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Isomorphisms between L_p-Function Spaces when $p < 1$

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We prove a number of results concerning isomorphisms between spaces of the type $L_p(X)$, where X is a separable p-Banach space and $0 < p < 1$. Our results imply that the quotient of $L_p([0, 1] \times [0, 1])$ by the subspace of functions depending only on the first variable is not isomorphic to L_p, answering a question of N. T. Peck. More generally if $\mathscr{B}_0$ is a sub-σ-algebra of the Borel sets of $[0, 1]$, then $L_p([0, 1])/L_p([0, 1], \mathscr{B}_0)$ is isomorphic to L_p if and only if $L_p([0, 1], \mathscr{B}_0)$ is complemented. We also show that L_p has, up to isomorphism, at most one complemented subspace non-isomorphic to L_p and classify completely those spaces X for which $L_p(X) \cong L_p$. In particular if $\mathscr{L}(L_p, X) = \{0\}$ and $L_p(X) \cong L_p$ then $X \cong l_p$ or is finite-dimensional. If X has trivial dual and $L_p(X) \cong L_p$ then $X \cong L_p$.

1. INTRODUCTION

A quasi-Banach space X is an F-space on which the topology is given by a quasi-norm $x \mapsto \|x\|$, which satisfies

$$\|x\| > 0 \qquad x \neq 0, \quad x \in X, \tag{1.0.1}$$

$$\|tx\| = |t|\, \|x\| \qquad t \in \mathbb{R}, \quad x \in X, \tag{1.0.2}$$

$$\|x + y\| \leqslant k(\|x\| + \|y\|) \qquad x, y \in X, \tag{1.0.3}$$

where k is a constant independent of x and y. If, in addition, for some $p > 0$, we have

$$\|x + y\|^p \leqslant \|x\|^p + \|y\|^p \qquad x, y \in X, \tag{1.0.4}$$

then X is a p-Banach space. If X can be equivalently re-normed to be a p-Banach space then X is said to be p-convex.

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For any quasi-Banach space X we define $l_p(X)$ $(0 < p < \infty)$ to be the space of sequences (x_n) with $x_n \in X$ such that

$$\|(x_n)\| = \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{1/p} < \infty, \tag{1.0.5}$$

$l_p(X)$ is a quasi-Banach space, which is a p-Banach space when X is a p-Banach space (when $0 < p \leqslant 1$). Also we define $L_p(X)$ to be the space of Borel maps f: $[0, 1] \to X$ such that

$$\|f\| = \left\{\int_0^1 \|f(s)\|^p \, ds\right\}^{1/p} < \infty. \tag{1.0.6}$$

Identifying, as usual, functions equal a.e., $L_p(X)$ is also a quasi-Banach space which is a p-Banach space when $0 < p \leqslant 1$. We remark here that $[0, 1]$ with Lebesque measure can be replaced with any Polish space with a diffuse probability measure, but for the purposes of the introduction, the above definition suffices.

If X and Y are two quasi-Banach spaces then $\mathscr{L}(X, Y)$ denotes the space of all operators T: $X \to Y$ with quasi-norm

$$\|T\| = \sup_{\|x\| \leqslant 1} \|Tx\|.$$

If $X = Y$ we abbreviate this to $\mathscr{L}(X)$.

In the remainder of the introduction we sketch some of the problems with motivated this research and then summarise our results.

This paper arises out of problems suggested by [6]. In [7] it was shown that if H is the subspace of $L_p([0, 1] \times [0, 1])$ consisting of all functions depending only on the second variable, then H is uncomplemented in $L_p([0, 1] \times [0, 1])$ when $0 < p < 1$. (The case $p = 0$ had earlier been proved by Berg *et al.* [2].) A natural question (suggested by Peck) is:

PROBLEM 1.1. *Is* $L_p([0, 1] \times [0, 1])/H \cong L_p$?

Of course if $1 \leqslant p < \infty$, H is complemented and the answer to Problem 1.1 is yes. A number of problems we consider are illuminated by comparison with the case $p = 1$ in particular, and in that case one can prove easily the following theorem (basically due to Lindenstrauss [10]).

THEOREM 1.2. *If X is a Banach space and N is a closed subspace such that $N \cong L_1$ and $X/N \cong L_1$ (or is even an $\mathscr{L}_1$-space), then N is complemented in X.*

This suggests a companion problem to Problem 1.1.

PROBLEM 1.3. *If X is a p-Banach space $(0 < p < 1)$ and N is a closed subspace of X isomorphic to L_p such that $X/N \cong L_p$, is N complemented in X? What about the case $X = L_p$?*

Also Theorem 1.2 suggested an approach to Problem 1.1 by examining whether $L_p([0, 1]^2)/H$ is an $\mathscr{L}_p$-space when $0 < p < 1$. We intend to develop the theory of $\mathscr{L}_p$-spaces when $p < 1$ in a separate paper. For the purposes of resolving 1.1, however, the approach proved to be a failure, since $L_p([0, 1]^2/H$ is an $\mathscr{L}_p$-space (and therefore could be isomorphic to L_p).

An alternative approach is to treat $L_p([0, 1]^2)/H$ as the space $L_p(L_p/1)$, where $L_p/1$ is the quotient of L_p by a subspace of dimension one [8]. As shown in [8], $L_p/1 \not\cong L_p$, and clearly one would wish to somehow exploit this non-isomorphism. Thus we state a new problem:

PROBLEM 1.4. *Characterize those p-Banach spaces such that $L_p(X) \cong L_p$.*

or more generally:

PROBLEM 1.5. *If X and Y are two p-Banach spaces such that $L_p(X) \cong L_p(Y)$ what can one say about X and Y?*

Again comparison with the $p = 1$ is in order. If $p = 1$ there is a natural projection from $L_1(X)$ onto a subspace isomorphic to X, namely,

$$Pf(s) = \int_0^1 f(t)\, dt \qquad 0 \leqslant s \leqslant 1.$$

From this and the Pelczynski decomposition technique one quickly gets that $L_1(X) \cong L_1$ if and only if X is isomorphic to a complemented subspace of L_1. The problem of characterizing such X is still open. It is known (the Lewis–Stegall theorem [9, 14]) that if X is infinite-dimensional and has the Radon–Nikodym property then $X \cong l_1$, and it is still open whether, in general, $X \cong l_1$ or $X \cong L_1$. In Problem 1.5 one can conclude X is isomorphic to a complemented subspace of $L_1(Y)$ and Y to a complemented subspace of $L_1(X)$, and little else.

For $0 < p < 1$, the absence of such a projection radically changes the problem. Clearly $L_p(l_p) = L_p$ but l_p is not isomorphic to a complemented subspace of L_p (or even more simply $L_p(\mathbb{R}) \cong L_p$). Clearly one must also consider:

PROBLEM 1.6. *Is L_p prime when $0 < p < 1$? (if X is isomorphic to a complemented subspace of L_p is $X \cong L_p$?).*

We now sketch the main results of this paper and how they affect Problems 1.1–1.6.

Sections 2–4 are preparatory; we gather in many more or less elementary lemmas and definitions which are required later. Our first major result is Theorem 5.2 which gives a representation theorem for operators T: $L_p \to L_p(X)$, where X is a p-Banach space, generalizing the scalar case proved in [7]. We use Theorem 5.2 to then establish a lifting theorem (5.3) which yields the conclusion that if X is a p-Banach space and N is a closed subspace of X which is q-convex for some $q > p$ (e.g., if $\dim N < \infty$), then $L_p(X/N) \cong L_p$ implies $L_p(N)$ is complemented in $L_p(X)$ (in its natural embedding). Combined with the fact from [7] (also proved in more generality in Section 6) that $H = L_p(\mathbb{R})$ is uncomplemented in $L_p([0,1]^2) \cong {}_p(L_p)$ we can deduce that the answer to Problem 1.1 is no (Corollary 5.4), i.e., $L_p([0,1]^2)/H \not\cong L_p$. This is not the only proof of this fact given in the paper, however.

In Section 6 we consider operators T: $L_p(X) \to L_p(Y)$, where X and Y are two separable p-Banach spaces. We introduce the notion of diagonal maps and use them to show that if X is a subspace of Y and $L_p(X)$ is complemented in $L_p(Y)$ (in its natural embedding) then X is complemented in Y (this proof is valid for $0 < p \leqslant 1$). Of course this gives another proof that H is uncomplemented in $L_p([0,1]^2)$.

Our main result in Section 7 is Theorem 7.3 Here we consider the quotient $\Lambda(\mathscr{B}_0)$ of $L_p([0,1])$ $(0 < p < 1)$ by a closed subspace $L_p([0,1], \mathscr{B}_0)$, where $\mathscr{B}_0$ is some sub-σ-algebra of the Borel sets. In [6] we gave a complete characterization of those $\mathscr{B}_0$ for which this subspace is complemented. Here we show that $\Lambda(\mathscr{B}_0) \cong L_p$ implies that $L_p([0,1], \mathscr{B}_0)$ is complemented, generalizing Corollary 5.4. On the other hand, $\Lambda(\mathscr{B}_0)$ always contains a complemented copy of L_p. We contrast this with Example 8.7, where we construct a proper subspace of L_p, N, say, so that $N \cong L_p$ but L_p/N contains no complemented copy of L_p. Thus N cannot be moved by any automorphism into a space $L_p([0,1], \mathscr{B}_0)$. This is somewhat akin to the recent result of Bourgain [3] that L_1 contains an uncomplemented subspace isomorphic to L_1.

In Section 8 we prove our main results. Here the critical assumption is that a p-Banach space X is *p-trivial*, i.e., $\mathscr{L}(L_p, X) = \{0\}$. This definition was introduced in [7] and it was shown to be an appropriate analogue when $p < 1$ to the assumption that X has the Radon–Nikodym property. In Theorem 8.3, we show that if $L_p(X)$ is isomorphic to a complemented subspace of $L_p(Y)$, where Y is separable and p-trivial, then $X \cong X_1 \oplus X_2$, where X_1 is a complemented subspace of $l_p(Y)$ and X_2 is a complemented subspace of $L_p(Y)$ (either X_1 or X_2 can be $\{0\}$). Note that in the case $p = 1$ this is a trivial conclusion since X is complemented in $L_1(X)$. We then deduce a nice partial solution to Problem 1.5, namely, that if $0 < p < 1$ and X and Y are two separable p-Banach spaces which are p-trivial and $L_p(X) \cong L_p(Y)$, then $l_p(X) \cong l_p(Y)$. Specializing to $Y = \mathbb{R}$, we obtain that

$L_p(X) \cong L_p$ with X p-trivial, implies X is finite-dimensional or $X \cong l_p$. This result is strikingly similar to the Lewis–Stegall theorem, which implies that if $L_1(X) \cong L_1$ and X has the Radon–Nikodym Property then X is finite-dimensional or $X \cong l_1$.

In Section 9 we use these techniques to study the dual space of $\mathscr{L}(L_p)$. The problem of determining whether $\mathscr{L}(L_p)$ has trivial dual was suggested to the author by J. H. Shapiro. We characterize operators $S \in \mathscr{L}(L_p)$ so that $\chi(S) = 0$ for every $\chi = \mathscr{L}(L_p)^*$ as the small operators introduced in [7]. Using this, we show that if L_p has a complemented subspace Z non-isomorphic to L_p, then $L_p(Z) \not\cong L_p$. Hence we obtain (Theorem 9.6) that if $L_p(X) \cong L_p$ and $X^* = \{0\}$, then $X \cong L_p$.

Finally we somewhat illuminate Problem 1.6 by showing that if L_p fails to be prime, then there is, up to isomorphism, a unique complemented subspace $Z \not\cong L_p$. In particular Z must be prime. We believe that no such Z can exist.

We note here that we have some corresponding results for the case $p = 0$, but the techniques are completely different and we proposed to publish these separately.

2. Preliminaries from Measure Theory

Let Ω be a topological space, then we denote by $\mathscr{B}$ (or $\mathscr{B}(\Omega)$ where more precision is required) the σ-algebra of Borel subsets of Ω.

We start by giving some essentially known results on Borel measurable maps. Suppose Ω and K are Polish spaces and ν is a σ-finite Borel measure in Ω. Then a Borel measurable map $\sigma: \Omega \to K$ will be called *anti-injective* if $B \in \mathscr{B}(\Omega)$ and $\sigma | B$ is an injection then $\nu(B) = 0$.

Lemma 2.1. *In order for σ to fail to be anti-injective it is necessary and sufficient that there exists $B \in \mathscr{B}(\Omega)$ with $\nu(B) > 0$, such that if $C \in \mathscr{B}(\Omega)$ and $C \subset B$, then there exists $A \in \mathscr{B}(K)$ with*

$$\nu([\sigma^{-1}(A) \cap B] \Delta C) = 0.$$

Proof. If σ fails to be anti-injective then there exists $B \in \mathscr{B}(\Omega)$ with $\nu(B) > 0$ such that $\sigma | B$ is an injection. By Lusin's theorem we can find $B_n \subset B$ which are compact so that $\sigma | B_n$ is continuous and injective and $\nu(B \backslash \bigcup B_n) = 0$. If $C \subset B$ is a Borel set then $A = \sigma(\bigcup_{n=1}^{\infty} (C \cap B_n)) \in \mathscr{B}(K)$ and $C \Delta (\sigma^{-1}(A) \cap B) \subset B \backslash \bigcup B_n$.

