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Upper Values of Differential Games

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Let G be a differential game of prescribed duration [0, 1] with control sets Y and Z which are compact metric spaces. The *dynamics* of G are given by ($x \in R^m$)

$$dx/dt = f(t, x(t), y(t), z(t)), \tag{1}$$

where f: $[0, 1] \times R^m \times Y \times Z \to R^m$ is a continuous function satisfying a Lipschitz condition in x of the form

$$\| f(t, x_1, y, z) - f(t, x_2, y, z) \| \leqslant k(t) \| x_1 - x_2 \|,$$

where

$$\int k(t)\, dt < \infty.$$

The *pay-off* is given by

$$P = \mu(x(t)) + \int_0^1 h(t, x(t), y(t), z(t))\, dt, \tag{2}$$

where h: $[0, 1] \times R^m \times Y \times Z \to R$ is continuous and μ is a continuous functional on the Banach space of all possible trajectories in R^m. We suppose that player J_1 controlling the y-variable aims to maximize P while J_2 controlling the z-variable aims to minimize P.

In [4] we studied the problem of existence of value for such games. Using the same notation we review some of the results of [4]. An upper value U is introduced in Section 2, and a further upper value V^+ in Section 3; this latter value is that employed by Friedman [9]. It is also shown in Section 3 that $U \leqslant V^+$.

Let us now suppose that f, h are uniformly Lipschitz in (t, x) and that

$$P = \int_0^1 h(t, x, y, z) + g(x(1)), \tag{3}$$

where g is twice continuously differentiable. Then in Section 5, a concept of value is introduced which is due to Fleming, and we define the Fleming upper value W^+.

In Theorem 8.1 it is concluded that $V^+ \leqslant W^+$. Later Friedman [10] (quoted in [3]) proved that $V^+ = W^+$; however we believe his proof to be fallacious, as Lemma 1 involves an unjustifiable interchange of order of expectation and "inf sup." In this paper we supply a proof that $U = W^+$, and it follows that in general $U = V^+$.

In order to simplify the argument we shall suppose at first that P takes the form

$$P = g(x(1)), \tag{4}$$

where g is twice continuously differentiable, and its derivatives $\partial g/\partial t$, $\partial^2 g/\partial x_i$, $\partial^2 g/\partial x_i\, \partial x_j$ are all Lipschitz continuous in (t, x). We shall suppose also that f is Lipschitz continuous in (t, x) and K will serve as the Lipschitz constant in all cases. We shall also suppose that f, g vanish outside some bounded set (see [4, Section 9]).

For $(t, x, p) \in [0, 1] \times R^m \times R^m$ we define

$$H(t, x, p) = \min_z \max_y (p \cdot f(t, x, y, z)). \tag{5}$$

The following result is quoted by Fleming [7] from results of Friedman [8] or Oleinik and Kruzkhov [11].

Theorem A. *For $\epsilon > 0$ there is a unique solution ϕ of the equation*

$$(\epsilon^2/2)\, \nabla^2\phi + (\partial\phi/\partial t) + H(t, x, \nabla\phi) = 0 \tag{6}$$

subject to

$$\phi(1, \xi) = g(\xi), \tag{7}$$

and ϕ has the property that $\partial\phi/\partial t$ and $\partial^2\phi/\partial x_i\, \partial x_j$ are bounded and satisfy Hölder inequalities of the form $(0 < \gamma < 1)$

$$|\psi(t, x) - \psi(t', x')| \leqslant Q[|t - t'|^{\gamma/2} + |x - x'|^{\gamma}]. \tag{8}$$

From this Fleming [7] proves that

Theorem B. *If ϕ^ϵ is the solution of* (6) *subject to* (7) *then*

$$\lim_{\epsilon\to 0} \phi^\epsilon(0, 0) = W^+.$$

The other results needed for the proof are given in [5, Lemmas 4.2 and 4.3]. We summarize them in

