Endomorphisms of Symmetric Function Spaces

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1. Introduction and statement of main theorems. We recall that a quasi-Banach space is a complete metrizable (real) vector space whose topology is given by a quasi-norm $x \mapsto \|x\|$ satisfying

$$\|x\| > 0 \qquad x \neq 0$$
$$\|\alpha x\| = |\alpha|\|x\| \qquad \alpha \in \mathbf{R}, x \in X$$
$$\|x_1 + x_2\| \leq C(\|x_1\| + \|x_2\|) \qquad x_1, x_2 \in X$$

where C is independent of x_1 and x_2.

We let $L_0 = L_0[0,1]$ denote the space of all Borel-measurable real-valued functions on $[0,1]$ with the topology of convergence in Lebesgue measure λ; as is customary we shall not distinguish between a function and its equivalence class. If $f \in L_0$ we define

$$d_f(x) = \lambda(|f| > x) \qquad 0 \leq x < \infty$$

to be its distribution function. A *symmetric*, or *re-arrangement-invarant* function space is a quasi-Banach space X which is algebraically an order-ideal of L_0, containing $L_\infty[0,1]$ and such that

(a) If $f \in X$, $g \in L_0$ with $d_g \leq d_f$ then $g \in X$ and $\|g\| \leq \|f\|$.
(b) If $0 \leq f_n \leq 1$ and $f_n \downarrow 0$ a.e. then $\|f_n\| \downarrow 0$.
(c) If $0 \leq f_n \uparrow f$ and $f \in X$ then $\|f_n\| \uparrow \|f\|$.
(d) X is either minimal or maximal as described below (cf. [8] p. 119).

X is *minimal* if L_∞ is dense in X (equivalently if X is separable) and *maximal* if every (quasi-) norm bounded increasing sequence is bounded above in X. In general for arbitrary X, its maximal hull $X_{\max}$ can be defined to be the space of all $f \in L_0$ so that

$$\|f\|_{\max} = \sup(\|g\| : g \in X, |g| \leq |f|) < \infty.$$

We shall largely be concerned with L-convex function spaces. These are discussed in [6] and [7], but for our purposes it suffices to define X to be L-convex if there exists $r > 0$ and a constant $C < \infty$ so that for $f_1, \ldots, f_n \in X$

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$$\left\|\left(\sum_{i=1}^{n}|f_i|^r\right)^{1/r}\right\|\le C\left(\sum_{i=1}^{n}\|f_i\|^r\right)^{1/r}$$

i.e. X is *r-convex* for some $r > 0$. Equivalently X is L-convex if there is a Banach function space Y and $r > 0$ so that $X = \{f:|f|^r \in Y\}$. As is shown in [6] most naturally arising function spaces are L-convex; as an example we note that if ℓ_∞ is not finitely representable in X then X is L-convex.

We define the dilation operators $D_t:X \to X$ by

$$\begin{aligned} D_tf(s) &= f(st^{-1}) \qquad 0 \le s \le \min(1,t) \\ &= 0 \qquad \min(1,t) < s \le 1. \end{aligned}$$

The *Boyd indices* of X are given by

$$p_X = \lim_{s\to\infty}\frac{\log s}{\log\|D_s\|}$$

$$q_X = \lim_{s\to 0}\frac{\log s}{\log\|D_s\|}.$$

Here $0 < p_X \le q_X \le \infty$. For a discussion see [1] or [8].

We now state the main theorem of the paper.

Theorem 1.1. *Let X be an L-convex symmetric function space not contained in $L_1[0,1]$. Then $p_X \le 1$ and if $p_X \le r \le \min(1,q_X)$ there exists a constant C so that if $T:X \to X$ is a linear operator then T extends to a linear operator $T:L_r \to L_r$ with $\|T\|_r \le C\|T\|$.*

We remark that if $r = 1$ the constant C can always be taken to be 1. We also remark that the hypothesis that X is not contained in L_1 is equivalent to the hypothesis that the closure X_0 of L_∞ in X has trivial dual (Lemma 3.2 below). In particular if X is minimal (or separable) then we may equivalently say that X has trivial dual. However the space $L(1,\infty)$ (weak L_1) is an example of a symmetric L-convex function space with non-trivial dual satisfying the hypotheses of the theorem (cf. [2]).

A close relative of Theorem 1.1 is:

Theorem 1.2. *Let X be an L-convex symmetric function space not contained in $L_1[0,1]$. Suppose $T:X \to X$ is a linear operator. Then there exist Borel maps $a_n:[0,1] \to \mathbf{R}$ and $\sigma_n:[0,1] \to [0,1]$ so that*

(i) $\sigma_m t \ne \sigma_n t \qquad m \ne n$

(ii) $|a_n(t)| \ge |a_{n+1}(t)| \qquad n \in \mathbf{N}$

(iii) $\sum |a_n(t)| < \infty \qquad$ a.e.