Conversely suppose B satisfies the conditions of the lemma. Let (U_n) be a countable base for the topology in Ω. For each n pick $A_n \in \mathscr{B}(K)$ with $\nu((\sigma^{-1}(A_n) \cap B) \Delta (U_n \cap B)) = 0$. Let $F = \bigcup [(\sigma^{-1}(A_n) \cap B) \Delta (U_n \cap B)]$. Then $\nu(F) = 0$. If $\omega_1, \omega_2 \in B \backslash F$ and $\omega_1 \neq \omega_2$ there exists $n \in N$ so that

$\omega_1 \in U_n$ but $\omega_2 \notin U_n$. Hence $\sigma\omega_1 \in A_n$ but $\sigma\omega_2 \notin A_n$, i.e., $\sigma\omega_1 \neq \sigma\omega_2$. Thus $\sigma | B \backslash F$ is injective.

Now if $\sigma\colon \Omega \to K$ is a Borel map there is an induced measure $\sigma^*\nu$ or K defined by

$$\sigma^*\nu(B) = \nu(\sigma^{-1}B) \qquad B \in \mathscr{B}(K).$$

Our next proposition is in reality a form of Maharam's theorem on homogeneous measure algebras [11]; but is stated in the language necessary for this paper. We refer also to Semadeni [15, pp. 471–477].

PROPOSITION 2.2. *Suppose Ω and K are Polish spaces and ν is a probability measure on Ω. Suppose $\sigma\colon \Omega \to K$ is an anti-injective Borel map. Then there is a compact metric space M, a diffuse probability measure π on M and a Borel map $\tau\colon \Omega \to M$ such that*

(i) *There is a Borel map $\rho\colon K \times M \to \Omega$ with $\rho(\sigma \times \tau)(\omega) = \omega$ ν – a.e., $\omega \in \Omega$, where $(\sigma \times \tau)(\omega) = (\sigma\omega, \tau\omega)$.* (2.2.1)

(ii) $$(\sigma \times \tau)^* \nu = \sigma^*\nu \times \pi. \tag{2.2.2}$$

Proof. Let (U_n) be a base for the open sets of Ω, where each set is repeated infinitely often. Let $\mathscr{B}_0$ be the sub-σ-algebra of $\mathscr{B}(\Omega)$ of all sets of the form $(\sigma^{-1}B;\, B \in \mathscr{B}(K))$.

We shall show how to construct a sequence of finite sets F_n and a sequence of Borel maps $\xi_n\colon \Omega \to F_n$ such that (2.2.3)–(2.2.5) hold:

$$|F_n| \geqslant 2 \text{ and } A \subset F_n \text{ then } \nu(\xi_n^{-1}(A)) = \pi_n(A), \tag{2.2.3}$$

where π_n is the probability measure on F_n defined by $\pi_n(A) = |A|/|F_n|$ ($|C| =$ the cardinality of C).

If $\mathscr{B}_n$ is the smallest sub-σ-algebra of $\mathscr{B}$ such that $\sigma, \xi_1, \ldots, \xi_n$ are measurable with respect to $\mathscr{B}_n$, then for $n \geqslant 1$, ξ_n is independent of $\mathscr{B}_{n-1}$, i.e., if $A \subset F_n$ and $B \in \mathscr{B}_{n-1}$ then (2.2.4)

$$\nu(\xi_n^{-1}(A) \cap B) = \pi_n(A)\,\nu(B).$$

There exists $A \in \mathscr{B}_n$ with $\nu(A \,\Delta\, U_n) < 1/n, \qquad n \geqslant 1.$ (2.2.5)

The argument is an easy induction. If $(\xi_k : k \leqslant n-1)$ have been constructed, where $n \geqslant 1$, then the map $\sigma \times \xi_1 \times \cdots \times \xi_{n-1}\colon \Omega \to K \times F_1 \times \cdots \times F_{n-1}$ is anti-injective, by the finiteness of $F_1 \times \cdots \times F_{n-1}$ (some obvious rewording is necessary when $n = 1$). By Lemma 2.1 this means $\mathscr{B}_{n-1}$ (or more exactly the measure algebra induced by $\mathscr{B}_{n-1}$) *induces no ideal* in $(\mathscr{B}, \nu)$ as defined on p. 473 of Semadeni [15]. Now as in Lemmas 26.5.13 and 26.5.14 of [15, pp. 475, 476], we can find F_n and ξ_n to satisfy (2.2.3)–(2.2.5).

Now let M be the product $\prod_n F_n$ with the product measure $\bigotimes_n \pi_n = \pi$ which is diffuse and define $\tau: \Omega \to M$ by $\tau(\omega) = (\xi_n(\omega))_{n=1}^{\infty}$. For each open set $U \in (U_n)_{n=1}^{\infty}$, it is clear that there is a Borel set $A \in \mathscr{B}(M)$ with

$$\nu((\sigma \times \tau)^{-1}(A) \cap U) = 0.$$

Hence the same is true for every Borel set $U \subset \Omega$. Now Lemma 2.1 (or more precisely its proof) implies that there is a Borel subset Ω_0 of Ω with $\nu(\Omega \backslash \Omega_0) = 0$ such that $\sigma \times \tau | \Omega \backslash \Omega_0$ is injective: We may suppose Ω_0 is an F_σ-set, $(\sigma \times \tau)(\Omega_0)$ is Borel and $(\sigma \times \tau)^{-1} | (\sigma \times \tau)\, \Omega_0$ is Borel; this follows easily from applying Lusin's theorem. Defining $\rho = (\sigma \times \tau)^{-1}$ on $(\sigma \times \tau)\, \Omega_0$ and arbitrarily (subject to being Borel) off $(\sigma \times \tau)\, \Omega_0$ we can satisfy (2.2.1).

It is easy to see that (2.2.2) is automatic from the construction.

Suppose now X is separable quasi-Banach space. Then it is easy to see that if Ω is a Polish space, then a Borel map $g: \Omega \to X$ is a uniform limit of countably-valued Borel maps $g_n: \Omega \to X$.

If X and Y are separable quasi-Banach spaces then a map $\Phi: \Omega \to \mathscr{L}(X, Y)$ is *strongly (Borel) measurable* if it is a Borel map for the strong-operator topology on $\mathscr{L}(X, Y)$, i.e., for $x \in X$, the map $\omega \mapsto \Phi(\omega)\, x$ is a Borel map into Y. From the preceding paragraph it can be shown that if $g: \Omega \to X$ is a Borel map and $\Phi: \Omega \to \mathscr{L}(X, Y)$ is strongly measurable then $\omega \mapsto \Phi(\omega)(g(\omega))$ is also a Borel map.

The following lemma is one that we shall require later for the special case $X = L_p$ when the conditions are satisfied for $c = 1 + \varepsilon$ for any $\varepsilon > 0$ (this follows quickly from results of Rolewicz [13, pp. 253–254].

LEMMA 2.3. *Let Ω be a Polish space and let ν be a σ-finite Borel measure as Ω. Suppose X is a separable quasi-Banach space with the property*

> *For some fixed $c > 0$, if $x, y \in X$ with $\|x\| = \|y\| = 1$ there exists an invertible $T \in \mathscr{L}(X)$ with $\|T\|$, $\|T^{-1}\| \leqslant c$ and $Tx = y$.* (2.3.1)

Suppose further $g: \Omega \to X$ is a Borel map such that $0 < \alpha \leqslant \|g(\omega)\| \leqslant \beta < \infty$ for $\omega \in \Omega$. Then given $u \in X$ with $\|u\| = 1$ there exists a strongly measurable map $\omega \mapsto T_\omega$ such that:

$$\|T_\omega\| \leqslant c\beta \qquad \omega \in \Omega, \tag{2.3.2}$$

$$\|T_\omega^{-1}\| \leqslant c\alpha^{-1} \qquad \omega \in \Omega, \tag{2.3.3}$$

$$T_\omega u = g(\omega) \qquad \nu - \text{a.e.} \tag{2.3.4}$$

Proof. Let $G \subset \mathscr{L}(X)$ the subset of $\mathscr{L}(X) \times \mathscr{L}(X)$ of all (S, T) with $TS = ST = I$, $\|S\| \leqslant c\alpha^{-1}$ and $\|T\| \leqslant c\beta$. Then in the product strong-operator topology G is closed. For if $(S_n, T_n) \to (S, T)$ then $\|S\| \leqslant c\alpha^{-1}$ and $\|T\| \leqslant c\beta$ and since $S_n \to S$ uniformly as compact sets we have

$$\begin{aligned} STx &= \lim_{n\to\infty} S(T_n x) \\ &= \lim_{m\to\infty} \lim_{n\to\infty} S_m(T_n x) \\ &= \lim_{n\to\infty} S_n(T_n x) \\ &= x. \end{aligned}$$

Now let $V \subset X$ be the set $\{Tu\colon (S, T) \in G\}$. Assumption (2.3.1) guarantees that $V \supset \{x\colon \alpha \leqslant \|x\| \leqslant \beta\}$.

Now G is a Polish space (it is bounded and closed in $\mathscr{L}(X) \times \mathscr{L}(X)$). The map $(S, T) \mapsto Tu$ is continuous onto V. Hence by the von Neumann selection theorem (Diestel [4, p. 270]; see also [1; 12, p. 448]), there is a universally measurable map $\phi(v) = (\phi_1(v), \phi_2(v))$, where $\phi_2(v)\, u = v$.

Now let

$$T'_\omega = \phi_2(g(\omega)).$$

Then $\omega \mapsto T'_\omega$ is universally measurable and by modifying it on a set of measure zero we get the result of the lemma.

Remark. Inversion is continuous on the set G so the map $\omega \mapsto T_\omega^{-1}$ is also strongly Borel measurable.

3. p-Integral Operators

In this section K will be a compact metric space and X a p-Banach space, where $0 < p \leqslant 1$. We define a bounded linear operator $T\colon C(K) \to X$ to be p-integral if for some constant c, we have

$$\sum_{i=1}^{n} \|Tf_i\|^p \leqslant c \max_{s \in K} \sum_{i=1}^{n} |f_i(s)|^p \tag{3.0.1}$$

whenever $f_1, \ldots, f_n \in C(K)$. The best constant c arising in (3.0.1) is denoted by $\pi_p(T)$. In the case $p = 1$, this reduces to the standard definition of an integral operator and $\pi_1(T)$ is the integral or absolutely summing norm of T (cf. Diestel and Uhl [5, pp. 161–169]).

Let $\mathscr{M}_p(K; X)$ denote the subspace of $\mathscr{L}(C(K); X)$ of all p-integral operators. We observe that the map π_p: $\mathscr{M}_p(K; X) \to \mathbb{R}$ is lower-semi-continuous for the strong-operator topology.

LEMMA 3.1. *Suppose T: C(K) → X is p-integral. Then there is a unique positive Borel measure on K, which we denote by* $\mu = \mu(T)$ *so that*

$$\text{(i)} \quad \|Tf\| \leqslant \left\{\int_K |f(s)|^p \, d\mu(s)\right\}^{1/p} \qquad f \in C(K), \tag{3.1.1}$$

$$\text{(ii)} \quad \|\mu\| = \mu(K) = \pi_p(T), \tag{3.1.2}$$

(iii) *If ν is a positive Borel measure with*

$$\|Tf\| \leqslant \left\{\int_K |f(s)|^p \, d\nu(s)\right\}^{1/p} \qquad f \in C(K),$$

then $\mu \leqslant \nu$. (3.1.3)

Proof. Let P be the open positive cone in $C(K)$, i.e., $P = \{f \in C(K);$ $f(s) > 0$ for all $s \in K\}$. Let C be the cone of all functions of the form

$$\phi = \sum_{i=1}^{n} \|Tf_i\|^p - \pi_p(T) \sum_{i=1}^{n} |f_i|^p$$

for $f_1, ..., f_n \in C(K)$. By hypothesis $C \cap P = \varnothing$ and so, as P has non-empty interior, the Hahn–Banach theorem implies the existence of a positive Borel measure μ with $\|\mu\| = \pi_p(T)$ such that

$$\int \phi \, d\mu \leqslant 0 \qquad \phi \in C,$$

which quickly yields (3.1.1) and (3.1.2).

Uniqueness will follow from proving (3.1.3). Let $\rho = \nu + \mu$ and by the Radon–Nikodym theorem write $d\nu = \phi \, d\rho$ and $d\mu = \psi \, d\rho$, where ϕ, ψ are non-negative Borel functions. If ν does not satisfy $\nu \geqslant \mu$ then there is a compact subset K_0 of K of positive μ-measure such that

$$\psi(s) > \phi(s) \qquad s \in K_0.$$

Let h_n be any sequence of functions in $C(K)$ satisfying $0 \leqslant h_n \leqslant 1$, $h_n | K_0 = 1$ and $h_n(s) \to 0$ for $s \notin K_0$. Then for $f \in C(K)$

$$\begin{aligned} \|Tf\|^p &\leqslant \|T(h_n f)\|^p + \|T(f - h_n f)\|^p \\ &\leqslant \int_K |h_n f|^p \, \phi \, d\rho + \int_K |f - h_n f|^p \, \psi \, d\rho; \end{aligned}$$

and by the Dominated Convergence Theorem,

$$\|Tf\|^p \leqslant \int_{K_0} |f|^p \phi \, d\rho + \int_{K \setminus K_0} |f|^p \psi \, d\rho.$$

Thus, for $f_1,..., f_n \in C(K)$,

$$\sum_{i=1}^{n} \| Tf_i\|^p \leqslant c \max_{s \in K} \sum_{i=1}^{n} |f_i(s)|^p,$$

where

$$c = \int_{K_0} \phi \, d\rho + \int_{K \setminus K_0} \psi \, d\rho$$

$$< \int_K \psi \, d\rho = \|\mu\| = \pi_p(T).$$

This contradiction proves the lemma.