THEOREM C. *Let* $\rho: Y \times Z \to R$ *be a continuous real-valued function on* $Y \times Z$. *Then there is a strategy* α *for* J_1 *such that*

$$\rho(\alpha z(t), z(t)) \geqslant \min_{z \in Z} \max_{y \in Y} \rho(y, z) \qquad \text{a.e.} \quad 0 \leqslant t \leqslant 1$$

for $z \in \mathscr{M}_2$ *(the space of measurable control functions for* J_2*).*

For completeness we sketch the proof of Theorem C. Since Y is compact and metrizable there is a continuous surjection $\theta: K \to Y$ where K is the Cantor subset of the unit interval. For each $z \in Z$, let S_z be the set of $k \in K$ such that

$$\rho(\theta k, z) = \max_{y} \rho(y, z)$$

Then S_z is closed and we select $\gamma(z)$ to be the least member of S_z. We thus define $\alpha z(\cdot) = \theta, \gamma\{z(\cdot)\}$ for $z = z(\cdot) \in \mathscr{M}_2$. Since $z(\cdot) \in \mathscr{M}_2$, for each $\epsilon > 0$ there is a subset E of $[0, 1]$ such that $mE > 1 - \epsilon$ and z is continuous on E. It is easy to show that αz is then upper semicontinuous on E and therefore measurable. It follows that αz is measurable on $[0, 1]$ and so $\alpha z \in \mathscr{M}_1$ as required. It is also clear that α is a strategy.

Before proceeding to the proof of the main theorem, let us observe that the proof given below is very similar to that given by Fleming [6] or [7] for Theorem B. In order to avoid some problems of integrability we have adopted Fleming's approach in [6] (as opposed to [7]) of "discretizing" the probabilistic perturbation of the game. These problems are overcome in [7] by supposing strategies to be Borel functions; this is quite reasonable but leads to considerable technical difficulties. These are hidden in Fleming's statement [7, p. 992] that: "A proof by induction on the number of moves shows that $V_N(s, x)$ is the value of the game with initial data (s, x)." Even with discrete noise this is technically arduous to prove; compare the similar Lemma 1 given below. It should perhaps be observed that Lemma 1 is intuitively quite obvious; nevertheless we felt obliged to supply a strict proof, since this result is basic to the theory of differential games, and an analogous lemma is required in Fleming's theory. The remainder of the proof is essentially an amalgamation of Fleming's arguments in [6 and 7].

Let N be an integer and let $\delta = 1/N$; for $0 \leqslant j \leqslant N$ we write $t_j = j\delta$. Suppose $\{\eta_{ij}\,;\, i = 1, 2, ..., N,\, j = 1, 2, ..., m\}$ is a collection of independent random variables each taking the values ± 1 with probability $1/2$. Let η_i denote the vector η_{ij} in R^m and let $\eta = (\eta_1, ..., \eta_N)$; let L denote the lattice of possible values of η in $(R^m)^N$. We denote by $\pi_j(\eta)$ the sequence of vectors $(\eta_1, ..., \eta_j)$.

We let $\mathcal{M}_1(j)$ and $\mathcal{M}_2(j)$ be the spaces of control functions for J_1 and J_2 defined on $[t_j, 1]$. We denote by Γ_j the set of strategies on $[t_j, 1]$ for J_1, i.e., maps

$$\alpha: \mathcal{M}_2(j) \to \mathcal{M}_1(j)$$

such that if

$$z_1(t) = z_2(t) \qquad \text{a.e.} \qquad t_j \leqslant t \leqslant \tau$$

then

$$\alpha z_1(t) = \alpha z_2(t) \qquad \text{a.e.} \qquad t_j \leqslant t \leqslant \tau.$$

Then a *stochastic control* θ of order j for J_2 is map $\eta \to \theta_\eta$ $\theta: L \to \mathcal{M}_2(j)$ such that:

(i) $\theta_\eta(t)$ is independent of η for $t_j \leqslant t \leqslant t_{j+1}$

(ii) If $\eta_k = \eta_k^*$ $k = j+1, \ldots, l$ (where $l < N$) then $\theta_{\eta_k}(t) = \theta_{\eta_k^*}(t)$ a.e. $t_j \leqslant t \leqslant t_{l+1}$.

