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The Existence of Value in Differential Games of Pursuit and Evasion

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1. INTRODUCTION

A two-person zero-sum differential game can be considered as a control problem in which these are present two controllers or players, J_1 and J_2, with directly conflicting interests. The dynamics of the game are described by a system of differential equations and at the end of the game a (real-valued) quantity, called the payoff, is computed; this represents the cost of the game to J_2, or the amount received by J_1. During the course of the game the players can affect the outcome of the game, that is the final payoff, by choosing certain control variables. The greatest payoff that J_1 can force is called the lower value of the game whilst the least value the minimizing player J_2 can force is called the upper value. Our objective is to study when these values are the same, that is, when the game "has value."

In earlier papers [2, 3] we have studied differential games of fixed duration, that is the game ends after a predetermined length of time. We have shown that if the Isaacs condition [see (9) below] is satisfied then the game has value. In the present paper we consider the more complicated situation where the game ends when the trajectory enters a certain set F, called the terminal set. The time at which this occurs is called the capture time. In a pursuit–evasion game the payoff is just the length of time elapsed up to the capture time. This includes the case of a pursuer P chasing an evader E in some finite-dimensional space $\mathbf{R}^p$. Considering the dynamics of the two players as one system of equations, the trajectory of the game lies in $\mathbf{R}^{2p}$. In terms of the trajectory variables the terminal set F, that is, the set of points where capture occurs, is the diagonal subspace of $\mathbf{R}^{2p}$ described by the coordinates of the two players being the same.

In a generalized pursuit–evasion game, the same again ends when capture occurs, but the payoff is now the integral up to the capture time of some nonnegative function h, representing the energy used by the pursuer. A game of survival is even more complicated in that the payoff is now the integral up to capture time of a function h which may be positive or negative, possibly together with a terminal payoff. Mention should be made here of the work of Friedman [6–8] and Varaiya and Lin [12], on pursuit–evasion and survival games. However, our approach is an extension of ideas of our earlier papers [2, 3].

We study generalized pursuit–evasion games by considering certain approximating fixed-time games. The main result is that, if a slightly modified Isaacs condition is satisfied all the approximating games have value and the limit of their values is shown to be a certain "extended value" of the original game. Our definition of extended value is slightly similar to one of Friedman [5, 8], but the details are quite different.

Finally in Section 6 relaxed controls are introduced for survival games and, in terms of the related relaxed game, the upper and lower extended values are related to the upper and lower values.

2. Notation

A differential game G played by two players J_1 and J_2 for the time interval $I = [t_0, T_0]$ is considered. Many of the concepts defined below, and the related results, exist if the time interval is allowed to be infinite. As, however, some of the quantities might then be infinite the situation where $T_0 < \infty$ will be discussed.

Let Y and Z be two compact metric spaces. A map $\varphi : I \to Y$ (resp., Z) is *measurable* if for every continuous map $\psi : Y \to \mathbf{R}$ (resp., $\psi : Z \to \mathbf{R}$, the real numbers) the map $\psi \circ \varphi$ is measurable in the usual Lebesgue sense on I. At each time $t \in I$, J_1 picks an element $y(t)$ from Y and J_2 picks an element $z(t)$ from Z in such a way that the resulting functions, $y : I \to Y$ and $z : I \to Z$, are measurable. When selecting $y(t)$ J_1 is aware only of the history of the game up to time t, and a similar restriction applies to J_2. The set of all measurable functions $y : I \to Y$ is denoted by $\mathscr{M}_1$; the elements of $\mathscr{M}_1$ are called control functions for J_1. $\mathscr{M}_2$, the set of control functions for J_2, is similarly defined with Z replacing Y.

The dynamics of the differential game are given by the family of ordinary differential equations:

$$\dot{x} = f(t, x, y(t), z(t)), \tag{1}$$

together with an initial condition

$$x(t_0) = x_0 .$$

Here $x \in \mathbf{R}^m$ and $t \in I$. f is a continuous function from $I \times \mathbf{R}^m \times Y \times Z$ to $\mathbf{R}^m$ satisfying a Lipschitz condition

$$\| f(t, x_1, y, z) - f(t, x_2, y, z)\| \leqslant k(t) \| x_1 - x_2 \|, \tag{2}$$

whenever $x_1, x_2 \in \mathbf{R}^m$, $t \in I$, $y \in Y$ and $z \in Z$. k is a Lebesgue measurable function on I satisfying

$$\int_{t_0}^{T_0} k(t)\, dt = A < \infty. \tag{3}$$

Throughout the game the player J_1 chooses $y(t)$ so that the final payoff, defined below, is as large as possible, whilst the player J_2 tries to make the final payoff as small as possible. We distinguish three types of differential game according to the form of the payoff P. First we have games of fixed duration where

$$P = \mu(x(\cdot)) + \int_{t_0}^{T_0} h(t, x(t), y(t), z(t))\, dt. \tag{4}$$

Here $h : I \times \mathbf{R}^m \times Y \times Z \to \mathbf{R}$ is continuous and μ is a (nonlinear) real-valued continuous functional on the Banach space $C(I)^m$ of trajectories in $\mathbf{R}^m$. These games are treated in our paper [3].