LEMMA 3.2. *For $\phi \in C(K)$ and $T \in \mathscr{L}(C(K), X)$ define $T_\phi \in \mathscr{L}(C(K), X)$ by*

$$T_\phi(f) = T(\phi f) \qquad f \in C(K). \tag{3.2.1}$$

Then if $T \in \mathscr{M}_p(K, X)$, $T_\phi \in \mathscr{M}_p(K, X)$ and

$$d\mu(T_\phi) = |\phi|^p \, d\mu(T). \tag{3.2.2}$$

Proof. That $T_\phi \in \mathscr{M}_p(K, X)$ is trivial. Note that if $\mu = \mu(T)$ then

$$\|T_\phi(f)\|^p \leqslant \int |\phi f|^p \, d\mu$$

so that $d\mu(T_\phi) \leqslant |\phi|^p \, d\mu$. If ϕ does not vanish, (3.2.2) now follows by reversing the reasoning. If $\phi \geqslant 0$

$$d\mu(T_{\phi+\alpha}) \leqslant d\mu(T_\phi) + \alpha^p \, d\mu$$

when α is a positive constant. Thus

$$|\phi + \alpha|^p \, d\mu \leqslant d\mu(T_\phi) + \alpha^p \, d\mu;$$

and letting $\alpha \to 0$ we obtain (3.2.2). For general ϕ note that

$$d\mu(T_{\varphi^2}) \leqslant |\varphi|^p \, d\mu(T_\varphi)$$

and apply a similar argument.

Now let $\mathscr{M}(K)$ be the dual space of $C(K)$ (i.e., the space of finite signed Borel measures in K).

LEMMA 3.3. *The map* μ: $\mathscr{M}_p(K;X) \to \mathscr{M}(K)$ (*where* $\mathscr{M}_p(K, X)$ *has the strong-operator topology and* $\mathscr{M}(K)$ *the weak*-topology*) *is Borel measurable.*

Proof. It will suffice to show that the map

$$T \mapsto \int_K \phi \, d\mu(T)$$

is Borel, where $\phi \in C(K)$ and $\phi \geqslant 0$. Let $\phi = \psi^p$, where $\psi \geqslant 0$. Then

$$\int \phi \, d\mu(T) = \int \psi^p \, d\mu(T)$$
$$= \pi_p(T_\psi).$$

As $T \mapsto T_\psi$ is clearly continuous, and we have already observed that π_p is lower-semi-continuous the result follows.

Now if $T \in \mathscr{M}_p(K;X)$ and $\mu = \mu(T)$, we can extend T to a linear operator (which we still denote by T) T: $L_p(K, \mu) \to X$ with $\|T\| = 1$. In particular Tf can be defined uniquely for any bounded Borel function f and the map $T \mapsto Tf$ is a Borel map from $\mathscr{M}_p(K;X)$ into X. Furthermore, in place of (3.0.1) we have

$$\sum_{i=1}^{n} \|Tf_i\|^p \leqslant \pi_p(T) \sup_{s \in K} \sum_{i=1}^{n} |f_i(s)|^p \tag{3.3.1}$$

for $f_1, ..., f_n$ bounded Borel functions in K.

To conclude this section we establish a lifting-type result which we need later. We recall [8] that if Y is a p-Banach space and N is a closed subspace of Y which is either q-convex for some $q > p$ or a pseudo-dual space then a bounded linear operator S: $L_p \to Y/N$ can be "lifted" to an operator S_1: $L_p \to Y$ so that $QS_1 = S$, where Q: $Y \to Y/N$ is the quotient map. (Here N is pseudo-dual if there is a Hausdorff vector topology in N for which the unit ball is relatively compact.) Furthermore there is a constant c independent of S so that $\|S_1\| \leqslant c \|S\|$. The same conclusions for l_p are valid with no restriction on N and with $c = 1$. Applying this we have:

LEMMA 3.4. *Suppose X is a p-Banach space and N is a closed subspace of X which is either pseudo-dual or q-convex for some $q > p$. Suppose*

$T: C(K) \to X/N$ is p-integral. Then there exists a p-integral operator $T_1 : C(K) \to X$ and there is a constant c depending only on X and N so that

$$\pi_p(T_1) \leqslant c\pi_p(T), \tag{3.4.1}$$

$$\mu(T_1) \leqslant c\mu(T). \tag{3.4.2}$$

Proof. Simply consider the induced map T: $L_p(K, \mu) \to X/N$ and since $L_p(\mu) \cong l_p$, L_p, $l_p \oplus L_p$, or $l_p^n \oplus L_p$ there exists a lift $T_1 : L_p(K, \mu) \to X$ with $\|T_1\| \leqslant c\|T\|$. Restricted to $C(K)$, T_1 is p-integral and (3.4.1) and (3.4.2) are immediate with c replaced by c^p.

Remark. If μ is diffuse then $T_1$is unique.

The form that we shall require later is the following:

LEMMA 3.5. *Let Ω be a Polish space and ν a σ-finite Borel measure on Ω. Suppose K is a compact metric space. Suppose X is a separable p-Banach space and N is closed subspace of X which is either pseudo-dual or q-convex for some $q > p$. Let $\omega \mapsto T_\omega$ be a strongly Borel measurable map from Ω into $\mathscr{M}_p(K; X/N)$. Then there is a constant $c > 0$ and a strongly measurable map $\omega \mapsto S_\omega$ of Ω into $\mathscr{M}_p(K; X)$ such that*

$$QS_\omega = T_\omega \qquad \nu\text{-a.e.}, \tag{3.5.1}$$

$$\mu(S_\omega) \leqslant c\mu(T_\omega) \qquad \nu\text{-a.e.}, \tag{3.5.2}$$

where Q: $X \to X/N$ is the quotient map.

Proof. This is again an application of the von Neumann selection theorem. Let G be the set of $S \in \mathscr{M}_p(K; X)$ such that $\mu(QS) \leqslant c\mu(S)$. It is readily verified that $\mathscr{L}(C(K), X)$ is a Souslin spaces (for the strong-operator topology) and that $\mathscr{M}_p(K; X)$ is a Borel subset (in fact an F_σ-set). It follows easily that G also is Borel (use Lemma 3.3 and the fact the positive cone in $\mathscr{M}(K)$ is a Borel set).

Now the map $S \mapsto QS$ maps G onto $\mathscr{M}_p(K; X/N)$ by Lemma 3.4 and so there is a universally measurable map $\theta: \mathscr{M}_p(K, X/N) \to G$ so that $Q\theta(S) = S$ (Theorem 2.2 of [7]).

To complete the proof let $S_\omega = \theta(T_\omega)$ ν-a.e. simply modifying on a set of measure zero to ensure Borel measurability.

4. The spaces $L_p(X)$

Let Ω be a Polish space and let ν be a σ-finite measure on Ω. Then if X is separable p-Banach space, we define $L_p(\Omega, \nu; X)$ to be the space of all Borel maps $f: \Omega \to X$ such that

$$\|f\| = \left\{\int_\Omega \|f(\omega)\|^p \, d\nu(\omega)\right\}^{1/p} < \infty.$$

After the usual identification of functions equal almost everywhere $L_p(\Omega, \nu; X)$ is a p-Banach space. We now list without proof several easy facts.

Fact 4.0.1. *The simple functions are dense in $L_p(\Omega, \nu; X)$ and hence $L_p(\Omega, \nu; X)$ is separable.*

Fact 4.0.2. *If ν_0 is a finite measure having the same sets of measure 0 as ν and $d\nu = \phi^p \cdot d\nu_0$, where ϕ is a positive Borel function, then the map $f \mapsto \phi \cdot f$ is an isometry of $L_p(\Omega, \nu; X)$ onto $L_p(\Omega, \nu_0; X)$.*

Fact 4.0.3. *For any two diffuse measures ν_1 and ν_2 on Polish spaces Ω_1 and Ω_2, respectively, the quasi-Banach spaces $L_p(\Omega_1 \nu_1; X)$ and $L_p(\Omega_2 \nu_2; X)$ are isometric. In cases where only the isomorphism class of the space matters we denote this space $L_p(X)$.*

Fact 4.0.4. *If N is a closed subspace of X then $L_p(\Omega, \nu; N)$ is a closed subspace of $L_p(\Omega, \nu; X)$ and $L_p(\Omega, \nu; X)/L_p(\Omega, \nu; N) \cong L_p(\Omega, \nu; X/N)$ under the natural quotient map.*

Fact 4.0.5. *Suppose Ω_1 and Ω_2 are Polish spaces, ν_1 is a diffuse σ-finite measure on Ω_1 and ν_2 is a diffuse σ-finite measure on Ω_2. Then $L_p(\Omega_1 \times \Omega_2, \nu_1 \times \nu_2; X)$ is naturally isometric to $L_p(\Omega_1, \nu_1; L_p(\Omega_2, \nu_2; X))$ under the isomorphism*

$$Tf(\omega_1)(\omega_2) = f(\omega_1, \omega_2) \qquad \omega_1 \in \Omega_1, \quad \omega_2 \in \Omega_2.$$

It is also convenient to observe that $L_p(\Omega, \nu; X)$ can be interpreted as the p-convex tensor product of $L_p(\Omega, \nu; \mathbb{R})$ $(= L_p(\Omega, \nu))$ and X (Vogt [17]). In fact if $\phi \in L_p(\Omega, \nu)$ and $x \in X$ we shall write $f = \phi \otimes x$, where

$$f(\omega) = \phi(\omega)\, x. \tag{4.0.6}$$

A conclusion from Vogt's results [17] is the following:

PROPOSITION 4.1. *Let Z be a p-Banach space. Then there is a natural isometry $T \mapsto \hat{T}$ between $\mathscr{L}(L_p(\Omega, \nu; X), Z)$ and $\mathscr{L}(X, \mathscr{L}(L_p(\Omega, \nu), Z))$ given by*

$$\hat{T}(x)(\phi) = T(\phi \otimes x). \tag{4.1.1}$$

We shall conclude this short introductory section by considering a very special class of operators on $L_p(X)$ spaces.

We define an operator $T\colon L_p(\Omega, \nu; X) \to L_p(\Omega, \nu; Y)$ to be *diagonal* if for every $B \in \mathscr{B}(\Omega)$ if

$$f(\omega) = 0 \qquad \omega \in B,$$

then

$$Tf(\omega) = 0 \qquad \omega \in B.$$

THEOREM 4.2. *Suppose X and Y are separable p-Banach spaces and T: $L_p(\Omega, \nu; X) \to L_p(\Omega, \nu; Y)$ is a diagonal operator. Then there is a strongly Borel measurable map $s \mapsto A_s$ $(\Omega \to \mathscr{L}(X, Y))$ such that*

$$Tf(s) = A_s(f(s)) \qquad \nu\text{-a.e.}, \tag{4.2.1}$$

$$\|T\| = \nu\text{-ess. sup}\, \|A_s\|. \tag{4.2.2}$$

Furthermore the map $s \mapsto A_s$ is unique up to ν-null sets.

Conversely if $s \mapsto A_s$ is a strongly Borel measurable map with ν-ess. $\sup \|A_s\| < \infty$, *then* (4.2.1) *defines a bounded linear operator.*

Proof. First we observe that the converse is pretty well automatic once one notes that $A_s(f(s))$ is Borel; see the remarks preceding Lemma 2.3.

For the direct part of the theorem, choose F_n to be an increasing sequence of finite-dimensional subspaces of X whose union F is dense in X. Let (x_n) be any sequence dense in the unit ball of F, such that $\{x_n : x_n \in F_k\}$ is dense in the unit ball of F_k.

By picking a Hamel basis of F and extending linearly we may determine linear maps $A_s : F \to Y$ so that

$$T(1_\Omega \otimes x)(s) = A_s(x) \qquad \nu\text{-a.e.} \quad x \in F.$$

From the diagonal property it is easy to see that

$$T(1_B \otimes x)(s) = 1_B(s) \cdot A_s(x) \qquad \nu\text{-a.e.} \quad x \in F,$$

and hence if $\phi \in L_p(\Omega, \nu)$,

$$T(\phi \otimes x)(s) = \phi(s) A_s(x) \qquad \nu\text{-a.e.} \quad x \in F.$$

The proof will be completed by showing $\|A_s\| \leqslant \|T\|$ ν-a.e. and since

$x \mapsto A_s x$ is a Borel map for $x \in F$, it will follow by density that, after defining suitably on a set of measure zero, we have (4.2.1) and (4.2.2). Uniqueness is trivial, by considering $T(1_\Omega \otimes x_n)$.

For each $n \in \mathbb{N}$, let

$$\theta_n(s) = \max_{1 \leqslant i \leqslant n} \|A_s x_i\|.$$

Note that the quasi-norm is Borel measurable on X and so θ_n is a Borel function. Define

$$f_n(s) = x_i,$$

where $i = i(s)$ is the least i such that

$$\|A_s x_i\| = \theta_n(s).$$

Then if $B \in \mathscr{B}(\Omega)$ and $\nu(B) < \infty$, $1_B \otimes f_n \in L_p(\Omega, \nu; X)$ and

$$\| 1_B \otimes f_n\|^p \leqslant \nu(B).$$

However,

$$\| T(1_B \otimes f_n)\|^p = \int_B \theta_n(s)^p \, d\nu(s).$$

We conclude

$$\theta_n(s) \leqslant \| T\| \qquad \nu\text{-a.e.},$$

and hence

$$\sup_n \theta_n(s) \leqslant \| T\| \qquad \nu\text{-a.e.},$$

i.e.,

$$\|A_s | F_k\| \leqslant \| T\| \qquad \nu\text{-a.e.},$$

for each k (Of course A_s is continuous on F_k). Thus as required

$$\|A_s\| \leqslant \| T\| \qquad \nu\text{-a.e.}$$

5. Operators on L_p

We now give a representation theorem for operators from L_p into $L_p(X)$ which generalizes the representation theorem given in [7]. We give the result in two parts.