(iv) $Tf(t) = \sum_{n=1}^{\infty} a_n(t)f(\sigma_n t) \qquad$ a.e. $f \in L_\infty$.

This theorem should be compared with the results of [7]. In [7] it is shown that an operator $T:X \to Y$ satisfies the conclusions of Theorem 1.2 for arbitrary Y if $X \supset L(1,\infty)$; it is also shown that the condition $X \supset L(1,\infty)$ is "almost" necessary. Here, however, by restricting consideration to endomorphisms we obtain a significantly stronger conclusion. As an example where Theorem 1.2 applies, but the results of [7] do not we mention the Lorentz spaces $L(1,p)$ $(1 < p < \infty)$ or Orlicz spaces L_ϕ where $\phi(x) = O(x)$ as $x \to \infty$ but

$$\int_1^\infty \frac{\phi(x)}{x^2}\,dx = \infty.$$

By way of contrast we mention the Boyd interpolation theorem ([1], [8]). We shall prove this theorem in our setting as follows.

Theorem 1.3. *Suppose $T:L_p \to L_p$ is a bounded linear operator which restricts to a bounded operator $T:L_q \to L_q$ where $0 < p < q < \infty$. Suppose X is a symmetric L-convex function space with $p < p_X \leq q_X < q$. Then T maps X into X and is bounded there.*

Thus Theorem 1.3 is in a certain sense opposite to our Theorem 1.1.

The proofs of the main results will be developed in Sections 2-7. As will be seen Theorems 1.1 and 1.2 are intimately related and cannot really be proved separately. In Section 8 we give some elementary applications, generalizing results known for L_p if $0 < p < 1$ to any minimal L-convex symmetric function space with trivial dual.

Finally we note that [0,1] with Lebesgue measure could be replaced by any Polish space K with a positive measure μ on K satisfying $\mu(K) \leq 1$ in the statement of the main theorems. Indeed $X(K,\mu)$ is then defined to be the space of Borel-measurable functions $f:K \to \mathbf{R}$ such that if f^* is the decreasing rearrangement of f i.e. $f^*:[0,1] \to \mathbf{R}$ and

$$f^*(t) = \inf_{\mu(E)=t} \sup_{K\setminus E} |f(s)|$$

then $f^* \in X = X[0,1]$ and we put

$$\|f\| = \|f^*\|.$$

We shall let $\mathcal{B}(K)$ denote the Borel sets of K and abbreviate $\mathcal{B}[0,1]$ to $\mathcal{B}$.

2. Operators on symmetric function spaces. If X and Y are quasi-Banach spaces, we denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators $T:X \to Y$. For $T \in \mathcal{L}(X,Y)$ we define

$$\|T\| = \sup(\|Tx\| : \|x\| \leq 1).$$

We write $\mathcal{L}(X) = \mathcal{L}(X,X)$.

If X and Y are symmetric function spaces then $T \in \mathcal{L}(X,Y)$ is *regular* if there exist positive operators $P_i:X \to Y (i = 1,2)$ so that $T = P_1 - P_2$. We say that T is *controllable* if there exists $h \in L_0$ so that

$$|Tf| \leq h \qquad \text{a.e.}$$

for every $f \in X$ with $|f| \leq 1$ a.e. Clearly every regular operator is controllable but the converse is false (cf. [7]).

Let $\mathcal{M}[0,1]$ denote the space of regular Borel measures on $[0,1]$ with the weak $*$-topology. An *abstract kernel* is a Borel measurable map $t \to \mu_t([0,1] \to \mathcal{M}[0,1])$ so that whenever B is a Borel subset of $[0,1]$ with $\lambda(B) = 0$ then $|\mu_t|(B) = 0$ a.e. If $T \in \mathcal{L}(X,Y)$ is a controllable operator then there is an essentially unique abstract kernel $t \mapsto \mu_t$ so that

$$Tf(t) = \int f d\mu_t \qquad \text{a.e.}$$

for $f \in L_\infty$. See [7], [9] or [10] for details. We say that $\{\mu_t\}$ is the abstract kernel representing T.

If μ_t is an atomic measure for almost every $t \in [0,1]$ then $\{\mu_t\}$ is an *atomic kernel*. In this case there exist Borel maps $a_n : [0,1] \to \mathbf{R}$ and $\sigma_n : [0,1] \to [0,1]$ so that

$$\mu_t = \sum_{n=1}^{\infty} a_n(t) \delta(\sigma_n t) \qquad \text{a.e.}$$

and $\sigma_n(t) \neq \sigma_m(t)$ whenever $m \neq n$, and $|a_n(t)| \geq |a_{n+1}(t)|$ for all $n \in \mathbf{N}$. See [7] or [10] Theorem 6.2. In [7] Theorem 4.4 it is shown that if X is a minimal symmetric function space with trivial dual, and $T \in \mathcal{L}(X,Y)$ is controllable then its representing kernel is atomic so that we can write

$$Tf(t) = \sum_{n=1}^{\infty} a_n(t) f(\sigma_n t)$$

for $f \in L_\infty$.