Secondly, suppose F is a closed subset of $\mathbf{R}^{m+1}$ such that

$$[T_0, \infty) \times \mathbf{R}^m \subset F. \tag{5}$$

F is called the *terminal set*. Given a control function $y(t)$ for J_1 and a control function $z(t)$ for J_2 the conditions (2) and (3) imply there is a trajectory $x(t)$ which satisfies (1) almost everywhere, and the condition (5) implies there is a smallest value of t, $t_F(x)$, such that $(t, x(t)) \in F$. $t_F(x)$ will be called the *capture time* of the trajectory $x(t)$. Suppose now that P is of the form

$$P = g(t_F, x(t_F)) + \int_{t_0}^{t_F(x)} h(t, x(t), y(t), z(t))\, dt, \tag{6}$$

where as before h is continuous and $g : \mathbf{R}^{m+1} \to \mathbf{R}$ is continuous. Such a game is called a *game of survival* (Friedman [7]).

A particular case of a game of survival is obtained if we take $g = 0$ and $h \equiv 1$ in (6). The payoff P is then just the capture time $t_F(x)$ so the player J_1, called the evader, is trying to avoid capture and make $t_F(x)$ as large as possible whilst J_2, the pursuer, is trying to make $t_F(x)$ as small as possible. Under these conditions G is called a *pursuit–evasion game*.

Finally, if $g = 0$ and $h \geqslant 0$ then G is called a *generalized pursuit–evasion game*.

3. Strategies and Values

The problem of assigning a value to a differential game has been approached in at least three different ways [3–5]. We shall follow a method first suggested by Roxin [11] and adopted in our previous paper [3]. However, the reader should also be referred to the alternative definition given by Friedman [5]. It is not certain, except in the special case of fixed-duration games, that Friedman's concept of value coincides with the one used here.

A map $\alpha : \mathscr{M}_2 \to \mathscr{M}_1$ is a *strategy* for J_1 if whenever

$$z_1(t) = z_2(t) \qquad \text{a.e.} \quad t_0 \leqslant t \leqslant t_1$$

then

$$\alpha z_1(t) = \alpha z_2(t) \qquad \text{a.e.} \quad t_0 \leqslant t \leqslant t_1 .$$

The set of strategies for J_1 is denoted by Γ. We say $\alpha \in \Gamma$ is a *delay strategy* if there exists $s > 0$ such that whenever

$$z_1(t) = z_2(t) \qquad \text{a.e.} \quad t_0 \leqslant t \leqslant t_1$$

then

$$\alpha z_1(t) = \alpha z_2(t) \qquad \text{a.e.} \quad t_0 \leqslant t \leqslant \min(T_0 , t_1 + s).$$

The set of delay strategies is denoted by Γ_0. The value for $\alpha \in \Gamma$ is defined as

$$u(\alpha) = \inf(P(\alpha z, z); z \in \mathscr{M}_2).$$

We then define

$$U = \sup(u(\alpha); \alpha \in \Gamma),$$

$$V^- = \sup(u(\alpha); \alpha \in \Gamma_0).$$

Note that V^- as defined here corresponds to $V^-(0)$ or $U^+(0)$ in [3, Section 2]. It is not clear that Theorem 3.4 applies to survival games so it is not obvious that V^- coincides with the V^- defined by Friedman [5, 6, 8].

Similar definitions are made for the set of strategies Δ and delay strategies Δ_0 for J_2. For $\beta \in \Delta$ we define

$$v(\beta) = \sup(P(y, \beta y); y \in \mathscr{M}_1),$$

$$V = \inf(v(\beta); \beta \in \Delta),$$

and

$$V^+ = \inf(v(\beta); \beta \in \Delta_0).$$

From the definitions it is clear that $V^- \leqslant U$; it also follows that $U \leqslant V^+$. For if $\alpha \in \Gamma$ and $\beta \in \Delta_0$ then by an inductive construction one can find controls $y(t)$ and $z(t)$ such that

$$\alpha z = y, \qquad \beta y = z.$$

[As β is a delay strategy one can initially construct $z(t)$ for $t_0 \leqslant t \leqslant t_0 + s$, for some s. From this $y(t)$ for $t_0 \leqslant t \leqslant t_0 + s$ can be determined. This process is then repeated.]

Thus

$$P(\alpha z, z) = P(y, \beta y)$$

so

$$u(\alpha) \leqslant v(\beta).$$

Thus

$$U \leqslant V^+.$$

A more detailed argument is given in [3, Section 2] for fixed-time games. It follows that $V^- \leqslant U$, $V \leqslant V^+$. G is said to have *value* if $V^- = V^+$ and *weak value* if $U = V$.

Finally we quote the principal result from our paper [3] on games of fixed duration. (See also [2] and Friedman [8].) We assume, therefore that the payoff is given by (4).

Define

$$F^+(t, x, p) = \min_{z \in Z} \max_{y \in Y} \left(\sum_{i=1}^{m} p_i f_i(t, x, y, z) + h(t, x, y, z) \right)$$

$$= \min_{z} \max_{y} (p \cdot f + h) \tag{7}$$

for $p \in \mathbf{R}^m$, $x \in \mathbf{R}^m$ and $t \in I$.

Similarly

$$F^-(t, x, p) = \max_{y} \min_{z} (p \cdot f + h), \tag{8}$$

for $p \in R^m$, $x \in R^m$ and $t \in I$.