Note that θ is independent of $\eta_1, \ldots, \eta_j$. The set of stochastic controls of order j will be denoted by Θ_j.

A *stochastic strategy* A for J_1 is a map $\eta \to A_\eta$, $A: L \to \Gamma_j$ such that

(i) If $z_1(t) = z_2(t)$ a.e. $t_j \leqslant t \leqslant \tau < t_{j+1}$ and $\eta, \eta^* \in L$ then

$$A_\eta(z_1)(t) = A_{\eta^*}(z_2)(t) \qquad \text{a.e.} \qquad t_j \leqslant t \leqslant \tau.$$

(ii) If $\eta_k = \eta_k^*$, $k = j+1, \ldots, l (l < N)$

$$z_1(t) = z_2(t) \qquad \text{a.e.} \qquad t_j \leqslant t \leqslant \tau$$

where $\tau \leqslant t_{l+1}$ then

$$A_\eta(z_1)(t) = A_{\eta^*}(z_2)(t) \qquad \text{a.e.} \qquad t_j \leqslant t \leqslant \tau.$$

Note again that A is independent of $\eta_1, \ldots, \eta_j$. The set of all stochastic strategies of order j will be denoted by $\mathcal{A}_j$.

Now for $\zeta \in R^m$ and $\epsilon > 0$ we describe a game $G_\epsilon^\delta(t_j, \zeta)$. Let $y \in \mathcal{M}_1(j)$ and $z \in \mathcal{M}_2(j)$; the *trajectory* corresponding to (y, z) and $\eta \in L$ is the (discontinuous) solution of the equation

$$x_\eta(t) = \zeta + \int_{t_j}^{t} f(\tau, x_\eta(\tau), y(\tau), z(\tau))\, d\tau + \epsilon\delta^{1/2} \sum_{t_j < t_k \leqslant t} \eta_k$$

The *pay off* is given by

$$P_\eta^j(\zeta; y, z) = g(x_\eta(1)).$$

For a given stochastic strategy $A \in \mathcal{A}_j$ and stochastic control $\theta \in \Theta_j$ we define

$$P^j(\zeta; A, \theta) = \mathcal{E}(P_\eta^j(\zeta; A_\eta\theta_\eta, \theta_\eta))$$

We define the value of A to J_1 by

$$u_i(\zeta; A) = \inf_{\theta \in \Theta_j} P^j(\zeta; A, \theta)$$

and the value of the game $G_\epsilon^\delta(t_j, \zeta)$ to J_1 by

$$U_\epsilon^\delta(t_j, \zeta) = \sup_{A \in \mathcal{A}_j} u_j(\zeta; A).$$

Clearly, we have

$$U_\epsilon^\delta(1, \zeta) = g(\zeta).$$

LEMMA 1. *For* $j < N$

$$U_\epsilon^\delta(t_j, \zeta) = \sup_{\alpha \in \Gamma_j} \inf_{z \in \mathcal{M}_2(j)} \mathcal{E}(U_\epsilon^\delta(t_{j+1}, x_\eta(t_{j+1})))$$

where x *is the trajectory corresponding to* $(\alpha z, z)$.

Proof. Let $A \in \mathcal{A}_j$; then the behaviour of A on (t_j, t_{j+1}) is that of a fixed strategy $\alpha \in \Gamma_j$ (condition (i)). Suppose $z_0(t) \in \mathcal{M}_2(j)$; then $(\alpha z_0, z_0)$ induce a trajectory $x_\eta(t)$ for $\eta \in L$. In fact $x_\eta(t_{j+1})$ depends on η_{j+1} only. We denote the possible values in R^m of η_{j+1} by S; to $\sigma \in S$ there corresponds a value ζ_σ of $x_\eta(t_{j+1})$. We define $A^\sigma \in \mathcal{A}_{j+1}$ by