PROPOSITION 5.1. *Suppose* $0 < p \leqslant 1$ *and* (5.1.1)–(5.1.5) *hold*:

$$K \text{ is a compact metric space and } \lambda \text{ is a probability measure on } K. \tag{5.1.1}$$

$$\Omega \text{ is a Polish space and } \nu \text{ is a } \sigma\text{-finite Borel measure on } \Omega. \tag{5.1.2}$$

$$X \text{ is a separable } p\text{-Banach space.} \tag{5.1.3}$$

$$\omega \mapsto T_\omega(\Omega \to \mathscr{M}_p(K;X)) \text{ is a strongly Borel measurable map.} \tag{5.1.4}$$

If $\mu_\omega = \mu(T_\omega)$ (*as in Lemma* 3.1) *then for some* $c < \infty$,

$$\int_\Omega \mu_\omega(B)\, d\nu(\omega) \leqslant c^p \lambda(B) \qquad B \in \mathscr{B}(K). \tag{5.1.5}$$

Then we conclude

(i) *If* $f \in L_p(K;\lambda)$ *then* ν*-a.e.*, $f \in L_p(K, \mu_\omega)$. (5.1.6)

(ii) *The formula*

$$Tf(\omega) = T_\omega f \qquad f \in L_p(K, \lambda) \tag{5.1.7}$$

defines a bounded linear operator $T\colon L_p(K,\lambda) \to L_p(\Omega, \nu; X)$ *with* $\|T\| \leqslant c$.

Proof. By (5.1.5) and Theorem 3.1 of [7] there is a bounded linear operator $U\colon L_1(K;\lambda) \to L_1(\Omega,\nu)$ defined by

$$Uf(\omega) = \int f\, d\mu_\omega \qquad \nu\text{-a.e.}$$

[The extension of [7] to the σ-finite case is easy.] In particular if $f \in L_p(K,\lambda)$ then $|f|^p$ is μ_ω-integrable ν-a.e., i.e., (5.1.6) holds. Thus the formula in (5.1.7) makes sense ν-a.e. and in fact $\omega \mapsto T_\omega f$ is Borel by approximating by bounded simple functions. Finally

$$\begin{aligned}\int_\Omega \|T_\omega f\|^p\, d\nu(\omega) &\leqslant \int_\Omega \int_K |f|^p\, d\mu_\omega\, d\nu(\omega)\\ &= \|U(|f|^p)\|_1\\ &\leqslant c^p \int_\Omega |f|^p\, d\lambda.\end{aligned}$$

Thus $\|T\| \leqslant c$.

The converse is naturally much more interesting.

THEOREM 5.2. *Suppose we have* (5.1.1)–(5.1.3) *and that* $T: L_p(K, \lambda) \to L_p(\Omega, \nu; X))$ *is a bounded linear operator.*

Then there is a strongly Borel measurable map $\omega \mapsto T_\omega(\Omega \to \mathscr{M}_p(K; X))$ *such that* (5.1.5) *and* (5.1.6) *hold, with* $c = \|T\|$ *and*

$$Tf(\omega) = T_\omega f \qquad \nu\text{-a.e.}, \quad f \in L_p(K, \lambda). \tag{5.2.1}$$

Proof. As in Theorem 3.1 of [7] we may suppose that K is totally disconnected. To do this it is necessary only to check that if $\sigma: K \to K_1$ is a Borel isomorphism onto a totally disconnected compact metric space then the induced map $\sigma^*: \mathscr{M}_p(K; X) \to \mathscr{M}_p(K_1, X)$ is also Borel, where

$$\sigma^* T(f) = T(f \circ \sigma) \qquad f \in C(K_1).$$

This is clear (cf. remarks following Lemma 3.3).

Now suppose that for each n, $\mathscr{A}_n$ is a partitioning of K into clopen sets so that $\mathscr{A}_{n+1}$ refines $\mathscr{A}_n$ for every n and if $\mathscr{A}_n = (U_{n,k}: 1 \leqslant k \leqslant l(n))$ then diam $U_{n,k} \leqslant n^{-1}$. Let $\chi_{n,k}$ be the characteristic function of $A_{n,k}$, and let E be the linear span of $(\chi_{n,k}: 1 \leqslant k \leqslant l(n),\ 1 \leqslant n < \infty)$. E is dense in $C(K)$ by the Stone-Weierstrass theorem. We may further suppose that the map $f \mapsto Tf(\omega)$ is everywhere linear on E (by picking a Hamel basis of E and extending). Thus we may define $T_\omega: E \to X$ by

$$T_\omega f = Tf(\omega).$$

For any n, k,

$$\int_\Omega \|T_\omega \chi_{n,k}\|^p \, d\nu \leqslant \|T\|^p \lambda(U_{n,k}),$$

and hence

$$\int_\Omega \sum_{k=1}^{l(n)} \|T_\omega \chi_{n,k}\|^p \, d\nu \leqslant \|T\|^p.$$

Now if

$$\phi(\omega) = \sup_n \sum_{k=1}^{l(n)} \|T_\omega \chi_{n,k}\|^p,$$

then from the Monotone Convergence Theorem,

$$\int \phi(\omega) \, d\nu(\omega) \leqslant \|T\|^p,$$

and so $\phi(\omega) < \infty$ ν-a.e. Let Ω_0 be a Borel set with $\nu(\Omega \backslash \Omega_0) = 0$ such that

$\phi(\omega) < \infty$ for $\omega \in \Omega_0$. Then if $\omega \in \Omega_0$ and $f_1, \ldots, f_m \in E$ we can find a large enough n so that

$$f_i = \sum_{j=1}^{l(n)} c_{ij} \chi_{n,j}, \qquad i = 1, 2, \ldots, m.$$

Thus

$$\| T_\omega f_i \|^p \leqslant \sum_{j=1}^{l(n)} |c_{ij}|^p \, \| T_\omega \chi_{n,j} \|^p,$$

so that

$$\begin{aligned} \sum_{i=1}^{m} \| T_\omega f_i \|^p &\leqslant \sum_{j=1}^{l(n)} \left(\sum_{i=1}^{m} |c_{ij}|^p \right) \| T_\omega \chi_{n,j} \|^p \\ &\leqslant \phi(\omega) \max_{s \in K} \sum_{i=1}^{m} |f_i(s)|^p. \end{aligned}$$

It follows easily that T_ω extends to a p-integral operator, $T_\omega : C(K) \to X$, and $\pi_p(T_\omega) = \phi(\omega)$. After suitable definition on $\Omega \backslash \Omega_0$, $\omega \mapsto T_\omega$ can be assumed strongly Borel measurable (clearly $f \mapsto T_\omega f$ is Borel for $f \in E$).

Now a measure μ'_ω can be defined so that

$$\mu'_\omega(U_{n,k}) = \sup_{m \geqslant n} \sum_{U_{m,h} \subset U_{n,k}} \| T_\omega \chi_{m,h} \|^p$$

for $\omega \in \Omega_0$. It is clear than, first for $f \in E$ and then in general

$$\| T_\omega f \|^p \leqslant \int |f|^p \, d\mu'_\omega.$$

However, $\|\mu'_\omega\| = \phi(\omega) = \pi_p(T_\omega)$ and hence $\mu(T_\omega) = \mu'_\omega = \mu_\omega$, say. Thus

$$\int \mu_\omega(U_{n,k}) \, dv \leqslant \| T \|^p \, \lambda(U_{n,k}),$$

and so we obtain (5.1.5).

Now it follows easily from the preceding Proposition 5.1 that T can be given by formula (5.2.1).

Remark 5.2.3. The map $\omega \mapsto T_\omega$ is unique up to sets of ν-measure zero given only that

$$T_\omega f = Tf \qquad \nu\text{-a.e.},$$

for $f \in C(K)$. Simply check against a dense countable subset of $C(K)$.

THEOREM 5.3. *Suppose X is a separable p-Banach space and N is a closed subspace of X which is either pseudo-dual or q-convex for some $q > p$. Then if $T: L_p \to L_p(X/N)$ is a bounded linear operator, there is a bounded operator $T_1: L_p \to L_p(X)$ so that $QT_1 = T$, where $Q: L_p(X) \to L_p(X/N)$ is the natural quotient map.*

Proof. Write $L_p = L_p(K, \lambda)$, where K is compact metric, and $L_p(X/N) = L_p(\Omega, \nu; X/N)$. Then the operator T may be represented as above

$$Tf(\omega) = T_\omega f,$$

where $\omega \mapsto T_\omega(\Omega \to \mathscr{M}_p(K, X/N))$ is a Borel map satisfying

$$\int \mu(T_\omega)|B|\, d\nu(\omega) \leqslant \|T\|^p \lambda(B) \qquad B \in \mathscr{B}(K).$$

By Lemma 3.5 there is a Borel map $\omega \mapsto S_\omega$ of Ω into $\mathscr{M}_p(K, X)$ so that (3.5.1) and (3.5.2) hold. (We use Q for both the quotient $X \to X/N$ and $L_p(X) \to L_p(X/N)$.) Now if

$$Sf(\omega) = S_\omega f$$

then by Proposition 5.1, S is a bounded linear operator from $L_p(K, \lambda)$ into $L_p(\Omega, \nu; X)$ and clearly $QS = T$.

COROLLARY 5.4. *$L_p(L_p|1)$ is not isomorphic to L_p for $0 < p < 1$.*

Proof. If $T: L_p \to L_p(L_p|1)$ is an isomorphism then there is a lift $S: L_p \to L_p(L_p)$. Now $I - ST^{-1}Q$ is a projection of $L_p(L_p)$ onto its subspace $L_p(\mathbb{R})$, where $\mathbb{R} \subset L_p$ is the space of constants. The non-existence of such a projection is shown in Corollary 4.5 of [7]. (Another proof will be given later with more general results.)

6. DIAGONAL PROJECTIONS

In Section 4 we introduced diagonal operators $T: L_p(\Omega, \nu; X) \to L_p(\Omega, \nu: Y)$ and gave a representation theorem for them (Theorem 4.2). We now slightly extend our definition. Suppose K is a compact metric space and λ is a probability measure on K; let Ω be a Polish space and ν be a σ-finite Borel measure on Ω. Let $\sigma: \Omega \to K$ be a Borel map. If X and Y are separable p-Banach spaces we shall show that a bounded linear operator T: $L_p(K, \lambda; X) \to L_p(\Omega, \nu; Y)$ is *σ-elementary* if, wherever $B \in \mathscr{B}(K)$,

$$f(s) = 0 \qquad s \in B,$$

implies

$$Tf(\omega)=0 \qquad \omega\in\sigma^{-1}B.$$

Clearly if $\Omega=K$ and $\nu=\lambda$ and σ is the identity we obtain by this the definition of a diagonal operator. Let us remark that it is possible for suitable σ that the only σ-elementary operator is $T=0$.

Now for each $n\in\mathbb{N}$ let $\{B_{n,1},\dots,B_{n,l(n)}\}$ be a partitioning of K into disjoint Borel sets of diameter at most $1/n$. For $T\in\mathscr{L}=\mathscr{L}(L_p(K,\lambda;X),$ $L_p(\Omega,\nu;Y))$ we define

$$\Pi_n(T)=\sum_{k=1}^{l(n)}P_{\sigma^{-1}(B_{n,k})}TP_{B_{n,k}}.$$

Here, if A is a Borel subset of Ω, then $P_A\colon L_p(\Omega,\nu;Y)\to L_p(\Omega,\nu;Y)$ is defined by

$$\begin{aligned}P_Af(\omega)&=f(\omega) \qquad &\omega\in A,\\ &=0 &\omega\notin A,\end{aligned}$$

and similarly on $L_p(K,\lambda;X)$.

Now $\Pi_n\colon\mathscr{L}\to\mathscr{L}$ is a projection and $\|\Pi_n\|=1$. Our main result is then

Proposition 6.1.

(i) *For $f\in L_p(K,\lambda;X)$,*

$$\lim_{n\to\infty}\Pi_n(T)f=\Delta_\sigma(T)f \quad \textit{exists.} \tag{6.1.1}$$

(ii) *$\Delta_\sigma\colon\mathscr{L}\to\mathscr{L}$ is a projection. $\|\Delta_\sigma\|\leqslant 1$ and $\Delta_\sigma(\mathscr{L})$ is the set of σ-elementary operators* (6.1.2)

(iii) *If $\phi\in L_p(K,\lambda)$ and $x\in X$ then*

$$\Delta_\sigma(T)(\phi\otimes x)(\omega)=\phi(\sigma\omega)\,\hat{T}(x)_\omega\,(e_{\sigma\omega}) \qquad \nu\text{-a.e.,} \tag{6.1.3}$$

where $\hat{T}$ is defined by (4.1.1), *$\omega\mapsto\hat{T}(x)_\omega$ is the representation of $\hat{T}(x)$ determined in Theorem* 5.2 *and if $t\in K$*

$$\begin{aligned}e_t(s)&=1 \qquad &s=t,\quad s\in K\\ &=0 &s\neq t,\quad s\in K.\end{aligned}$$

Proof. For $\phi\in L_p(K,\lambda)$ and $x\in X$

$$\widehat{\Pi_n(T)}(x)(\phi)(\omega)=\hat{T}(x)_\omega\,(\phi\cdot 1_{B_{n,k}}),$$

where $\omega \in \sigma^{-1}B_{n,k}$, $1 \leqslant k \leqslant l(n)$. Thus $k = k_n(\omega)$. Letting $n \to \infty$ we see that for ν-a.e., $\omega \in \Omega$, $\phi \cdot 1_{B_{n,k}} \to \phi(\sigma\omega)\, e_{\sigma\omega}$ in $L_p(\mu(\hat{T}(x)_\omega))$. Thus

$$\lim_{n\to\infty} \widehat{\Pi_n(T)}(x)(\phi)(\omega) = \phi(\sigma\omega)\, \hat{T}(x)_\omega\, (e_{\sigma\omega}) \qquad \nu\text{-a.e.} \tag{6.1.4}$$

and in particular the right-hand side is Borel measurable (when suitably extended on a set of measure zero).

