{C}_1^{(0)}$ is due to Weiss [15, 16]. Weiss also characterizes a special type of operator in the class $[\mathscr{C}_2, \mathscr{C}_2]$ in [15, 17] and these ideas are extended in Anderson and Vaserstein [2] and Anderson [1]. Some related results on discontinuous traces were obtained by Figiel but were never published.

8. Concluding Remarks

First we recall [7] that a quasi-Banach space X is *logconvex* if there is a constant C so that for $x_1, ..., x_n \in X$

$$\left\| \sum_{j=1}^{n} x_j \right\| \leqslant C \sum_{j=1}^{n} \|x_j\| \left(1 + \log \frac{S}{\|x_j\|}\right),$$

where $S = \sum \|x_j\|$. Equivalently X is logconvex if for some C we have

$$\left\| \sum_{k=1}^{n} x_k \right\| \leqslant C \sum_{j=1}^{n} \|x_k\| (1 + \log k)$$

(cf. [7]). Logconvex spaces arise in several natural situations in analysis (cf. [7, 14]).

Theorem 8.1. *The spaces h_1^{sym}, H_1^{sym}, and $\mathscr{C}h_1$ are all logconvex quasi-Banach spaces.*

Proof. The space H_1^{sym} is defined in [8] and is included in the theorem for completeness. We will treat only the case of $\mathscr{C}h_1$—the others being similar. We first note that for $\phi \in \mathscr{L}$, the maps $\tau_\phi : \mathscr{C}_1 \to \mathbf{C}$ are quasi-additive with uniform constant C (Theorem 6.8). Thus the maps

$$\begin{aligned} \tau_\phi^*(A) &= \|A\|_1 \, \tau_\phi\left(\frac{A}{\|A\|_1}\right), \qquad & A \neq 0 \\ &= 0, & A = 0 \end{aligned}$$

are quasi-linear with uniform constant C (Theorem 3.5 of [9] shows that the τ_ϕ^* are real quasi-linear and the relation $\tau_\phi(\lambda A) = \lambda \tau_\phi(A)$ for $|\lambda| = 1$ shows that each τ_ϕ^* is complex quasi-linear).

Now by Theorem 7.1 of [7] and its proof we have the existence of a constant C independent of ϕ so that

$$\left| \tau_\phi^*(A_1 + \cdots + A_n) - \sum_{k=1}^{n} \tau_\phi^*(A_k) \right| \leqslant C \sum_{k=1}^{n} \|A_k\|_1 \left(1 + \log \frac{S}{\|A_k\|_1}\right),$$

where $S = \sum \|A_k\|_1$.

Now by Lemma 3.4 of [9] we have

$$\| \tau_\phi^*(A) - \tau_\phi(A) \|_1 \leqslant C \| A \|_1,$$

where C is independent of ϕ. Thus

$$\left| \tau_\phi \left(\sum A_k \right) - \sum \tau_\phi(A_k) \right| \leqslant C \sum \| A_k \|_1 \left(1 + \log \frac{S}{\| A_k \|_1} \right).$$

From this we deduce

$$\left| \tau_\phi \left(\sum A_k \right) \right| \leqslant C \left(\sum \| A_k \|_h + \sum \| A_k \|_1 \left(1 + \log \frac{S}{\| A_k \|_1} \right) \right)$$
$$\leqslant C \left(\sum \| A_k \|_h \left(1 + \log \frac{S'}{\| A_k \|_h} \right) \right),$$

where $S' = \sum \| A_k \|_h$. Here we utilize the fact that the map $(\xi_1, ..., \xi_n) \to \sum \xi_i + \sum \xi_i \log(\sum \xi_j / \xi_i)$ is monotone in each ξ_i so that we can replace $\| A_k \|_1$ by $\| A_k \|_h$. Finally, we take suprema over all $\phi \in \mathscr{L}$ and deduce

$$\left\| \sum A_k \right\|_h \leqslant C \left(\sum \| A_k \|_h \left(1 + \log \frac{S'}{\| A_k \|_h} \right) \right).$$

We state now two simple corollaries.