$$A_\eta^\sigma(z)(t) = A_{\eta\sigma}(\hat{z})(t) \qquad t_{j+1} \leqslant t \leqslant 1$$

where

(a) $\eta_k^\sigma = \eta_k, \qquad k \neq j+1,$

$\eta_{j+1}^\sigma = \sigma,$

(b) $\hat{z} \in \mathcal{M}_2(j)$ is defined by

$$\begin{aligned} \hat{z}(t) &= z_0(t) \qquad t_j \leqslant t \leqslant t_{j+1} \\ &= z(t) \qquad t_{j+1} \leqslant t \leqslant 1. \end{aligned}$$

Clearly for each $\sigma \in S$, A^σ is independent of $\eta_1, \ldots, \eta_{j+1}$ and belongs to $\mathcal{A}_{j+1}$. Now

$$u_{j+1}(\zeta_\sigma; A^\sigma) \leqslant U_\epsilon^\delta(t_{j+1}, \zeta_\sigma)$$

and hence, given $\nu > 0$, there exists $\theta^\sigma \in \Theta_{j+1}$ such that

$$P^{j+1}(\zeta_\sigma; A^\sigma, \theta^\sigma) \leqslant U_\epsilon^\delta(t_{j+1}, \zeta_\sigma) + \nu.$$

Now define $\theta \in \Theta_j$ by

$$\begin{aligned} \theta_\eta(t) &= z_0(t), \qquad t_j < t \leqslant t_{j+1}, \\ &= \theta_\eta^\sigma(t), \qquad t_{j+1} < t \leqslant 1 \quad \text{and} \quad \eta_{j+1} = \sigma. \end{aligned}$$

Then

$$\begin{aligned} P^j(\zeta; A, \theta) &= \frac{1}{2^m} \sum_{\sigma \in S} P^{j-1}(\zeta_\sigma; A^\sigma, \theta^\sigma) \\ &\leqslant \mathscr{E}(U_\epsilon^\delta(t_{j+1}, x_\eta(t_{j+1})) + \nu. \end{aligned}$$

Hence $u_j(\zeta; A) \leqslant \mathscr{E}(U_\epsilon^\delta(t_{j+1}, x_\eta(t_{j+1})))$ and so as $z_0 \in \mathscr{M}_2(j)$ is arbitrary

$$u_i(\zeta; A) \leqslant \inf_{z_0 \in \mathscr{M}_2(j)} \mathscr{E}(U_\epsilon^\delta(t_{j+1}, x_\eta(t_{j+1})))$$

and hence,

$$U_\epsilon^\delta(t_j, \zeta) \leqslant \sup_{\alpha \in \Gamma_j} \inf_{z \in \mathscr{M}_2(j)} \mathscr{E}(U_\epsilon^\delta(t_{j+1}, x_\eta(t_{j+1}))).$$

Conversely fix $\alpha \in \Gamma_j$; then for $z \in \mathscr{M}_2(j)$ there is a trajectory $x_\eta^{(z)}(t)$ corresponding to $(\alpha z, z)$ and $\eta \in L$. Now let $\zeta_\sigma^{(z)} = x_\eta^{(z)}(t_{j+1})$ when $\eta_{j-1} = \sigma$ (note $x_\eta^{(z)}(t_{j+1})$ depends only on η_{j+1}). For $\zeta \in R^m$ and $\nu > 0$ there exists $A(\zeta) \in \mathscr{A}_{j+1}$ such that

$$u_{j+1}(\zeta, A(\zeta)) \geqslant U|_\epsilon^\delta(t_{j+1}, \zeta) - \nu.$$

Define $A \in \mathscr{A}_j$ by

$$\begin{aligned} A_\eta z(t) &= \alpha z(t), \qquad t_j \leqslant t \leqslant t_{j+1}, \\ &= A(\zeta_\sigma^{(z)})\, \hat{z}(t), \qquad t_{j+1} \leqslant t \leqslant 1,\ \eta_{j+1} = \sigma. \end{aligned}$$

where $\hat{z}$ is the restriction of z to $\mathscr{M}_2(j+1)$. It is easy (but tedious!) to check that $A \in \mathscr{A}_j$. For $\theta \in \Theta_j$ define $\theta^\sigma \in \Theta_{j+1}$ by

$$\theta_\eta^\sigma(t) = \theta_{\eta\sigma}(t), \qquad t_{j+1} \leqslant t \leqslant 1.$$