Now

$$\| \widehat{\Pi_n(T)}(x)(\phi)(\omega)\|^p \leqslant \int_K |\phi|^p\, d\mu(\hat{T}(x)_\omega)$$

for every n and

$$\int_\Omega \int_K |\phi|^p\, d\mu(\hat{T}(x))_\omega\, d\nu(\omega) \leqslant \| \hat{T}(x)\|^p\, \|\phi\|^p$$
$$\leqslant \|T\|^p\, \|x\|^p\, \|\phi\|^p.$$

Thus we can employ the dominated convergence theorem to show that

$$\int_\Omega \|\phi(\sigma\omega)\, \hat{T}(x)_\omega\, (e_{\sigma\omega}) - \widehat{\Pi_n(T)}(x)(\phi)(\omega)\|^p\, d\nu \to 0.$$

Thus we define $\Delta_\sigma(T)$ by

$$\widehat{\Delta_\sigma(T)}(x)(\phi) = \lim_{n\to\infty} \widehat{\Pi_n(T)}(x)(\phi),$$

for $x \in X$ and $\phi \in L_p(K, \lambda)$, then $\Delta_\sigma(T)$ is a well-defined member of $\mathscr{L}$, (6.1.3) holds and

$$\|\Delta_\sigma(T)\| \leqslant \liminf_{n\to\infty} \|\Pi_n(T)\| \leqslant \|T\|.$$

By a density argument we quickly obtain (6.1.1). The fact that Δ_σ is a projection is trivial, and we have $\|\Delta_\sigma\| \leqslant 1$. Clearly if T is σ-elementary then $\Delta_\sigma(T) = T$. Conversely suppose $\Delta_\sigma(T) = T$ and $f \in L_p(K, \lambda; X)$ satisfies

$$f(s) = 0 \qquad s \in B,$$

where $B \in \mathscr{B}(K)$. Then there exists a sequence of functions f_n of the form

$$f_n = \sum_{i=1}^{m} \phi_i \otimes x_i$$

(where $\phi_i \in L_p(K,\lambda)$, $x_i \in X$) such that $f_n \to f$. In fact $f_n \cdot 1_{K\setminus B} \to f$ and

$$f_n \cdot 1_{K\setminus B} = \sum_{i=1}^{m} \phi_i \cdot 1_{K\setminus B} \otimes x_i .$$

Now,

$$\begin{aligned} \Delta_\sigma(T)(f_n \cdot 1_{K\setminus B})(\omega) &= \sum_{i=1}^{m} \phi_i(\omega)\, 1_{K\setminus B}(\omega)\, \hat{T}(x_i)_\omega\, (e_{\sigma\omega}) \\ &= 0 \qquad \text{if} \quad \omega \in B \ (\nu\text{-a.e.}). \end{aligned}$$

Hence T is σ-elementary and (6.1.2) is established.

Remarks. Of course the projection Δ_σ defined above is independent of the choice of $(B_{n,k})$. In the case $K=\Omega$, $\lambda=\nu$ and $\sigma(s)=s$, $s\in K$, we shall denote Δ_σ by simply Δ. Δ is thus a projection onto the diagonal operators.

Let us call a subspace W of $L_p(K,\lambda;X)$ *diagonal* if for $\phi\in W$, then $P_B\phi\in W$ for every $B\in\mathscr{B}$. Note that if T: $L_p(K,\lambda;X)\to L_p(K,\lambda;Y)$ is diagonal, then the range of T, $\mathscr{R}(T)$, and the kernel of T, $\mathscr{N}(T)$ are diagonal subspaces.

PROPOSITION 6.2. *Suppose W is a complemented diagonal subspace of $L_p(K,\lambda;X)$. Then there is a diagonal projection onto W.*

Proof. If T: $L_p(K,\lambda;X)\to W$ is a projection then so is $\Pi_n(T)$ for each n, where $\Pi_n(T)$ is defined as in Theorem 6.1. Hence so is $\Delta(T)$, since $\Pi_n(T)(f)\to\Delta(T)(f)$ for every $f\in L_p(X)$.

THEOREM 6.3. *Let Y be a closed subspace of the separable p-Banach space X. Suppose $L_p(K,\lambda;Y)$ is complemented in $L_p(K,\lambda;X)$. Then Y is complemented in X.*

Proof. By Proposition 6.2, there is a diagonal projection T: $L_p(K,\lambda;X)\to L_p(K,\lambda;Y)$. By Theorem 4.2 there is a strongly measurable map $s\mapsto A_s$ from K into $\mathscr{L}(X)$ so that

$$Tf(s) = A_s(f(s)) \qquad \lambda\text{-a.e.}$$

Now

$$T^2f(s) = A_s^2(f(s)) \qquad \lambda\text{-a.e.},$$

and by considering a dense countable subset of X we see

$$A_s^2 = A_s \qquad \lambda\text{-a.e.},$$

i.e., A_s is a projection λ-a.e. Clearly we also have λ-a.e., $A_s(X) \subset Y$ and $A_s(y) = y$ for $y \in Y$. Thus λ-a.e. A_s is a projection of X onto Y.

EXAMPLE. The fact that $L_p(\mathbb{R})$ is uncomplemented in $L_p(L_p)$ $(p < 1)$ (where $\mathbb{R}$ is the subspace of constants), which we used in the preceding section follows immediately. As noted where this is equivalent to the absence of a projection from $L_p([0, 1] \times [0, 1])$ onto the subspace of functions depending only on the first variable.

7. ELEMENTARY OPERATORS

Again in this section we suppose Ω is a Polish space and ν is a σ-finite Borel measure on Ω, while K is a compact metric space and λ is a probability measure on K. X and Y will be separable p-Banach spaces.

THEOREM 7.1. *Suppose σ: $\Omega \to K$ is a Borel map which is anti-injective (see Section 2), and that T: $L_p(K, \lambda; X) \to L_p(\Omega, \nu; Y)$ is a σ-elementary bounded linear operator.*

Then there a linear operator J: $L_p(\Omega, \nu; Y) \to L_p(K, \lambda; L_p(Y))$ and a Borel subset A of Ω such that

(i) *$J = JP_A$ and $J|L_p(A, \nu; Y)$ is an isometry.* (7.1.1)

(ii) *There is a Borel subset C of K such that $L_p(C, \lambda, L_p(Y)) = \mathscr{R}(J)$.* (7.1.2)

(iii) *JT is diagonal and $\mathscr{R}(T) \subset L_p(A, \nu; Y)$.* (7.1.3)

Proof. It will be convenient to suppose ν is a probability measure. The reduction to this case is immediate from Fact 4.0.2.

We now appeal to Lemma 2.2, to determine a compact metric space M, a probability measure π on M and a Borel map τ: $\Omega \to M$ such that (2.2.1) and (2.2.2) are satisfied. Let ρ: $K \times M \to \Omega$ be as in (2.2.1).

The measure $\sigma^*\nu$ may be decomposed into a part absolutely continuous with respect to λ and a part singular with respect to λ, i.e.,

$$d(\sigma^*\nu) = \theta\, d\lambda + d\eta,$$

where θ: $K \to \mathbb{R}$ is a non-negative Borel map and η is singular with respect to λ.

We consider the space $L_p(K, \lambda; L_p(M, \pi, Y)) \cong L_p(K \times M, \lambda \times \pi, Y)$ (Fact 4.0.5). Define J: $L_p(\Omega, \nu, Y) \to L_p(K \times M, \lambda \times \pi, Y)$ by

$$Jf(s, t) = \theta(s)^{1/p} f(\rho(s, t)) \qquad s \in K, \quad t \in M.$$

Thus

$$\begin{aligned}\|Jf\|^p &= \int_{K\times M} \theta(s)\,\|f(\rho(s,t))\|^p\, d(\lambda\times\pi)(s,t)\\ &\leqslant \int_{K\times M} \|f(\rho(s,t))\|^p\, d(\sigma^*\nu\times\pi)(s,t)\\ &= \int_\Omega \|f(\omega)\|^p\, d\nu(\omega)\\ &= \|f\|^p.\end{aligned}$$

Hence $\|J\|\leqslant 1$. However, $\|Jf\|=\|f\|$ if and only if

$$\int_{K\times M} \|f(\rho(s,t))\|^p\, d(\eta\times\pi)(s,t)=0 \tag{7.1.4}$$

and $Jf=0$ if and only if

$$\theta(s)\, f(\rho(s,t))=0 \qquad (\lambda\times\pi)\text{-a.e.} \tag{7.1.5}$$

There exists $B_0\in\mathscr{B}(K)$ with $\eta(B_0)=\lambda(K\backslash B_0)=0$. Let $A=\sigma^{-1}(B_0)$. Then $(\eta\times\pi)(\rho^{-1}A)=0$ and so if $f=P_Af=0$, then $f(\rho(s,t))=0$ $(\lambda\times\pi)$-a.e. This establishes (7.1.1).

To prove (7.1.2) let $C=\{s\colon \theta(s)>0\}$ and observe that if $f\in L_p(C,\lambda,L_p(Y))$, then $f=Jg$, where

$$\begin{aligned} g(\omega) &= \theta(\sigma\omega)^{-1/p} f(\sigma\omega,\tau\omega) && \text{if}\quad \theta(\sigma\omega)>0\\ &=0 && \text{if}\quad \theta(\sigma\omega)=0.\end{aligned}$$

Finally if $f\in L_p(K,\lambda;X)$ and $B\in\mathscr{B}(K)$ then

$$f(s)=0 \qquad \lambda\text{-a.e. } s\in B$$

implies

$$Tf(\omega)=0 \qquad \nu\text{-a.e. } \omega\in\sigma^{-1}B$$

so that JT is diagonal.

We now turn to the general case.

THEOREM 7.2. *Suppose $\sigma\colon \Omega\to K$ is any Borel map and $T\colon L_p(K,\lambda;X)\to L_p(\Omega,\nu;Y)$ is a σ-elementary bounded linear operator.*

Then there is a linear operator J: $L_p(\Omega, \nu; X) \to L_p(K, \lambda, l_p(Y) \oplus L_p(Y))$, *and a Borel subset* A *of* Ω *such that*:

(i) $J = JP_A$ *and* $J|L_p(A, \nu; Y)$ *is an isometry.* (7.2.1)

(ii) $\mathscr{R}(J)$ *is closed and there is a diagonal projection* Q *onto* $\mathscr{R}(J)$. (7.2.2)

(iii) JT *is diagonal and* $\mathscr{R}(T) \subset L_p(A, \nu; Y)$. (7.2.3)

Remark. Here the direct sum is the l_p-sum.

Proof. Choose a maximal family $(\Omega_n;\ n = 1, 2, \ldots)$ of disjoint compact subsets of Ω of positive finite ν-measure, and such that $\sigma|\Omega_n$ is injective and continuous. Such a family is clearly at most countably infinite. Let $\Omega_\infty = \Omega \backslash \bigcup \Omega_n$. Then Ω_∞ is a G_σ-set and $\sigma|\Omega_\infty$ is anti-injective by Lusin's theorem. For convenience of exposition we shall suppose $(\Omega_n)_{n=1}^\infty$ is infinite and $\nu(\Omega_\infty) > 0$; only minor modifications are required for the other cases.