DEFINITION 3.1. The game is said to satisfy the Isaacs condition if

$$F^+(t, x, p) = F^-(t, x, p). \tag{9}$$

THEOREM 3.2. *If the Isaacs condition is satisfied the game G, defined by* (1), (2), *and* (4) *has value, i.e.*,

$$V^+ = V^- \qquad (= V \text{ say}).$$

This is the main result, Theorem 10.1, of [3].

4. EXTENDED VALUES

In this section we discuss an alternative concept of value convenient for games of survival. Our approach is slightly related to that of Friedman [5]; however, the details differ considerably from those of Friedman.

Consider a game G defined by (1), (2), and (6). Define the real-valued function $\mathscr{x}$ on I by

$$\mathscr{x}(t) = \int_{t_0}^{t} h(s, x(s), y(s), z(s))\, ds \tag{10}$$

and consider the map $t \to (x(t), \mathscr{x}(t))$ from I into $\mathbf{R}^{m+1}$. Lemma 3.3 of [3] tells us that the set of all possible trajectories in $\mathbf{R}^{m+1}$ is relatively compact in the Banach space of continuous $\mathbf{R}^{m+1}$-valued functions on I.

An approximate strategy A for player J_1 is a sequence (α_n) of delay strategies, i.e., $\alpha_n \in \Gamma_0$ for all n. Similarly, an approximate strategy B for J_2 is a sequence of delay strategies. Corresponding to a pair (A, B) we may determine a payoff as follows. Since α_n and β_n are each delay-strategies there exist unique $z_n \in \mathscr{M}_2$, $y_n \in \mathscr{M}_1$ with $\alpha_n z_n = y_n$ and $\beta_n y_n = z_n$ (see [3, Section 11]). The controls $(y_n(t), z_n(t))$ induce a trajectory $(x_n(t), \mathscr{x}_n(t))$ in $\mathbf{R}^{m+1}$. By the relative compactness of the set of possible trajectories the sequence $(x_n(t), \mathscr{x}_n(t)$ has at least one accumulation point $(\bar{x}(t), \bar{\mathscr{x}}(t))$. For each such accumulation point we determine the payoff

$$P = \bar{\mathscr{x}}(\bar{t}) + g(\bar{x}(\bar{t})),$$

where $\bar{t}$ is the first time $\bar{t} \geqslant t_0$ with $\bar{x}(\bar{t}) \in F$; the set of such payoffs is called the *payoff set* $P(A, B)$ of the pair A, B. Now define

$$V_e^+ = \inf_B \sup_A (\sup P(A, B)), \tag{11}$$

$$V_e^- = \sup_A \inf_B (\inf P(A, B)). \tag{12}$$

G is said to have *extended value* if $V_e^+ = V_e^-$ $(= V_e)$. Further, a pair of approximate strategies (A^*, B^*) is an *approximate saddle point* if for any other pair (A, B)

$$P(A, B^*) \leqslant P(A^*, B^*) \leqslant P(A^*, B)$$

(where we say $S_1 \leqslant S_2$ for two subsets S_1, S_2 of $\mathbf{R}$ if $s_1 \leqslant s_2$ whenever $s_1 \in S_1$ and $s_2 \in S_2$).

We now restrict attention to generalized pursuit–evasion games with a payoff given by

$$P = \int_{t_0}^{t_F} h(t, x(t), y(t), z(t)) \, dt,$$

where $h \geqslant 0$. We shall relate the extended value to properties of the ordinary value when the terminal set F is varied. For $\epsilon > 0$ let F_ϵ denote the set of $(t, x) \in \mathbf{R}^{m+1}$ with $\rho(t, x) =$ distance of (t, x) from $F \leqslant \epsilon$. F_ϵ is then a closed subset of $\mathbf{R}^{m+1}$. Consider the generalized pursuit–evasion game G_ϵ with terminal set F_ϵ replacing F. The upper and lower values of G_ϵ will be denoted by $V^+(F_\epsilon)$ and $V^-(F_\epsilon)$.

THEOREM 4.1.

$$\lim_{\epsilon \to 0} V^+(F_\epsilon) = V_e^+$$

and

$$\lim_{\epsilon \to 0} V^-(F_\epsilon) = V_e^-.$$

Proof. Clearly, as $\epsilon \to 0$, $V^+(F_\epsilon)$ and $V^-(F_\epsilon)$ increase to some limits l_1 and l_2, respectively. It is, therefore, sufficient to consider the sequence $V^+(F_{1/n})$ and $V^-(F_{1/n})$. For each n there exists a delay strategy $\beta_n{}^*$ for the pursuer J_2 such that

$$v_n(\beta_n{}^*) \leqslant V^+(F_{1/n}) + 1/n$$

(where v_n denotes the value in the game $G_{1/n}$). Consider the approximate strategy $B^* = (\beta_n{}^*)$ and suppose $A = (\alpha_n)$ is an approximate strategy for the evader. Then there is a unique sequence $y_n \in \mathscr{M}_1$ with

$$\alpha_n \beta_n{}^* y_n = y_n$$

and $(y_n, \beta_n{}^* y_n)$ induces a trajectory $(x_n(t), \varkappa_n(t))$ in $\mathbf{R}^{m+1}$. Let t_n be the time at which $x_n(t)$ first enters $F_{1/n}$ so that we have