As $\theta \in \Theta_j$, $\theta_\eta(t)$ is independent of η on $[t_j, t_{j+1}]$, e.g. $\theta_\eta(t) = z(t)$. If we condition $\eta_{j+1} = \sigma$ we have

$$P_\eta^j(\zeta; A_\eta\theta_\eta, \theta_\eta) = P_\eta^{j+1}(\zeta_\sigma^{(z)}; A_\eta(\zeta_\sigma^{(z)})\, \theta_\eta^\sigma, \theta_\eta^\sigma)$$

and since $A(\zeta_\sigma^{(z)})$ and θ^σ are independent of η_{j+1} we obtain

$$\begin{aligned} \mathscr{E}(P_\eta^j(\zeta; A_\eta\theta_\eta, \theta_\eta) \mid \eta_{j+1} = \sigma) &= \mathscr{E}(P_\eta^{j+1}(\zeta^{(z)}; A_\eta(\zeta_\sigma^{(z)})\, \theta_\eta^\sigma, \theta^\sigma) \\ &= P^{j+1}(\zeta_\sigma^{(z)}; A(\zeta_\sigma^{(z)}), \theta^\sigma) \\ &\geqslant u_{j+1}(\zeta_\sigma^{(z)}; A(\zeta_\sigma^{(z)})) \\ &\geqslant U_\epsilon^\delta(t_{j+1}, \zeta_\sigma^{(z)}) - \nu. \end{aligned}$$

Hence,

$$P^j(\zeta; A, \theta) \geqslant \mathscr{E}_\sigma(U_\epsilon^\delta(t_{j+1}, \zeta_\sigma^{(z)})) - \nu$$
$$= \mathscr{E}(U_\epsilon^\delta(t_{j+1}, x_\eta^{(z)}(t_{j+1}))) - \nu.$$

Therefore

$$U_\epsilon^\delta(t_j, \zeta) \geqslant \sup_{\alpha\in\Gamma_j} \inf_{z\in\mathscr{M}_2(j)} \mathscr{E}(U_\epsilon^\delta(t_{j+1}), x_\eta(t_{j+1}))$$

as required. The lemma is thus proved.

LEMMA 2. *For any* δ

$$U_0^\delta(t_j, \zeta) = U(t_j, \zeta).$$

Proof. By Lemma 1

$$U_0^\delta(t_j, \zeta) = \sup_{\alpha\in\Gamma_j} \inf_{z\in\mathscr{M}_2(j)} U_0^\delta(t_{j+1}, x(t_{j+1})),$$

and since U satisfies the same conditions with the same terminal condition, [5] Theorem 3.1, we obtain the result by induction.

LEMMA 3. *Let* x_η^ϵ *and* x *be the trajectories corresponding to the controls* (y, z) *and* $\eta \in L$ *in* $G_\epsilon^\delta(t_j, \zeta)$ *and* $G_0^\delta(t_j, \zeta)$. *Then*

$$\| x_\eta^\epsilon(1) - x(1) \| \leqslant \epsilon\delta^{1/2} e^K \| w \|$$

where

$$\| w \| = \sup_{j+1\leqslant k\leqslant N} \left\| \sum_{j+1}^{k} \eta_i \right\|.$$