By the preceding theorem there is a linear map $J_\infty : L_p(\Omega_\infty, \nu; Y) \to L_p(K, \lambda; L_p(Y))$ such that for some Borel subset A_∞ of Ω_∞,

$J_\infty = J_\infty P_{A_\infty}$ and $J|L_p(A_\infty, \nu; Y)$ is an isometry. (7.2.4)

There is a Borel subset C_∞ of K such that $\mathscr{R}(J_\infty) = L_p(C_\infty, \lambda, L_p(Y))$. (7.2.5)

$J_\infty P_\infty T$ is diagonal and $\mathscr{R}(P_\infty T) \subset L_p(A_\infty, \nu; Y)$. (Here $P_\infty = P_{\Omega_\infty}$ is the natural projection of $L_p(\Omega, \nu; Y)$ onto its subspace $L_p(\Omega_\infty, \nu; Y)$.) (7.2.6)

For $n < \infty$, $\sigma|\Omega_n$ has a continuous inverse on $\sigma(\Omega_n)$. Let $\nu_n = \nu|\Omega_n$ and suppose

$$d(\sigma^* \nu_n) = \theta_n \, d\lambda + d\eta_n$$

(as in 7.1), where θ_n is a non-negative Borel function and η_n is singular with respect to λ. Define

$$J_n : L_p(\Omega_n, \nu; Y) \to L_p(K, \lambda, Y)$$

by

$$\begin{aligned} J_n f(s) &= \theta_n(s)^{1/p} f(\sigma^{-1} s) \qquad && s \in \sigma(\Omega_n) \\ &= 0 && s \notin \sigma(\Omega_n). \end{aligned}$$

Then $\|J_n\| \leqslant 1$, and arguing as in Theorem 7.1 we can find a subset A_n of Ω_n so that $J = JP_{A_n}$ and $J|L_p(A_n, \nu; Y)$ is an isometry. Furthermore, $\mathscr{R}(J) =$

$L_p(C_n, \lambda; Y)$, where $C_n = (s; \theta_n(s) > 0)$, $\mathscr{R}(P_n T) \subset L_p(A_n, \nu; Y)$ and $J_n P_n T$ is diagonal.
Now define

$$J: L_p(\Omega, \nu; Y) \to L_p(K, \lambda, l_p(Y) \oplus L_p(Y))$$

by

$$Jf(s) = ((J_n P_n f(s))_{n=1}^{\infty} J_\infty P_\infty f(s)).$$

Let $A = \bigcup_{n=1}^{\infty} A_n$ and define a diagonal projection $Q: L_p(K, \lambda; l_p(Y) \oplus L_p(Y)) \to L_p(K, \lambda, l_p(Y) \oplus L_p(Y))$ by

$$Qf(s) = ((P_{C_n} f_n(s))_{n=1}^{\infty}, P_{C_\infty} f_\infty(s)),$$

where

$$f(s) = ((f_n(s))_{n=1}^{\infty}, f_\infty(s)).$$

Then conditions (7.2.1)–(7.2.3) are satisfied. We omit the easy verification.

THEOREM 7.3. *Suppose K is a compact metric space and λ is a diffuse probability measure on K. Let $\mathscr{B}_0$ be a sub-σ-algebra of $\mathscr{B} = \mathscr{B}(K)$, and let $L_p(K, \mathscr{B}_0, \lambda)$ be the closed subspace of $L_p(K, \lambda)$ of all $\mathscr{B}_0$-measurable functions. Assume $L_p(K, \mathscr{B}_0, \lambda)$ is a non-trivial subspace and let $\Lambda(\mathscr{B}_0)$ be the quotient space $L_p(K, \lambda)/L_p(K, \mathscr{B}_0, \lambda)$. Then*

(i) *$\Lambda(\mathscr{B}_0)$ contains a complemented copy of L_p.* (7.3.1)

(ii) *If $\Lambda(\mathscr{B}_0) \cong L_p$, then $L_p(K, \mathscr{B}_0, \lambda)$ is complemented in L_p.* (7.3.2)

Remarks. In [7] we showed that $L_p(K, \mathscr{B}_0, \lambda)$ is complemented in $L_p(K, \lambda)$ if and only if there exist $A \in \mathscr{B}$, $\varepsilon > 0$ such that:

$$\lambda(B \cap A) \geqslant \varepsilon \lambda(B) \qquad B \in \mathscr{B}_0. \tag{7.3.3}$$

If $C \subset A$, $C \in \mathscr{B}$, then there exists $B \in \mathscr{B}_0$ with $\lambda((B \cap A) \Delta C) = 0$. (7.3.4)

In fact, conditions (7.3.3) and (7.3.4) imply the existence of an automorphism of L_p taking $L_p(A, \lambda)$ into $L_p(K, \mathscr{B}_0, \lambda)$. Precisely $P_A | L_p(K, \mathscr{B}_0, \lambda)$ is an isomorphism onto $L_p(A, \lambda)$ and has inverse $V: L_p(A, \lambda) \to L_p(K, \mathscr{B}_0, \lambda)$, say; then the automorphism is given by $U = P_{K \setminus A} + V P_A$. In particular, if $L_p(K, \mathscr{B}_0, \lambda)$ is complemented then $\Lambda(\mathscr{B}_0) \cong L_p$.

Proof. We may assume that $\mathscr{B}_0$ is generated up to sets of measure zero by a sequence $(B_n)_{n=1}^{\infty}$ of Borel sets. Define $\sigma: K \to 2^{\mathbb{N}} = M$, say, by

$$(\sigma s)_n = 1_{B_n}(s) \qquad s \in K.$$

Induce a probability measure π on M by $\pi = \sigma^* \lambda$. Then $L_p(K, \mathscr{B}_0, \lambda) = \mathscr{R}(T)$ when T is the σ-elementary operator

$$Tf(s) = f(\sigma s).$$

Now by Theorem 7.2 we can find a map J: $L_p(K, \lambda) \to L_p(M, \pi, l_p \oplus L_p)$ and a Borel subset A of K such that $J = JP_A$, $J | L_p(A, \lambda)$ is an isometry, $\mathscr{R}(J)$ is complemented, JT is diagonal and $\mathscr{R}(T) \subset L_p(A, \lambda)$. Since $1_K \in \mathscr{R}(T)$, we must have $\lambda(K \backslash A) = 0$, i.e., J is an isometric embedding. Of course by construction T is also an isometric embedding.

Thus

$$JT\phi(s) = \phi(s)\, g(s) \qquad \pi\text{-a.e.} \quad s \in M,$$

where g: $M \to l_p \oplus L_p$ satisfies $\| g(s) \| = 1$, $s \in M$. To prove (7.3.1) let us first assume that for some $n < \infty$, and some Borel subset C of M of positive π-measure,

$$| g_n(s) | \geqslant \delta > 0, \qquad s \in C.$$

If we defined P: $\mathscr{R}(J) \to \mathscr{R}(JT)$ by

$$\begin{aligned} Pf(s) &= g_n(s)^{-1} f_n(s)\, g(s) && s \in C \\ &= 0 && s \notin C, \end{aligned}$$

then P is a projection. $J^{-1}PJ$ is clearly seen to be a projection of $L_p(\sigma^{-1}C, \lambda)$ onto $L_p(\sigma^{-1}C, \mathscr{B}_0, \lambda)$. Clearly $L_p(\sigma^{-1}C, \lambda)/L_p(\sigma^{-1}C, \mathscr{B}_0, \lambda)$ is isomorphic to a complemented subspace of $\Lambda(\mathscr{B}_0)$. Thus if $\Lambda(\mathscr{B}_0)$ does not contain a complemented copy of L_p, we conclude

$$L_p(\sigma^{-1}C, \mathscr{B}_0, \lambda) = L_p(\sigma^{-1}C, \lambda).$$

Since M cannot be a countable family of such sets, there is a compact set $A \subset M$ such that $g_n(s) = 0$, $s \in A$, $1 \leqslant n < \infty$ and $\pi(A) > 0$.

This means $\| g_\infty(s) \| = 1$, $s \in A$. By observing the form of the projection onto $\mathscr{R}(J)$ we note that $\Lambda(\mathscr{B}_0)$ contains a complemented copy of $L_p(A, \pi; L_p)/\mathscr{R}(JTP_A)$.

Now by Lemma 2.3 we can find a strongly measurable map $s \mapsto V_s$ $(A \to \mathscr{L}(L_p))$ such that $V_s g_\infty(s) = 1_\Omega$ (where $L_p = L_p(\Omega)$) and $\| V_s \|$,

$\|V_s^{-1}\| \leqslant 2$. [Here V_s^{-1} is constructed in Lemma 2.3.] Using this we construct an automorphism

$$V: L_p(A, \pi; L_p) \to L_p(A, \pi; L_p)$$
$$Vf(s) = V_s f \qquad s \in A,$$

and conclude

$$L_p(A, \pi; L_p)/\mathscr{R}(JTP_A) \cong L_p(A, \pi; L_p|1).$$

As $L_p|1 \cong L_p \oplus L_p|1$ [8], (7.3.1) follows.

Now we turn to (7.3.2). First we embed $l_p \oplus L_p$ isometrically in L_p by an isometry U, say, and thus induce a diagonal isometry

$$\hat{U}: L_p(M, \pi, l_p \oplus L_p) \to L_p(M, \pi, L_p),$$
$$\hat{U}f(s) = U(f(s)) \qquad s \in M.$$

Let $h = \hat{U}g$. As above we find a diagonal automorphism V of $L_p(M, \pi, L_p)$ so that $\|V\|, \|V^{-1}\| \leqslant 2$ and $Vh(s) = 1_\Omega$ for all s. Thus $\mathscr{R}(V\hat{U}JT) = L_p(M, \pi, \mathbb{R} \cdot 1_\Omega)$. Now $\Lambda(\mathscr{B}_0) \cong \mathscr{R}(J)/\mathscr{R}(JT) \cong \mathscr{R}(V\hat{U}J)/\mathscr{R}(V\hat{U}JT)$ and so we have an embedding

$$R: \Lambda(\mathscr{B}_0) \to L_p(M, \pi, L_p)/\mathscr{R}(V\hat{U}JT),$$

where $RQ_1 = Q_2 V\hat{U}J$ if $Q_1: L_p(K, \lambda) \to \Lambda(\mathscr{B}_0)$ and $Q_2: L_p(M, \pi, L_p) \to L_p(M, \pi; L_p)/\mathscr{R}(V\hat{U}JT)$ are the quotient maps.

Now $L_p(M, \pi; L_p)/\mathscr{R}(V\hat{U}JT) \cong L_p(M, \pi; L_p|1)$ and so by Theorem 5.3, there is a lift $\tilde{R}: \Lambda(\mathscr{B}_0) \to L_p(M, \pi; L_p)$ so that $Q_2\tilde{R} = R$. Since $\mathscr{R}(Q_2\tilde{R}) \subset \mathscr{R}(Q_2 V\hat{U}J)$ we conclude $\mathscr{R}(\tilde{R}) \subset \mathscr{R}(V\hat{U}J)$ and can define $S = (V\hat{U}J)^{-1}\tilde{R}$. $S: \Lambda(\mathscr{B}_0) \to L_p(K, \lambda)$ is an embedding and $RQ_1S = Q_2\tilde{R} = R$ so that $Q_1S = I$ on $\Lambda(\mathscr{B}_0)$. Hence $L_p(K, \mathscr{B}_0, \lambda)$ is complemented.

8. Isomorphisms between $L_p(X)$-Spaces

We recall that a p-Banach space X is p-trivial [6] if $\mathscr{L}(L_p, X) = \{0\}$ when $0 < p < 1$. As shown in [6], p-triviality is an appropriate non-locally convex analogue of the Radon–Nikodym Property for Banach spaces. Note that, for example, if X has a separating dual or is q-convex for some $q > p$, then X is p-trivial.

THEOREM 8.1. *Suppose K is an infinite compact metric space and λ is a probability measure on K. Suppose X and Y are separable p-Banach spaces, where $0 < p < 1$, and that Y is p-trivial. Then if $T: L_p(K, \lambda; X) \to L_p(K, \lambda; Y)$*

is a bounded linear operator, we can find a Polish space Ω and a σ-finite Borel measure ν on Ω so that $T = ST_0$, where

(i) $S: L_p(\Omega, \nu; Y) \to L_p(K, \lambda, Y)$ *is a bounded linear surjection.* (8.1.1)

(ii) $T_0: L_p(K, \lambda; X) \to L_p(\Omega, \nu; Y)$ *is an elementary operator.* (8.1.2)

Proof. For let $\{x_n\}$ be a dense countable subset of X. For each n consider V_n: $L_p(K, \lambda) \to L_p(K, \lambda; Y)$ given by $V_n = \hat{T}(x_n)$ as in Proposition 4.1. Now there is a strongly Borel measurable map $s \mapsto V_{n,s}$ $(\Omega \to \mathscr{M}_p(K; Y))$ so that

$$V_{n,s} f = V_n f(s) \qquad \lambda\text{-a.e.,} \quad s \in K,$$

and if $\mu_{n,s} = \mu(V_{n,s})$ then $s \mapsto \mu_{n,s}$ is also a Borel map $(K \to \mathscr{M}(K))$ and $V_{n,s}$: $L_p(K, \mu_{n,s}) \to Y$ is continuous. But Y is p-trivial and so

$$\| V_{n,s} f \|^p \leqslant \int_K |f(s)|^p \, d\mu^a_{n,s}.$$

where $\mu^a_{n,s}$ is the purely atomic part of the measure $\mu_{n,s}$. By definition of $\mu(V_{n,s})$ we conclude that $\mu^a_{n,s} = \mu_{n,s}$ $(s \in K)$.

Now applying Theorem 2.10 of [7] there are universally measurable maps $a_{n,j}: K \to \mathbb{R}$, $t_{n,j}: K \to K$ so that

$$\mu_{n,s} = \sum_{j=1}^{\infty} a_{n,j}(s) \, \delta(\tau_{n,j}(s)), \tag{8.1.3}$$

where for every $s \in K$, and fixed $n \in \mathbb{N}$, $\tau_{n,j}(s) = \tau_{n,k}(s)$ implies $j = k$.

For convenience we can redefine $a_{n,j}$, $\tau_{n,j}$ on a set of λ-measure zero so that they are Borel and (8.1.3) holds λ-a.e. Now let $\{\tau^*_n : n \in \mathbb{N}\}$ be any sequential ordering of the maps $\{\tau_{n,j};\, n \in \mathbb{N}, j \in N\}$, and define inductively $\sigma_1 = \tau^*_1$, and if $n > 1$,

$$\sigma_n(s) = \tau^*_k(s),$$

where k is the least index such that $\tau^*_k(s) \notin \{\sigma_1(s), \ldots, \sigma_{n-1}(s)\}$. This process defines a sequence $\{\sigma_n\}$ of Borel maps $\sigma_n: K \to K$ such that $\sigma_n(s) = \sigma_m(s)$ implies $n = m$ and for any fixed k, $\{\tau_{k,j}(s)\}_{j=1}^{\infty} \subset \{\sigma_j(s)\}_{j=1}^{\infty}$.