$$\varkappa_n(t_n) \leqslant V^+(F_{1/n}^+) + 1/n.$$

Suppose $(x_{n_k}(t), \varkappa_{n_k}(t))$ is a convergent subsequence of the trajectories $(x_n(t), \varkappa_n(t))$ converging to the trajectory $(\bar{x}(t), \bar{\varkappa}(t))$; we may suppose, selecting a further subsequence, that $t_{n_k} \to \bar{t}$ for some $\bar{t}$. Then we clearly have $\bar{x}(\bar{t}) \in F$ so that the payoff $\bar{P}$ of $(\bar{x}, \bar{\mathscr{x}})$ satisfies

$$\begin{aligned} \bar{P} &\leqslant \bar{\mathscr{x}}(\bar{t}) \\ &= \lim_{k\to\infty} \mathscr{x}_{n_k}(t_{n_k}) \\ &\leqslant \lim_{k\to\infty} V^+(F_{1/n_k}) = l_1 . \end{aligned}$$

Thus

$$P(A, B^*) \leqslant l_1 ,$$

and so

$$V_e^+ \leqslant l_1 . \tag{13}$$

However, if $B = (\beta_n)$ is any approximate strategy then for a fixed ϵ we may select control functions $y_n \in \mathscr{M}_1$ with

$$P_\epsilon(y_n , \beta_n y_n) \geqslant V^+(F_\epsilon) - \epsilon. \tag{14}$$

Define A then to be the approximate strategy (α_n) where $\alpha_n z = y_n$ for any $z \in \mathscr{M}_2$.

Let $(x_n , \mathscr{x}_n)$ be the sequence of trajectories of (A, B) and suppose some subsequence $(x_{n_k} , \mathscr{x}_{n_k})$ converges to the trajectory $(\bar{x}, \bar{\mathscr{x}})$. If $\bar{t}$ is the first time at which $(\bar{t}, \bar{x}(\bar{t}))$ is in F then $\bar{P} = \bar{\mathscr{x}}(\bar{t})$. For large enough k, $(\bar{t}, x_{n_k}(\bar{t})) \in F_\epsilon$ so that

$$P_\epsilon(y_{n_k} , \beta_{n_k} y_{n_k}) \leqslant \mathscr{x}_{n_k}(\bar{t}).$$

So by (14)

$$\mathscr{x}_{n_k}(\bar{t}) \geqslant V^+(F_\epsilon) - \epsilon$$

and taking limits

$$V^+(F_\epsilon) - \epsilon \leqslant \bar{\mathscr{x}}(\bar{t}).$$

Thus

$$P(A, B) \geqslant V^+(F_\epsilon) - \epsilon$$

so

$$\begin{aligned} V_e^+ &\geqslant \lim_{\epsilon\to 0} (V^+(F_\epsilon) - \epsilon \\ &= l_1 . \end{aligned}$$

Combining this with (14) we obtain

$$V_e^+ = l_1 .$$

Consider now V_e^-. First observe that there exists a sequence $(\alpha_n^*) = A^*$ of delay strategies for the evader (J_1) with

$$u_n(\alpha_n^*) \geqslant V^-(F_{1/n}) - 1/n.$$

Consider, for a fixed k, the "constant" approximate strategy $A_k = (\alpha_k^*)$, i.e., $A_k = (\alpha_n)$ where $\alpha_k = \alpha_k^*$ for all n.

For any $B = (\beta_n)$ let $(x_n, \mathscr{x}_n)$ be the trajectories induced by A_k and B, and suppose for some subsequence $(x_{n_p}, \mathscr{x}_{n_p}) \to (\bar{x}, \bar{\mathscr{x}})$. Let $\bar{t}$ be the first time that $(\bar{t}, \bar{x}(\bar{t})) \in F$. For large enough p we have

$$(\bar{t}, x_{n_p}(\bar{t})) \in F_{1/k}$$

so that $u_k(\alpha_k^*) \leqslant \mathscr{x}_{n_p}(\bar{t})$. As $p \to \infty$ we obtain

$$u_k(\alpha_k^*) \leqslant \bar{\mathscr{x}}(\bar{t}).$$

Therefore

$$V^-(F_{1/k}) - 1/k \leqslant P(A_k, B)$$

so that

$$l_2 \leqslant V_e^-. \tag{15}$$

Conversely, given any approximate strategy $A = (\alpha_n)$ we may determine $z_n \in \mathscr{M}_2$ with

$$P_{1/n}(\alpha_n z_n, z_n) \leqslant V^-(F_{1/n}) + 1/n.$$

Defining $\beta_n y = z_n$ we obtain an approximate strategy $B = (\beta_n)$ for the pursuer. Then we may show as above that

$$\begin{aligned} P(A, B) &\leqslant \lim_{n\to\infty} P_{1/n}(\alpha_n z_n, z_n) \\ &\leqslant l_2 \end{aligned}$$

so that

$$V_e^- \leqslant l_2. \tag{16}$$

Combining with (15)

$$V_e^- = l_2.$$

COROLLARY 4.2.

$$\lim_{\epsilon\to 0} V_e^+(F_\epsilon) = V_e^+,$$

and

$$\lim_{\epsilon \to 0} V_e^-(F_\epsilon) = V_e^-.$$

Proof.

$$V_e^+(F_\epsilon) = \lim_{\delta \downarrow \epsilon} V^+(F_\delta)$$

by the theorem so the result is immediate; similarly for V_e^-.