Proof. (cf. [6, p. 204])

$$x_\eta^\epsilon(t) - x(t) = \int_{t_j}^{t} (f(t, x_\eta^\epsilon(\tau), y(\tau), z(\tau) - f(t, x(\tau), y(\tau), z(\tau))\, d\tau$$
$$+ \epsilon\delta^{1/2} w(t)$$

where

$$w(t) = \sum_{t_i < t_j \leqslant t} \eta_i$$

Hence

$$\| x_\eta^\epsilon(t) - x(t) \| \leqslant \int_{t_j}^{t} K \| x_\eta^\epsilon(\tau) - x(\tau) \|\, d\tau + \epsilon\delta^{1/2} \| w(t) \|$$

and therefore it is easy to show that

$$\int_{t_j}^{t} \| x_\eta^\epsilon(t) - x(\tau)\| \, d\tau \leqslant \psi(t)$$

where ψ is the solution of

$$\dot{\psi} = K\psi + \epsilon\delta^{1/2} \| w(t)\|.$$

However

$$\psi(t) = \epsilon\delta^{1/2} e^{Kt} \int_{t_j}^{t} \| w(\tau)\| \, e^{-K\tau} \, d\tau$$

$$\leqslant (\epsilon\delta^{1/2}/K) \| w \| \, (e^{K(t-t_j)} - 1)$$

and hence

$$\psi(t) \leqslant \left(\frac{\epsilon\delta^{1/2}}{K}\right) \| w \| \, (e^K - 1), \qquad t_j \leqslant t \leqslant 1$$

and

$$\| x_\eta^\epsilon(t) - x(t)\| \leqslant \epsilon\delta^{1/2} e^K \| w \|.$$

The lemma follows.

Lemma 4. $\mathscr{E}(\| w \|) \leqslant 2mN^{1/2}$.

Proof. (cf. [6, p. 203]).

For a fixed coordinate l, we have

$$\mathscr{E}\left(\max_k \left| \sum_{j+1}^{k} \eta_{il} \right|\right) \leqslant 2\mathscr{E}\left(\left| \sum_{j+1}^{N} \eta_{il} \right|\right).$$

This follows from the result of Doob [2, p. 106] by elementary calculations. Hence

$$\mathscr{E}\left(\max_k \left| \sum_{j+1}^{k} \eta_{il} \right|\right) \leqslant 2\left(\mathscr{E}\left\{\left(\sum_{j+1}^{N} \eta_{il}\right)^2\right\}\right)$$

$$= 2(N-j)^{1/2}$$

since $\operatorname{Var}(\eta_{il}) = 1$ and $\mathscr{E}(\eta_{il}) = 0$ for all i, l, and they are mutually independent.

Hence adding over coordinates

$$\mathscr{E}(\| w \|) \leqslant 2m(N-j)^{1/2}$$

$$\leqslant 2mN^{1/2}.$$

LEMMA 5. $|U_\epsilon^\delta(t_j, \xi) - U(t_j, \xi)| \leqslant 2m\epsilon Ke^K$.

Proof. Let $A \in \mathcal{A}_j$ and $\theta \in \Theta_j$; then given $\eta \in L$, (A, θ) induce trajectories $x_\eta^\epsilon(t)$ and $x(t)$ in $G_\epsilon^\delta(t_j, \xi)$ and $G_0^\delta(t_j, \xi)$, respectively. By Lemmas 3 and 4

$$\mathscr{E}(\| x_\eta^\epsilon(1) - x_\eta(1)\|) \leqslant 2m\epsilon e^K,$$

and hence

$$\mathscr{E}(| g(x_\eta^\epsilon(1)) - g(x_\eta(1))|) \leqslant 2m\epsilon Ke^K.$$

It follows then easily that for given $A \in \mathcal{A}_j$ with values u^ϵ and u in G_ϵ^δ and G_0^δ, respectively

$$| u^\epsilon - u | \leqslant 2m\epsilon Ke^K,$$

and hence that

$$| U_\epsilon^\delta(t_j, \xi) - U_0^\delta(t_j, \xi)| \leqslant 2m\epsilon Ke^K.$$

Now apply Lemma 2.