COROLLARY 8.2. *Let V_n be a sequence of quasi-nilpotent operators in $\mathscr{C}_1$ so that $\sum \| V_n \|_1 < \infty$ and*

$$\sum \| V_n \|_1 \log_+ \frac{1}{\| V_n \|_1} < \infty$$

(or equivalently $\sum \| V_n \|_1 \log n < \infty$ when V_n is arranged in decreasing order of norm). Then $\sum_{n=1}^\infty V_n \in \mathscr{C}h_1$, i.e., $\sum_{n=1}^\infty V_n$ can be written as a finite linear combination of quasi-nilpotent operators.

COROLLARY 8.3. *Let (A_n), (B_n) be two sequences of Hilbert–Schmidt operators and suppose $\sum \| A_n \|_2 \| B_n \|_2 < \infty$ and*

$$\sum_{n=1}^\infty \| A_n \|_2 \| B_n \|_2 \log_+ \frac{1}{\| A_n \|_2 \| B_n \|_2} < \infty.$$

Then $\sum_{n=1}^\infty [A_n, B_n]$ can be written as a finite linear combination of commutators of Hilbert–Schmidt operators.

Proofs. For (8.2) just note that $\| V_n \|_1 = \| V_n \|_h$. For (8.3) note that the bilinear form $[A, B] \to AB - BA$ maps $\mathscr{C}_2 \times \mathscr{C}_2$ boundedly into $\mathscr{C}h_1$ (apply the Uniform Boundedness Principle) and so

$$\| [A, B] \|_h \leqslant C \| A_n \|_2 \| B_n \|_2.$$

Finally, we give an application to quasi-nilpotent operators in $\mathscr{C}_2$. If $V \in \mathscr{C}_2$ is a quasi-nilpotent operator and $V = H + iK$, where H and K are self-adjoint, then there are a number of results in the literature which relate the singular values of H and K (e.g., Gohberg and Krein [5, p. 139] or [4, p. 209]).

For any operator $T \in \mathscr{K}$, let $n(r, T)$ denote the number of singular values of T greater than r.

THEOREM 8.4. *Let H and K be self-adjoint Hilbert–Schmidt operators such that $H + iK$ is quasi-nilpotent. Define*

$$\theta(t) = \int_0^t r(n(r, H) - n(r, K))\, dr.$$

Then

$$\int_0^\infty \frac{|\theta(t)|}{t}\, dt < \infty.$$

Proof. Clearly $(H + iK)^2 \in \mathscr{C}h_1$ and so $H^2 - K^2 \in \mathscr{C}h_1$. For any $\phi \in \mathscr{L}$.

$$\tau_\phi(H^2) = \sum_{n=1}^\infty s_n^2 \phi(2 \log |s_n|),$$

where $s_n = s_n(H)$ are the singular values of H. Thus

$$\tau_\phi(H^2) = \sum_{n=1}^\infty \int_0^{s_n} 2r(\phi(2 \log r) + \phi'(2 \log r))\, dr,$$

where ϕ' is defined a.e. and $|\phi'| \leqslant 1$. Now

$$\sum_{n=1}^\infty \left| \int_0^{s_n} 2r\phi'(2 \log r)\, dr \right| \leqslant \sum_{n=1}^\infty s_n^2 = \| H \|_2^2.$$

Thus

$$\left| \tau_\phi(H^2) - \int_0^\infty 2rn(r, H)\, \phi(2 \log r)\, dr \right| \leqslant \| H \|_2^2.$$

We argue similarly with K and note that

$$|\tau_\phi(H^2-K^2)-\tau_\phi(H^2)+\tau_\phi(K^2)| \leqslant C(\|H\|_2+\|K\|_2)^2.$$

Thus for every such ϕ

$$\left|\int_0^\infty 2r(n(r,H)-n(r,K))\,\phi(2\log r)\,dr\right| \leqslant \gamma,$$

where γ is a constant depending only on H, K.

Assume ϕ vanishes outside some compact set. Then an integration by parts gives

$$\int_0^\infty \frac{\theta(r)}{r}\,\phi'(2\log r)\,dr \leqslant \frac{1}{4}\gamma.$$

As this is valid for all $\phi \in \mathscr{L}$ with compact support it quickly follows that

$$\int_0^\infty \frac{|\theta(t)|}{t}\,dt < \infty$$

as required.

Note added in proof. After the initial preparation of this paper, the author learned from T. Figiel that Theorem 7.4 can also be obtained by an analysis of the invariant linear functionals on $\mathscr{C}_1$.

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