In the special case of an ordinary pursuit–evasion game, where $h \equiv 1$, we can interpret these results as follows. The game has upper value T^+ if the pursuer can in any time $T^+ + \epsilon$, $\epsilon > 0$, guarantee to force the evader into the terminal set. However, for the extended upper value T_e^+ the pursuer can in any time $T_e^+ + \epsilon$ only guarantee to force the evader arbitrarily close to the terminal set.

5. Extended Values in Generalized Pursuit–Evasion Games

In a generalized pursuit–evasion game the pursuer's problem is to minimize the amount of "energy"

$$\mathscr{x}(t) = \int_{t_0}^{t} h(s, x(s), y(s), z(s))\, ds$$

consumed in forcing $x(t)$ into the terminal set F. Let us now consider a related fixed-time game in which the pursuer is "given" a fixed quantity of energy to drive as close as possible to the terminal set. That is, we consider the game $\hat{G}_E$ defined by

$$\begin{aligned} dx/dt &= f(t, x, y, z), \\ d\mathscr{x}/dt &= h(t, x, y, z), \end{aligned} \tag{17}$$

with initial conditions

$$x(t_0) = x_0, \qquad \mathscr{x}(t_0) = 0$$

and payoff

$$\mu_E\{x(\cdot), \mathscr{x}(\cdot)\} = \inf\{\rho(t, x(t)) : \mathscr{x}(t) \leqslant E,\ t_0 \leqslant t \leqslant T_0\}, \tag{18}$$

where $E \geqslant 0$ is fixed.

We make two assumptions on h: first we assume that h satisfies a Lipschitz condition in x, i.e.,

$$| h(t, x_1, y, z) - h(t, x_2, y, z)| \leqslant K_1(t) \| x_1 - x_2 \|, \tag{19}$$

where, as in (2)

$$\int_{t_0}^{T_0} K_1(t)\, dt < \infty.$$

We also assume that h is bounded away from zero, i.e.,

$$h(t, x, y, z) \geqslant h_0 > 0 \tag{20}$$

for all $(t, x, y, z) \in I \times \mathbf{R}^m \times Y \times Z$. These two assumptions enable us to use results from the theory of fixed-time games on $\hat{G}_E$. Assumption (19) guarantees that the equation (17) satisfies a Lipschitz condition of type (2). From assumption (20) we obtain

Lemma 5.1. *μ_E is uniformly continuous on the set of possible trajectories in $\hat{G}_E$ [considered as a subset of $C(I)^{m+1}$].*

Proof. Let X denote the set of all possible trajectories; i.e., all solutions of (17) for measurable functions $y(t) \in \mathscr{M}_1$, $z(t) \in \mathscr{M}_2$. As observed in Section 4, X is relatively compact, so in particular the set of all attainable points in $\mathbf{R}^{m+1}$ is bounded. Therefore, there is a uniform bound for f, i.e., for $(x, \mathscr{x}) \in X$ and $y \in Y$, $z \in Z$

$$\| f(t, x(t), y, z)\| \leqslant M. \tag{21}$$

Also we have a lower bound for h, so that

$$d\mathscr{x}/dt \geqslant h_0 . \tag{22}$$

We now show that μ_E is uniformly continuous on X. For $\epsilon > 0$ take

$$\delta = h_0\epsilon/(h_0 + M + 1). \tag{23}$$

Suppose $(x_1, \mathscr{x}_1)$ and $(x_2, \mathscr{x}_2) \in X$ are such that

$$\sup_{t_0 \leqslant t \leqslant T_0} \| x_1(t) - x_2(t)\| + \sup_{t_0 \leqslant t \leqslant T_0} \| \mathscr{x}_1(t) - \mathscr{x}_2(t)\| \leqslant \delta. \tag{24}$$

Then

$$\mu_E(x_1, \mathscr{x}_1) = \rho(t_1, x(t_1)) \qquad t_0 \leqslant t_1 \leqslant T_0,$$

where

$$\mathscr{x}_1(t_1) \leqslant E.$$

By (24)

$$\mathscr{x}_2(t_1) \leqslant E + \delta$$

and by (22) it follows that there exists t_1' with

$$t_0 \leqslant t_1' \leqslant t_1$$

and

$$\begin{aligned} x_2(t_1') &\leqslant E \\ t_1 - t_1' &\leqslant h_0^{-1}\delta. \end{aligned} \tag{25}$$

Combining (25) and (21) we have

$$\| x_2(t_1) - x_2(t_1')\| \leqslant Mh_0^{-1}\delta. \tag{26}$$

However,

$$\| x_2(t_1) - x_1(t_1)\| \leqslant \delta,$$

and so

$$\begin{aligned} \rho(t_1', x_2(t_1')) &\leqslant \mu_E(x_1, x_1) + h_0^{-1}\delta + Mh_0^{-1}\delta + \delta \\ &\leqslant \mu_E(x_1, x_1) + \epsilon. \end{aligned}$$

Hence

$$\mu_E(x_2, x_2) - \mu_E(x_1, x_1) \leqslant \epsilon,$$

and by symmetry

$$| \mu_E(x_2, x_2) - \mu_E(x_1, x_1)| \leqslant \epsilon. \tag{27}$$

We now introduce the "reverse Isaacs condition":