LEMMA 6. *Let $\phi(t, x)$ be the solution of* (6)–(7) *in the strip* $0 \leqslant t \leqslant 1$. *Then* $\lim_{\delta\to 0} U_\epsilon^\delta(0, 0) = \phi(0, 0)$.

Proof. ϕ and its derivatives $\partial\phi/\partial t$, $\partial^2\phi/\partial x_i\, \partial x_j$ satisfy Hölder conditions of the form (8). Hence following Fleming [7, p. 998] we may write

$$\phi(t + \tau, x + \chi) = \phi(t, x) + (\partial\phi/\partial t)\tau + \nabla\phi \cdot \chi + \frac{1}{2}\sum_{i,j} \frac{\partial^2\phi}{\partial x_i \partial x_j} \chi_i\chi_j + \rho(t, x, \tau, \chi)$$

where $| \rho(t, x, \tau, \chi)| \leqslant M_1(\tau^{1+\nu/2} - \tau \| \chi \|^\nu + \| \chi \|^{2+\nu})$.

Now for fixed δ, let

$$\Psi(t_j, \xi) = \sup_{\alpha\in\Gamma_j} \inf_{z\in\mathscr{M}_2(j)} \mathscr{E}(\phi(t_{j+1}, x_\eta(t_{j+1}))).$$

We can by Theorem C select $\alpha \in \Gamma_j$ such that

$$p \cdot f(t_j, \xi, \alpha z(t), z(t)) \geqslant H(t_j, \xi, p) \qquad \text{a.e.} \qquad t_j \leqslant t \leqslant t_{j+1}$$

where $p = \nabla\phi(t_j, \xi)$, and $H(t, \xi, p) = \min_z \max_y (p \cdot f(t, \xi, y, z))$.

For $z \in \mathscr{M}_2(j)$ we obtain a trajectory $x_\eta(t)$ corresponding to $(\alpha z, z)$ and $\eta \in L$ with

$$x_\eta(t_{j+1}) = \xi + \chi$$

where

$$\chi = \int_{t_j}^{t_{j+1}} f(\tau, x_\eta(\tau), \alpha z(\tau), z(\tau))\, d\tau + \epsilon\delta^{1/2}\eta_{j+1}.$$

Then we have

$$\mathscr{E}(\chi_i) = \int_{t_j}^{t_{j+1}} f_i(\tau, x_n(\tau), \alpha z(\tau), z(\tau))\, d\tau$$

$$\operatorname{var}(\chi_i) = \epsilon^2\delta$$

and

$$\mathscr{E}(|\chi_i|) \leqslant ([\mathscr{E}(\chi_i)]^2 + \operatorname{var}(\chi_i))^{1/2} \leqslant M_3\delta^{1/2}.$$

We aise have that χ_i and χ_j are independent so that

$$|\mathscr{E}(\chi_i\chi_j)| = |\mathscr{E}(\chi_i)\,\mathscr{E}(\chi_j)| \leqslant M_4\,\delta^2, \qquad i \neq j,$$

and

$$|\mathscr{E}(\chi_i^2) - \epsilon^2\delta| \leqslant M_4\,\delta^2.$$

It can also be checked that

$$\mathscr{E}(\|\chi\|^{2+\gamma}) \leqslant M_5\,\delta^{1+\gamma/2}.$$

Finally we observe that

$$\mathscr{E}(\chi_i) = \int_{t_j}^{t_{j-1}} f_i(t_j\,, \xi, \alpha z(\tau), z(\tau))\, d\tau + r$$

where $|r| \leqslant M_6\,\delta^2$, and hence setting these estimates together we obtain

$$\begin{aligned}\mathscr{E}(\phi(t_{j+1}\,, x_n(t_{j+1}))) = \phi(t_j\,, \xi) + \delta\,\frac{\partial\phi}{\partial t} &+ \int_{t_j}^{t_{j+1}} p\cdot f(t_j\,, \xi, \alpha z(\tau), z(\tau))\, d\tau \\ &+ \frac{\epsilon^2}{2}\,\delta\nabla^2\phi + r,\end{aligned}$$

where $|r'| \leqslant \mathscr{M}_7\,\delta^{1+\gamma/2}$.