Now for each $j = 1, 2, \ldots$ let

$$\Delta_j(t) = \Delta_{\sigma_j}(T)$$

so that $\Delta_j(T)$: $L_p(K, \lambda; X) \to L_p(K, \lambda; Y)$ is σ_j-elementary. Let $\Omega = K \times \mathbb{N}$ with ν the product of λ and counting measure on $\mathbb{N}$ and define

$$T_0 : L_p(K, \lambda; X) \to L_p(\Omega, \nu; Y)$$

by

$$T_0 f(s, n) = \Delta_n(T) f(s) \qquad s \in K, \quad n \in \mathbb{N}.$$

We must first check that T_0 is a bounded linear operator. This is most easily done by checking its behavior on elements $\phi \otimes x$, and appealing to Proposition 4.1, to produce a bounded linear operator T_0' agreeing with T_0 on such elements. Then it is trivial that $T_0' = T_0$.

Now for fixed n,

$$\begin{aligned} T_0(\phi \otimes x)(s, n) &= \Delta_n(T)(\phi \otimes x)(s) \\ &= \phi(\sigma_n(s))\, \hat{T}(x)_s\, e_{\sigma_n(s)} \qquad \lambda\text{-a.e.} \end{aligned}$$

by (6.1.3). Hence

$$\| T_0(\phi \otimes x)(s, n)\|^p \leqslant |\phi(\sigma_n(s))|^p\, \mu(\hat{T}(x)_s)\{\sigma_n(s)\},$$

and

$$\begin{aligned} \int \| T_0(\phi \otimes x)(\omega)\|^p\, d\nu(\omega) &= \sum_{n=1}^{\infty} \int_K \| T_0(\phi \otimes x)(s, n)\|^p\, d\lambda(s) \\ &= \int_K \sum_{n=1}^{\infty} |\phi(\sigma_n(s))|^p\, \mu(T(x))_s\, \{\sigma_n(s)\}\, d\lambda(s) \\ &\leqslant \int_K \int_K |\phi(t)|^p\, d\mu(\hat{T}(x))_s\, d\lambda(s) \\ &\leqslant \| \hat{T}(x)\|^p \int_K |\phi(s)|^p\, d\lambda(s). \end{aligned}$$

This last step follows from (5.1.5) (with $c = \| \hat{T}(x)\|$). Thus

$$\| T_0(\phi \otimes x)\| \leqslant \| T\|\, \|\phi\|\, \|x\|.$$

as so T_0 is a bounded linear operator.

Note that T_0 is σ-elementary, where

$$\sigma(s, n) = \sigma_n(s) \qquad (s, n) \in \Omega.$$

Next define $S: L_p(\Omega, \nu; Y) \to L_p(K, \lambda; Y)$ by

$$Sf(s) = \sum_{n=1}^{\infty} f(s, n) \qquad s \in K.$$

Here the series converges λ-a.e. for any $f \in L_p(\Omega, \nu, Y)$ since

$$\int_K \sum_{n=1}^{\infty} \|f(s, n)\|^p \, d\lambda(s) = \|f\|^p$$

then S is an operator of norm one.

Now for $j \in \mathbb{N}$

$$ST_0(\phi \otimes x_j) = \sum_{n=1}^{\infty} \phi(\sigma_n s) \, V_{j,s}(e_{\sigma_n s}) \qquad \lambda\text{-a.e.}$$

For λ-almost every s, $\mu_{j,s}$ is supported on $\{\sigma_n s\}_{n=1}^{\infty}$ and $\phi \in L_p(\mu_{j,s})$. Thus

$$\begin{aligned} ST_0(\phi \otimes x_j) &= V_{j,s} \sum_{n=1}^{\infty} \phi(\sigma_n s) \, e_{\sigma_n s} \\ &= V_{j,s}(\phi) \\ &= T(\phi \otimes x_j) \end{aligned}$$

and by a density argument $ST_0 = T$.

THEOREM 8.2. *Under the same hypotheses as Theorem* 8.1, *there exist a diagonal operator*

$$D: L_p(K, \lambda; X) \to L_p(K, \lambda; l_p(Y) \oplus L_p(Y))$$

and a bounded linear operator

$$R: L_p(K, \lambda, l_p(Y) \oplus L_p(Y)) \to L_p(K, \lambda; Y) \text{ so that } T = RD.$$

Proof. We write $T = ST_0$ as in 8.1. Now by Theorem 7.2 we can find

$$J: L_p(\Omega, \nu; Y) \to L_p(K, \lambda, l_p(Y) \oplus L_p(Y))$$

and a Borel subset A of Ω such that (i) $\mathscr{R}(T_0) \subset L_p(A, \nu; Y)$, (ii) $J|L_p(A, \nu; Y)$ is an isometry onto $\mathscr{R}(J)$, (iii) $\mathscr{R}(J)$ is complemented by a projection Q and (iv) JT_0 is diagonal.

Let $D = JT_0$ and $R = SJ^{-1}Q$, where $J^{-1}: \mathscr{R}(J) \to L_p(A, \nu; Y)$, is the inverse of J.

THEOREM 8.3. *Suppose X and Y are separable p-Banach spaces with Y*

p-trivial, and such that $L_p(X)$ *is isomorphic to a complemented subspace of* $L_p(Y)$. *Then* $X \cong X_1 \oplus X_2$, *where* X_1 *is isomorphic to a complemented subspace of* $L_p(Y)$ *and* X_2 *is isomorphic to a complemented subspace of* $l_p(Y)$ [$X_1 = \{0\}$ *or* $X_2 = \{0\}$ *are possibilities here*].

Remarks. The converse of Theorem 8.3 is trivial.

Proof. Let T be an operator mapping $L_p(K, \lambda; X)$ isomorphically onto a complemented subspace of $L_p(K, \lambda; Y)$. Let $T = RD$ as in Theorem 8.2.

Here D and $R \mid \mathscr{R}(D)$ must be embeddings. If Q is a projection on $\mathscr{R}(T)$ then $R^{-1}QR$ is a projection on $\mathscr{R}(D)$, where $R^{-1}: \mathscr{R}(T) \to \mathscr{R}(D)$. However, $\mathscr{R}(D)$ is a diagonal subspace. Hence there is a diagonal projection P onto $\mathscr{R}(D)$ (by 6.2).

Now since D is embedding we can define an operator $E = D^{-1}P$ (where $D^{-1}: \mathscr{R}(D) \to L_p(K, \lambda; X)$) and E is also diagonal. Recalling Theorem 4.2, we get strongly Borel measurable maps $s \mapsto D_s$ $(K \to \mathscr{L}(X, l_p(Y) \oplus L_p(Y))$, $s \mapsto E_s$ $(K \to \mathscr{L}(l_p(Y) \oplus L_p(Y), X))$ so that

$$Df(s) = D_s(f(s)) \qquad \lambda\text{-a.e.},$$

$$Ef(s) = E_s(f(s)) \qquad \lambda\text{-a.e.}$$

Since ED is the identity on $L_p(K, \lambda; X)$, by the uniqueness of such a representation

$$E_s D_s = I \qquad \lambda\text{-a.e.}$$

This implies, a.e., that D_s maps X isomorphically onto a complemented subspace of $l_p(Y) \oplus L_p(Y)$.

To complete the proof suppose W is a complemented subspace of $l_p(Y) \oplus L_p(Y)$. Since $\mathscr{L}(L_p, Y) = 0$ we have $\mathscr{L}(L_p(Y), l_p(Y)) = 0$. Thus if Q is a projection onto W and P_1, P_2 are the canonical projections onto $l_p(Y)$ and $L_p(Y)$, respectively, we must have $P_1 Q P_2 = 0$.

It follows that QP_2 is a projection. Indeed $QP_2QP_2 = Q^2P_2 = QP_2$: $W_2 = \mathscr{R}(QP_2)$. Then W_2 is a complemented subspace of $L_p(Y)$ and $W = W_1 \oplus W_2$, where $W_1 = \mathscr{R}(QP_1Q)$. However, since QP_1 is on W a projection, $W_1 \cong \mathscr{R}(P_1QP_1)$, and P_1QP_1 is a projection. Hence W_1 is isomorphic to a complemented subspace of $l_p(Y)$. This will complete the proof.

THEOREM 8.4. *Suppose* X *and* Y *are p-trivial separable p-Banach spaces, with* $L_p(X) \cong L_p(Y)$. *Then* $l_p(X) \cong l_p(Y)$.

Proof. We have $X \cong X_1 \oplus X_2$, where X_1 is complemented in $l_p(Y)$ and X_2 is complemented in $L_p(Y)$. If X is p-trivial, $X_2 = \{0\}$, i.e., X is isomorphic to a complemented subspace of $l_p(Y)$. Thus $l_p(X)$ is isomorphic to a complemented subspace of $l_p(Y)$. Reversing the reasoning and applying the Pelczynski decomposition technique gives the result.

Remark. Of course $l_p(X) \cong l_p(Y)$ always implies $L_p(X) \cong L_p(Y)$ as $L_p(X) \cong L_p(l_p(X))$.

COROLLARY 8.5. *If X is p-trivial and $L_p(X) \cong L_p$ then X is finite-dimensional or $X \cong l_p$.*

This follows from a theorem of Stiles that every complemented infinite dimensional subspace of l_p is isomorphic to l_p [16].

COROLLARY 8.6. *If X is p-trivial and $L_p(X)$ contains a complemented copy of L_p then $X^* \neq \{0\}$.*

EXAMPLE 8.7. Let (e_n) be the cananical basis vectors of l_p and let $u_n = 2^{-1/p}(e_{2n-1} + e_{2n})$, $n = 1, 2, \ldots$. Then (u_n) is the basis for a subspace N_0 isomorphic to l_p. If $2^{1-1/p} < c < 1$, let $T: N_0 \to l_p$ be defined so that $Tu_n = ce_n$. As $\|T\| = c < 1$, $I - T: N_0 \to l_p$ is an isomorphism of N_0 onto a closed subspace N of l_p also isomorphic to l_p, with $(u_n - Tu_n)$ as a basis. By considering e_1, for example, it is not difficult to show N is a proper subspace. On the other hand, N is weakly dense. Indeed

$$u_n - Tu_n = c(Se_n - e_n),$$

where $Se_n = c^{-1} 2^{-1/p}(e_{2n-1} + e_{2n})$ so that S is bounded linear operator on l_1 with $\|S\|_1 < 1$. Thus $(I - S)$ is an automorphism of l_1 and $(u_n - Tu_n)$ is a weak basis of l_p.

Now l_p/N is p-trivial since $N \cong l_p$ by appealing to the lifting theorems of [8] (N is pseudo-dual). Thus $L_p(l_p/N)$ does not contain a complemented copy of L_p. However, $L_p(l_p/N) \cong L_p(l_p)/L_p(N)$. Hence we have found a subspace of L_p isomorphic to L_p but which cannot be moved by any automorphism into a subspace $L_p(K, \mathscr{B}_0, \lambda)$ for some sub-σ-algebra (use Theorem 7.3).

THEOREM 8.8. *Suppose X and Y are separable p-Banach spaces Y is p-trivial and $\mathscr{L}(X, Y) = 0$. Then if $L_p(X)$ is isomorphic to a complemented subspace of $L_p(Y)$, X is also isomorphic to a complemented subspace of $L_p(Y)$.*

The proof of Theorem 8.8 employs techniques similar to those of the proof of Theorem 8.4 and we omit it. Theorem 8.6 provides yet another proof that $L_p(L_p|1) \cong L_p$, for $L_p|1$ is not isomorphic to a complemented subspace of L_p (since, e.g., L_p is a K-space and $L_p|1$ is not [8]).

9. The Complemented Subspace Problem for L_p

Suppose K is a compact metric space and λ is a diffuse measure on K. Then by Theorem 3.2 of [7] every $T \in \mathscr{L}(L_p(K, \lambda))$ has a representation of the form

$$Tf(s) = \sum_{n=1}^{\infty} a_n(s) f(\sigma_n s) \qquad \lambda\text{-a.e.}, \quad s \in K, \tag{9.0.1}$$

where $a_n: K \to \mathbb{R}$ and $\sigma_n: K \to K$ are Borel maps satisfying

$$\sigma_m(s) \neq \sigma_n(s) \qquad s \in K, \quad m \neq n, \tag{9.0.2}$$

$$\sum_{n=1}^{\infty} |a_n(s)|^p < \infty \qquad \lambda\text{-a.e.}, \quad s \in K, \tag{9.0.3}$$

$$\sum_{n=1}^{\infty} \int_{\sigma_n^{-1}B} |a_n(s)|^p \, d\lambda(s) \leqslant \|T\|^p \lambda(B), \qquad B \in \mathscr{B}, \tag{9.0.4}$$

$$|a_n(s)| \geqslant |a_{n+1}(s)| \qquad s \in K, \quad n = 1, 2, \ldots. \tag{9.0.5}$$

Conversely if a_n, σ_n satisfy (9.0.2)–(9.0.5) then (9.0.1) defines a *bounded* linear operator. Of course this result also would follow from Theorem 5.2.

In [7], T was called *small* if each σ_n is anti-injective on the set $\{s: |a_n(s)| > 0\}$. This definition does not depend on the precise form of the representation (9.0.1) as long as (9.0.2) is satisfied.