DEFINITION 5.2. *G satisfies the reverse Isaacs condition if for* $p \in \mathbf{R}^m$

$$\min_{z \in Z} \max_{y \in Y} (p \cdot f - h) = \max_{y \in Y} \min_{z \in Z} (p \cdot f - h). \tag{28}$$

If G satisfies (9) (the Isaacs condition) and (28) we shall say that G satisfies the *extended Isaacs condition.*

LEMMA 5.3. *G satisfies the extended Isaacs condition if and only if for all* $p \in \mathbf{R}^m$, $q \in R$

$$\min_{z \in Z} \max_{y \in Y} (p \cdot f + qh) = \max_{y \in Y} \min_{z \in Z} (p \cdot f + qh). \tag{29}$$

Proof. For $q \neq 0$ (29) follows from (28) or (9) by multiplication by $|q|$. For fixed $p \in \mathbf{R}^m$ the functions

$$q \to \min_z \max_y (p \cdot f + qh),$$

$$q \to \max_y \min_z (p \cdot f + qh),$$

are continuous, so that (29) follows also for $q = 0$.

THEOREM 5.4. *If G is a generalized pursuit–evasion game satisfying* (19), (20), *and the extended Isaacs condition, then for each $E \geqslant 0$, $\hat{G}_E$ has value, denoted by $\hat{V}(E)$.*

Proof. By Lemma 5.1 and the Tietze extension theorem we may suitably define μ_E off the space of possible trajectories in $\hat{G}_E$ so that μ_E is continuous on $C(I)^{m+1}$. Thus $\hat{G}_E$ is a game of the type described by (1), (2), and (4). Also, the Isaacs condition for $\hat{G}_E$ takes the form (29) so that, by Lemma 5.3, we can quote Theorem 3.2 to assert that $\hat{G}_E$ has value, $\hat{V}(E)$.

The function h is bounded on the set of possible trajectories, and so for large enough E

$$\begin{aligned}\mu_E(x, \varkappa) &= \inf(\rho(t, x(t)); \varkappa(t) \leqslant E, t_0 \leqslant t \leqslant T_0) \\ &= 0.\end{aligned}$$

That is, $\hat{V}(E) = 0$ for large enough E. Clearly $\hat{V}(E)$ decreases in E, so we make the following definition.

DEFINITION 5.5.

$$E^* = \inf(E : \hat{V}(E) = 0). \tag{30}$$

THEOREM 5.6. *If G is a generalized pursuit–evasion game satisfying* (19) *and* (20) *and the extended Isaacs condition, then G has extended value E^*, i.e.,*

$$V_e^+ = V_e^- = E^*.$$

Proof. Suppose $\epsilon > 0$. Then there is an E_1 with $E_1 \leqslant E^* + \epsilon$ and

$$\hat{V}(E_1) = 0.$$

Hence, there exists a delay strategy β for the pursuer whose value $v_{E_1}(\beta)$ in $\hat{G}_{E_1}$ satisfies

$$v_{E_1}(\beta) \leqslant \epsilon.$$

For any control function $y(t) \in \mathscr{M}_1$ for the evader let $(x(t), \mathscr{x}(t))$ be the trajectory corresponding to $(y, \beta y)$. Then $\mu_{E_1}(x(t), \mathscr{x}(t)) \leqslant \epsilon$ and so for some t_ϵ, with $t_0 \leqslant t_\epsilon \leqslant T_0$,

$$\mathscr{x}(t_\epsilon) \leqslant E_1$$

and

$$\rho(t_\epsilon, x(t_\epsilon)) \leqslant \epsilon. \tag{31}$$

Now consider the delay strategy β in G_ϵ (see Section 4 above). By (31) the payoff P_ϵ corresponding to $(y, \beta y)$ satisfies

$$\begin{aligned} P_\epsilon &\leqslant \int_{t_0}^{t_\epsilon} h(s, x(s), y(s), z(s))\, ds \\ &= \mathscr{x}(t_\epsilon) \\ &\leqslant E_1, \end{aligned}$$

so that

$$v_\epsilon(\beta) \leqslant E_1 .$$

Hence

$$\begin{aligned} V^+(F_\epsilon) &\leqslant E_1 \\ &\leqslant E^* + \epsilon. \end{aligned}$$

Letting $\epsilon \to 0$ we obtain by Theorem 4.1

$$V_e^+ \leqslant E^*. \tag{32}$$

Conversely, if $0 < E < E^*$ then

$$\hat{V}(E) > 0$$

and so there exists a delay strategy α for the evader with

$$\hat{u}_E(\alpha) = 2\eta > 0. \tag{33}$$

If $z(t) \in \mathscr{M}_2$ is a control for the pursuer then $(\alpha z, z)$ induces a trajectory $(x(t), \mathscr{x}(t))$ satisfying

$$\rho(t, x(t)) \geqslant 2\eta$$

whenever

$$\mathscr{x}(t) \leqslant E.$$

Consider α as a delay strategy in G_η; then $u_\eta(\alpha) \geqslant E$ so that

$$V^-(F_\eta) \geqslant E.$$

Letting $\eta \to 0$

$$V_e^- \geqslant E^* \tag{34}$$

by Theorem 4.1. Hence, combining (32) and (34)