Hence by choice of α

$$\mathscr{E}(\phi(t_{j+1}\,, x_n(t_{j+1}))) \geqslant \phi(t_j\,, \xi) - M_7\,\delta^{1+\gamma/2}$$

and so

$$\Psi(t_j\,, \xi) \geqslant \phi(t_j\,, \xi) - M_7\,\delta^{1+\gamma/2}$$

Conversely by choosing $z(t) = z_0$ so that

$$\min_z \max_y p\cdot f = \max_y p\cdot f(t_j\,, \xi, y, z_0),$$

we may show by similar arguments that

$$\Psi(t_j, \xi) \leqslant \phi(t_j, \xi) + M_8 \delta^{1+\nu/2}.$$

Now let

$$\Lambda_j = \sup_{\xi} | \phi(t_j, \xi) - U_\epsilon^\delta(t_j, \xi)|.$$

Then clearly,

$$\Lambda_j \leqslant \Lambda_{j+1} + \sup_{\xi} | \phi(t_j, x) - \Psi(t_j, x)|$$

(using Lemma 1), and so

$$\Lambda_0 \leqslant M \delta^{1+\nu/2} . N = M \delta^{\nu/2},$$

where M is a constant independent of δ. This proves Lemma 6.

THEOREM (a). *Let G be a differential game defined by* (1) *and* (3); *then* $U = V^+ = W^+$.

(b). *If G is defined by* (1) *and* (2), *then* $U = V^+$.

Proof. (a) First we establish the result for the particular case of payoff of type 4. Then by Lemma 6 and Lemma 5

$$| \phi^\epsilon(0, 0) - U | \leqslant 2mKe^K \epsilon,$$

where ϕ^ϵ is the solution of (6)–(7). Hence

$$U = \lim_{\epsilon \to 0} \phi^\epsilon(0, 0) = W^+ \qquad \text{(Theorem B)}.$$

Now the extension to g which are merely continuous is easy by approximation. Next we may reduce the case of payoff of type (8) with h Lipschitz to this case by incorporating an extra coordinate

$$\dot{x}_{m+1} = h(t, x, y, x).$$

Finally by approximation we may assume h simply continuous.

(b) This is obtained by the approximation methods of [4, Section 10].

It is perhaps worth pointing out that without recourse to the results of Fleming concerning W^+ we may still prove that U is the "*Fleming solution*" of the Isaacs–Bellman equation.

$$(\partial \phi / \partial t) + \min_{x} \max_{y} (\nabla \phi \cdot f + h) = 0$$

as in [4, Section 5]. For after incorporating an extra coordinate x_{m+1} it is easy to see that for any δ, ϵ

$$U_\epsilon^\delta(t_j, \xi, \xi_{m+1}) = \xi_{m+1} + U_\epsilon^\delta(t_j, \xi, 0)$$

and it follows that (modifying Lemma 6 to prove uniform convergence of $U_\epsilon^\delta(t, x)$ to $\phi^\epsilon(t, x)$)

$$\phi^\epsilon(t, \xi, \xi_{m+1}) = \xi_{m+1} + \psi^\epsilon(t, \xi)$$

where ψ^ϵ is the unique solution of

$$(\epsilon^2/2)\, \nabla^2\phi + (\partial\phi/\partial t) + \min_z \max_y (\nabla\phi \cdot f + h) = 0$$

subject to $\psi^\epsilon(1, \xi) = g(\xi)$. Then $\lim_{\epsilon\to 0} \psi^\epsilon(t, \xi) = U(t, \xi)$.

We have been informed by J. Danskin that he has obtained similar results, by rather different methods, relating his own concept of value for $\sigma = 1$ (see [1]) to those adopted by Fleming and Friedman.

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