If T is not small, it is called large and large operators were characterized as follows (Theorem 5.6 of [7])

PROPOSITION 9.1. *The following conditions are equivalent*:

(i) *T is large* (9.1.1)

(ii) *There is a Borel subset of K with $\lambda(B) > 0$ such that $T|L_p(B, \lambda)$ is an isomorphism and $T(L_p(B, \lambda))$ is complemented in $L_p(K, \lambda)$* (9.1.2)

(iii) *There exist $S_1, S_2 \in \mathscr{L}(L_p(K, \lambda))$ such that $S_1 T S_2 = I$* (9.1.3)

COROLLARY 9.2. *If T is a large projection $\mathscr{R}(T) \cong L_p$.*

Proof. By (9.1), $\mathscr{R}(T)$ contains a complemented copy of L_p. Hence $\mathscr{R}(T) \cong L_p$ by the Pelczynski decomposition technique.

Now if T is small, returning to (9.0.1) it is possible to alter each σ_n to be anti-injective on K without changing T (simply redefine σ_n on $(s: a_n(s) = 0)$).

PROPOSITION 9.3. *Let* $U: L_p(K, \lambda) \to L_p(K, \lambda; L_p(K, \lambda))$ *be defined by*

$$Uf(s) = f(s) \cdot 1_k.$$

Then if T *is small there exists a bounded linear operator* $S: L_p(K, \lambda, L_p(K, \lambda)) \to L_p(K, \lambda)$ *such that* $T = SU$.

Proof. Clearly $T = S_0 E$, where $E: L_p(K, \lambda) \to L_p(K \times \mathbb{N}, \gamma)$ (γ is the product of λ and counting measure) is defined by

$$Ef(s, n) = a_n(s) f(\sigma_n s)$$

and $S_0: L_p(K \times \mathbb{N}, \gamma) \to L_p(K, \lambda)$ is given by

$$S_0 f(s) = \sum_{n=1}^{\infty} f(s, n), \qquad s \in K, \quad n \in \mathbb{N}.$$

Now E is σ-elementary, where σ is anti-injective. We now argue as in Theorem 8.2, but using Theorem 7.1 in place of Theorem 7.2 to write $T = S_1 V$, where $V: L_p(K, \lambda) \to L_p(K, \lambda; L_p)$ is diagonal and $S_1: L_p(K, \lambda; L_p) \to L_p(K, \lambda)$. It will be convenient to note $L_p(K, \lambda; L_p) \cong L_p(K, \lambda; L_p(K, \lambda))$. Thus

$$Vf(s) = f(s)\, g(s) \qquad \lambda\text{-a.e.},$$

where $g: K \to L_p$ is a Borel map with $\|g(s)\| \leqslant \|V\|$ λ-a.e. By Lemma 2.3 there is a strongly Borel measurable map $s \mapsto R_s$ of K into $\mathscr{L}(L_p)$ so that $\|R_s\| \leqslant 2$ and $R_s 1_K = g(s)$. If $R: L_p(K, \lambda; L_p) \to L_p(K, \lambda; L_p)$ is given by

$$Rf(s) = R_s(f(s)) \qquad s \in K,$$

then $T = S_1 RU$ as required.

THEOREM 9.4. *Let* $T \in \mathscr{L}(L_p)$; *then* T *is small if and only if* $\chi(T) = 0$ *for every* $\chi \in \mathscr{L}(L_p)^*$.

Proof. Suppose $\chi(T) = 0$ for every $\chi \in \mathscr{L}(L_p)^*$. If T is large then $I = S_1 T S_2$ for some $S_1, S_2 \in \mathscr{L}(L_p)$ and so $\chi(I) = 0$ for every $\chi \in \mathscr{L}(L_p)^*$. However, the diagonal projection Δ maps $\mathscr{L}(L_p)$ into a subspace (of diagonal operators) isomorphic to L_∞ and $\Delta(I) = I$. This is a contradiction.

Conversely suppose T is small. Then $T = SU$ as in 9.3. For each $\phi \in L_p(K, \lambda)$, define $U_\phi: L_p(K, \lambda) \to L_p(K, \lambda, L_p(K, \lambda))$ by

$$U_\phi f(s) = f(s)\, \phi, \qquad s \in K.$$

Then $\phi \mapsto U_\phi$ is linear and $\|U_\phi\| = \|\phi\|$. Hence there is a bounded linear map $\phi \mapsto SU_\phi$ of L_p into $\mathscr{L}(L_p)$ with $1 \to T$. Thus $\chi(T) = 0$ for every $\chi \in \mathscr{L}(L_p)^*$.

Remarks. If P is small projection with range Z, say, then $\mathscr{L}(Z)^* = \{0\}$, since if $\chi \in \mathscr{L}(Z)^*$, $S \in \mathscr{L}(Z)$ $T \to \chi(SPT|Z)$ is a linear functional as $\mathscr{L}(L_p)$ and $D \to \chi(S)$. Hence $\chi(S) = 0$.

COROLLARY 9.5. *If Z is a complemented subspace of L_p with $Z \not\cong L_p$ then $L_p(Z)$ is also isomorphic to a complemented subspace of L_p, and $L_p(Z) \not\cong L_p$.*

Proof. Clearly $L_p(Z)$ is also isomorphic to a complemented subspace of $L_p(L_p) = L_p$. Additionally if $\mathscr{L}(L_p(Z))^* \neq \{0\}$, then there exists $\chi \in \mathscr{L}(L_p(Z))^*$ with $\chi(I) \neq 0$. However, there is a natural injection $\mathscr{L}(Z) \to \mathscr{L}(L_p(Z))$, sending I_Z to $I_{L_p(Z)}$ using diagonal operators. Hence $\mathscr{L}(L_p(Z))^* = \{0\}$ and so $L_p(Z) \not\cong L_p$.

THEOREM 9.6. *Suppose X is a p-Banach space with trivial dual such that $L_p(X) \cong L_p$. Then $X \cong L_p$.*

Proof. By 8.3, X is isomorphic to a complemented subspace of L_p and by Corollary 9.5, $X \cong L_p$.

Our final result shows a complemented subspace of L_p must belong to one of at most two isomorphism classes. In fact, we strongly suspect L_p is prime.

PROPOSITION 9.7. *Suppose there is a complemented subspace Z of L_p with $Z \not\cong L_p$. Then*

(i) $Z \cong L_p(Z)$. (9.7.1)

(ii) *Every complemented subspace of L_p is isomorphic either to L_p or to Z.* (9.7.2)

(iii) *Z is prime.* (9.7.3)

Proof. Let us define $\mathscr{A}$ to be the collection of all operators $T \in \mathscr{L}(L_p)$ with the following property:

There is a constant $c < \infty$ and a Borel subset B of K with $\lambda(B) > 0$ such that whenever $A \subset B$ is a Borel set of positive measure, there exist operators $S_A : Z \to L_p$, $R_A : L_p \to Z$ with $R_A T P_A S_A = I$ on Z, and $\|R_A\| \|S_A\| < c$. (9.7.4)

Two properties of $\mathscr{A}$ are easy to establish and we omit the proofs.

$$\mathscr{A} \text{ is open (for the norm topology on } \mathscr{L}(L_p)). \tag{9.7.5}$$

$$\text{If } ST \in \mathscr{A} \quad \text{then } T \in \mathscr{A}. \tag{9.7.6}$$

The proof depends on the fact that a small enough perturbation of the identity on Z is invertible.

Let $Q: L_p \to Z$ be any projection. Q is certainly a small operator and so may be written $Q = SU$, where $U: L_p(K, \lambda) \to L_p(K, \lambda, L_p(K, \lambda))$ is given by

$$Uf(s) = f(s) \cdot 1_K.$$

We shall here identify U as an endomorphism of L_p, using the fact that $L_p(K, \lambda; L_p)$ is isometric to L_p; equally we may treat S as an endomorphism of L_p. Now $U(Z)$ is also complemented in L_p by the projection UQS, and is isomorphic to Z. Hence there exist operators $R_1: Z \to L_p$, $R_2: L_p \to Z$ such that $R_2 U R_1 = I$. Now for any Borel subset A of K, with positive λ-measure there are isometries $V_1: L_p \to L_p(A, \lambda)$, $V_2: L_p(A, \lambda; L_p) \to L_p(K, \lambda, L_p)$ such that $V_2 U V_1 = U$. Hence $R_2 V_2 P_A U P_A V_1 R_1 = 1$ and $\|R_2 V_2 P_A\| \cdot \|V_1 R_1\| \leqslant \|R_1\| \cdot \|R_2\|$. Thus $U \in \mathscr{A}$, with $c = \|R_1\| \|R_2\|$ and $B = K$.

The next step is that if $V: L_p(K, \lambda) \to L_p(K, \lambda; L_p)$ is a non-zero diagonal operator then $V \in \mathscr{A}$. Indeed

$$Vf(s) = f(s) g(s),$$

where $g: K \to L_p$ is a bounded Borel map. If for some $\delta > 0$ and $B \in \mathscr{B}$ of positive measure $\| g(s)\| \geqslant \delta$, $s \subset B$, then it is easy to show (via Lemma 2.3) that $UP_B = SV$ and so by (9.7.6), $V \in \mathscr{A}$. Appealing to Theorem 7.1 and (9.7.6), this shows that every non-zero σ-elementary operator for anti-injective σ is in $\mathscr{A}$.

Next we show that every non-zero small operator belongs to $\mathscr{A}$. Suppose $T \in \mathscr{L}(L_p)$ is small; then we may write

$$Tf(s) = \sum_{n=1}^{\infty} a_n(s) f(\sigma_n s) \qquad \lambda - \text{a.e.}$$

as in (9.0.1)–(9.0.5). Let

$$Sf(s) = a_1(s) f(\sigma_1 s) \qquad \lambda\text{-a.e.}$$

S is small and elementary and also non-zero (if $S1 = 0$ then $a_1 \equiv 0$ and hence $T = 0$).

We now repeat an argument from p. 371 of [7]. For each n, let $(B_{n,k}: 1 \leqslant k \leqslant l(n))$ be partitioning of K into Borel sets of diameter at most $1/n$. Let

$$T_n = \sum_{k=1}^{l(n)} P_{\sigma_1^{-1} B_{n,k}} T P_{B_{n,k}}.$$

Then as on p. 371 of [7], $\theta_p(T_n - S) \to 0$ and by Lemma 5.2 we can find a set A of positive λ-measure with $SP_B \neq 0$ and $\|(T_n - S) P_A\| \to 0$. Hence for large enough n, $T_n P_A \in \mathscr{A}$. Find $B \subset A$ so that (9.7.4) is satisfies for $T_n P_A$ and choose k so that $\lambda(B \cap B_{n,k}) > 0$. Then

$$T_n P_{B \cap B_{n,k}} \in \mathscr{A}, \qquad \text{i.e.,} \quad P_{\sigma_1^{-1} B_{n,k}} T P_{B \cap B_{nk}} \in \mathscr{A}.$$

Hence $TP_{B \cap B_{n,k}} \in \mathscr{A}$ and so $T \in \mathscr{A}$.

Now any small projection is in $\mathscr{A}$ and in particular it follows that if W is a complemented subspace of L_p, which is not isomorphic to L_p, then W contains a subspace isomorphic to Z and complemented in it. In particular Z contains a complemented copy of $L_p(Z)$; but the reasoning may be reversed to get a complemented copy of Z in $L_p(Z)$. The Pelczynski decomposition technique now yields (9.7.1) and (9.7.2). Statement (9.7.3) follows since Z cannot contain a complemented copy of L_p (again the Pelczynski technique!).

COROLLARY 9.8. *L_p has a prime complemented subspace.*

Proof. Either Z exists or it does not!

Remarks. We now finally state that $L_p(X) \cong L_p$ with X infinite-dimensional implies that X is one of the spaces l_p, L_p, $L_p \oplus \mathbb{R}^n$ $(n \geqslant 1)$, $L_p \oplus l_p$, $Z \oplus \mathbb{R}^n$ $(n \geqslant 1)$, $Z \oplus l_p$ provided Z exists. Otherwise the list is reduced by omitting those involving Z.

10. OPEN PROBLEMS

We list now the major problems left open in this paper.

Problem 10.1 (= Problem 1.6). Is L_p prime for $0 < p < 1$?

Two related problems are:

Problem 10.2. Does there exist a quasi-Banach space X such that $\mathscr{L}(X)^* = \{0\}$? Does there exist a quasi-Banach algebra with identity and a trivial dual?

We remark here that results of Zelazko [18] show that a commutative quasi-Banach algebra with identity has non-trivial dual.

Problem 10.3. If $0 < p < 1$, and X is a quasi-Banach space can there be a projection from $L_p(X)$ onto its subspace of constant functions?

One can show that Z would have the above property; in fact there is a clear relationship with Problem 10.2.

Problem 10.4. If $1 \leqslant p < \infty$ and X is a quasi-Banach space such that there is a projection from $L_p(X)$ into the subspace of constant functions, is X locally convex?

The author has a number of partial results on this problem which will be published elsewhere. In particular the answer to 10.4 is yes when X has a basis.

Problem 10.5 (= Problem 1.3). Let X be a closed subspace of L_p so that $X \cong L_p/X \cong L_p$. Is X complemented?

Problem 10.6 (See Theorem 7.3). If for two sub-σ-algebras $\mathscr{B}_0, \mathscr{B}_1$ of $\mathscr{B}$ we have $\Lambda(\mathscr{B}_0) \cong \Lambda(\mathscr{B}_1)$, what can one deduce about $\mathscr{B}_0$ and $\mathscr{B}_1$?

Problem 10.7. If X and Y are Banach space with the Radon–Nikodym property and $L_1(X) \cong L_1(Y)$ is $l_1(X) \cong l_1(Y)$?

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