$$V_e^- = V_e^+ = E^*.$$

THEOREM 5.7. *If G is a generalized pursuit–evasion game satisfying* (19) *and the extended Isaacs condition, then G has extended value, i.e.,*

$$V_e^+ = V_e^-.$$

Proof. For each n consider the game $G^{(n)}$ with the same dynamics and initial conditions as G, but with payoff

$$P^{(n)} = \int_{t_0}^{t_F} h^{(n)}(s, x(s), y(s), z(s))\, ds, \tag{35}$$

where

$$h^{(n)}(t, x, y, z) = h(t, x, y, z) + 1/n.$$

Then $G^{(n)}$ satisfies (19), (20), and the generalized Isaacs condition, so that by Theorem 5.6

$$V_e^+(G^{(n)}) = V_e^-(G^{(n)}).$$

For $\epsilon > 0$ we consider the game $G_\epsilon^{(n)}$ with terminal set F_ϵ and payoff

$$P_\epsilon^{(n)} = \int_{t_0}^{t_{F_\epsilon}} h^{(n)}(s, x(s), y(s), z(s))\, ds. \tag{36}$$

For fixed controls $y(s)$, $z(s)$

$$| P_\epsilon^{(n)} - P_\epsilon | \leqslant (1/n)(T_0 - t_0),$$

where P_ϵ is the payoff in G_ϵ. Hence it follows that

$$| V^+(G^{(n)}; F_\epsilon) - V^+(F_\epsilon)| \leqslant (1/n)(T_0 - t_0),$$

$$| V^-(G^{(n)}; F_\epsilon) - V^-(F_\epsilon)| \leqslant (1/n)(T_0 - t_0)$$

and so as $n \to \infty$

$$V^+(G^{(n)}; F_\epsilon) \to V^+(F_\epsilon),$$

$$V^-(G^{(n)}; F_\epsilon) \to V^-(F_\epsilon)$$

uniformly in $\epsilon > 0$. In particular

$$\lim_{n\to\infty}\lim_{\epsilon\to 0} V^+(G^{(n)}; F_\epsilon) = V_e^+$$

and

$$\lim_{n\to\infty}\lim_{\epsilon\to 0} V^-(G^{(n)}; F_\epsilon) = V_e^-.$$

Therefore

$$\begin{aligned} V_e^+ &= \lim_{\epsilon\to 0} V^+(F_\epsilon) \\ &= \lim_{\epsilon\to 0}\lim_{n\to\infty} V^+(G^{(n)}; F_\epsilon) \\ &= \lim_{n\to\infty}\lim_{\epsilon\to 0} V^+(G^{(n)}; F_\epsilon) \\ &= \lim_{n\to\infty} V_e^+(G^{(n)}) \\ &= \lim_{n\to\infty} V_e^-(G^{(n)}) \qquad \text{by Theorem 5.6.} \\ &= \lim_{n\to\infty}\lim_{\epsilon\to 0} V^-(G^{(n)}; F_\epsilon) \\ &= \lim_{\epsilon\to 0} V^-(F_\epsilon) = V_e^-. \end{aligned}$$

Remarks. In the special case of a pursuit–evasion game, that is, $h \equiv 1$, Theorem 5.6 has the following interpretation. For each fixed $T \leqslant T_0$ the fixed-time game $\hat{G}_T$ over the interval $[t_0, T]$ is considered, with dynamics given by (1) and payoff

$$P = \mu_T(x) = \inf_{t_0 \leqslant t \leqslant T} \rho(t, x(t)).$$

For such a game the Isaacs condition takes the form

$$\min_z \max_y p \cdot f = \max_y \min_z p \cdot f. \tag{37}$$

If (37) is satisfied the game $\hat{G}_T$ has a value $\hat{V}(T)$ and for large enough T $\hat{V}(T) = 0$.

Write

$$T^* = \inf(T : \hat{V}(T) = 0).$$

An immediate consequence of Theorem 5.6 is the following result:

COROLLARY 5.8. *If the Isaacs condition* (37) *is satisfied then in a pursuit–evasion game*

$$V_e^+ = V_e^- = T^*.$$

The conditions imposed on G in Theorem 5.7 do not seem the best possible. Condition (19), a Lipschitz condition on h, is only necessary to ensure that $\hat{G}$ is of the same type of game as that studied in [3]. It would seem likely that this condition may be removed altogether. Another, apparently superfluous, condition is the reverse Isaacs condition (28). This condition is intuitively unnecessary for the existence of value, although its removal seems to present severe technical difficulties. However, the extended Isaacs condition is not too restrictive. Two main practical cases are covered by this condition: the case of separated variables studied by Friedman [5, 6, 8] when

$$f(t, x, y, z) = f_1(t, x, y) + f_2(t, x, z),$$
$$h(t, x, y, z) = h_1(t, x, y) + h_2(t, x, z),$$

and the case of relaxed controls studied by the authors and Markus [1, 3] (see Section 6 below). In both these cases Theorem 5.7 is valid.

6. Applications of Relaxed Controls

In [1] the authors in collaboration with Markus applied the ideas of relaxed controls, already studied in control theory by Warga [15], to differential games. Essentially, one idealizes the possible controls available to each player by extending them to include probability measures on Y and Z. At any time t J_1 may choose an element from the space $\Lambda(Y)$ of regular probability measures on Y; similarly J_2 may choose from $\Lambda(Z)$. The spaces $\Lambda(Y)$ and $\Lambda(Z)$ are topologized in a natural way so that they become compact metric spaces (see [1, 15] for details); the functions f and h are extended thus

$$f(t, x, \sigma, \tau) = \int_Y \int_Z f(t, x, y, z)\, d\tau(z)\, d\sigma(y), \tag{38}$$

$$h(t, x, \sigma, \tau) = \int_Y \int_Z h(t, x, y, z)\, d\tau(z)\, d\sigma(y), \tag{39}$$

so that

$$f : I \times \mathbf{R}^m \times \Lambda(Y) \times \Lambda(Z) \to \mathbf{R}^m,$$
$$h : I \times \mathbf{R}^m \times \Lambda(Y) \times \Lambda(Z) \to \mathbf{R}$$

are continuous, and both satisfy a Lipschitz condition in x of the same type as satisfied by the original f and h. We shall refer to a game in which both players may use relaxed controls as a *relaxed* game.

Lemma 6.1. *In a relaxed game the extended Isaacs condition holds.*

Proof. For $p \in \mathbf{R}^m$, the equations (9) and (28) follow from the fundamental theorem of two-person games over compact convex sets due to Wald [14] (see also von Neumann [13]).

In the relaxed game the spaces $\mathscr{M}_1$ and $\mathscr{M}_2$ of available control functions are replaced by the spaces $\mathscr{M}_1^*$ and $\mathscr{M}_2^*$ of measurable maps $\sigma : I \to \Lambda(Y)$ and $\tau : I \to \Lambda Z()$. We may identify $\mathscr{M}_1^*$ as a compact subset of the dual of the Banach space $L^1(C(Y))$ of integrable functions $\varphi : I \to C(Y)$ with the weak* topology; we similarly topologize $\mathscr{M}_2^*$.

Lemma 6.2. *For fixed* $\tau(t) \in \mathscr{M}_2^*$ *the map* $\Sigma_\tau : \mathscr{M}_1^* \to (C(I))^{m+1}$ *is continuous, where* $C(I)^{m+1}$ *is the Banach space of continuous* $\mathbf{R}^{m+1}$*-valued functions on I and* $\Sigma_\tau(\sigma(t))$ *is the solution of*

$$dx/dt = f(t, x, \sigma(t), \tau(t)),$$

$$d\mathfrak{x}/dt = h(t, x, \sigma(t), \tau(t)),$$

subject to $x(t_0) = x_0$ *and* $\mathfrak{x}(t_0) = 0$.

Proof. See [1] or [15]. Note that $\mathscr{M}_1^*$ is metrizable.

Next we topologize the space of possible maps $\alpha : \mathscr{M}_2^* \to \mathscr{M}_1^*$ by the product topology $(\mathscr{M}_1^*)^{\mathscr{M}_2^*}$; the set of strategies for J_1 is then a subset of this space.

Lemma 6.3. *The set of strategies is closed and, therefore, compact.*

Proof. See [3, Theorem 11.2].

We now consider the effect of relaxed controls on extended values for games of survival.

Theorem 6.4. *Let G be a relaxed game of survival: then we have*

$$U \geqslant V_e^-,$$

$$V \leqslant V_e^+.$$

Proof. By the definition of V_e^- [Eq. (12)] there exists an approximate strategy $A^* = (\alpha_n^*)$ such that: $\inf_B \inf P(A^*, B) \geqslant V_e^- - \epsilon$. Let α^* be any cluster point of A^* in $(\mathscr{M}_2^*)^{\mathscr{M}_1^*}$. Then α^* is a strategy. Let $\tau = \tau(t)$ be any fixed control function in $\mathscr{M}_2^*$. Then $\Sigma_\tau(\alpha^*\tau)$ is a cluster point of $\Sigma_\tau(\alpha_n^*\tau)$ in $C(I)^{m+1}$. Define $B = (\beta_n)$ by $\beta_n(\sigma) = \tau$ for all n. Then B is an approximate strategy and $P(\alpha^*\tau, \tau) \in P(A^*, B)$. Therefore

$$u(\alpha^*) \geqslant \inf_B \inf P(A^*, B)$$

$$\geqslant V_e^- - \epsilon$$

so

$$U = \sup_{\alpha} u(\alpha) \geqslant V_e^-.$$

The other inequality is similarly proved.

COROLLARY 6.5. *In a relaxed game of survival* $V_e^- \leqslant V^+$ *and* $V_e^+ \geqslant V^-$. *If G has value and G has extended value, then they are equal.*

THEOREM 6.6. *Let G be a relaxed generalized pursuit–evasion game satisfying* (19). *Then G has extended value* V_e *and*

$$V_e = V^- = V \leqslant U \leqslant V^+.$$

Proof. By Theorem 5.7 and Lemma 6.1 G has extended value, i.e., $V_e^+ = V_e^- = V_e$, say. Then by Theorem 6.4 $V^- \leqslant V \leqslant V_e$. For any $\epsilon > 0$ $V^-(F_\epsilon) \leqslant V^-$ and so by Theorem 4.1,

$$\lim_{\epsilon \to 0} V^-(F_\epsilon) = V_e^- = V_e \leqslant V^-.$$

Hence

$$V_e = V^- = V$$

and the remaining inequalities follow.

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