An operator $T \in \mathcal{L}(X_1Y)$ is said to be *elementary* [4] (or a *Riesz homomorphism*) if it takes the form

$$Tf(t) = a(t) f(\sigma t)$$

where $a \in L_0$ and $\sigma : [0,1] \to [0,1]$ is a Borel map. T is said to be *locally elementary* (or a local *Riesz homomorphism*) if there exist disjoint Borel subsets $\{B_n\}_{n=1}^{\infty}$ of $[0,1]$ so that $\bigcup B_n = [0,1]$ and each TP_{B_i} is elementary. Here $P_A : X \to X$ is the natural projection $P_A f = 1_A \cdot f$ where 1_A is the characteristic function of A.

For any $\mu \in \mathcal{M}[0,1]$ we define for $x > 0$

$$\Delta_x(\mu) = \sum \{\delta(t) : |\mu|\{t\} > x\}$$

(cf. [4]). If $t \mapsto \mu_t$ is an atomic abstract kernel then the map $t \mapsto \Delta_x(\mu_t)$ is weak $*$-Borel and we can define positive measures α_x $(x \geq 0)$ on the Borel sets $\mathcal{B}$ by

$$\alpha_x(B) = \int_0^1 \Delta_x(\mu_t)(B) dt$$

α_x is then a positive (perhaps infinite) measure continuous with respect to λ. For each $x \in \mathcal{Q}_+$ let $t \mapsto w(t,x)$ be a Borel function so that

$$\int_B w(t,x)dt = \alpha_x(B) \qquad B \in \mathcal{B}.$$

Since $\alpha_x \geq \alpha_y$ for $x \leq y$ we can suppose $w(t,x)$ is monotone decreasing in x for each $t \in [0,1]$ and extend it to $[0,1] \times (0,\infty)$ by setting

$$w(t,x) = \sup_{y \geq x, y \in \mathcal{Q}} w(t,y).$$

It now may be checked for each $x \in (0,\infty)$ and $B \in \mathcal{B}$

$$\alpha_x(B) = \int_B w(t,x)dt.$$

This follows from the fact that for each $B \in \mathcal{B}$, $x \mapsto \alpha_x(B)$ is right-continuous (see [4] p. 331 for details).

Next for $0 \leq t \leq 1$ and $0 < s < \infty$ we define

$$\begin{aligned} F(t,s) &= \sup_{x \in \mathcal{Q}}\{x : w(t,x) \geq s\} \\ &= 0 \qquad \text{if } w(t,x) < s \text{ for all } x \in \mathcal{Q}_+. \end{aligned}$$

F is a Borel function on $[0,1] \times (0,\infty)$, which we shall call the *local distribution* of the kernel μ_t (note that this terminology deviates from that of [4]).

We shall now prove a special case of Theorem 9.6 of [4] which is appropriate to our needs in this paper.

Theorem 2.1. *Let $\{\mu_t\}$ be an atomic kernel and suppose $0 < p \leq 1$. In order that the operator*

$$Tf(t) = \int f d\mu_t$$

extend continuously to an endomorphism of L_p it is necessary and sufficient that

$$\operatorname*{ess\,sup}_{t \in [0,1]} \left\{ \int_0^\infty |F(t,s)|^p ds \right\}^{1/p} = C < \infty$$

and then $\|T\|_p = C$ (where $\|T\|_p$ denotes the norm of T in $\mathcal{L}(L_p)$).

Proof. Let

$$\mu_t = \sum_{n=1}^{\infty} a_n(t)\delta(\sigma_n t) \qquad \text{a.e.}$$

where $\sigma_m t \neq \sigma_n t$ for $m \neq n$. By Theorem 3.2 of [3] $T \in \mathcal{L}(L_p)$ if and only if

$$\|T\|_p^p = \sup_{\lambda(B)>0} \lambda(B)^{-1} \sum_{n=1}^{\infty} \int_{\sigma_n^{-1}B} |a_n(t)|^p dt < \infty.$$

Now

$$\sum_{n=1}^{\infty} \int_{\sigma_n^{-1}B} |a_n(t)|^p dt = \sum_{n=1}^{\infty} \int_{\sigma_n^{-1}B} \int_0^{|a_n(t)|} px^{p-1}\, dt dx$$
$$= \int_0^{\infty} px^{p-1} \int_0^1 \Delta_x(\mu_t)(B) dt dx$$
$$= \int_0^{\infty} px^{p-1}\, \alpha_x(B) dx$$
$$= \int_0^{\infty} px^{p-1} \int_B w(t,x) dt dx$$
$$= \int_B \int_0^{\infty} px^{p-1}\, w(t,x) dx dt$$
$$= \int_B \int_0^{\infty} F(t,s)^p ds dt$$

so that the result follows.

3. Initial observations on the main theorems. First we shall sketch the modifications necessary in the proof of the Boyd interpolation theorem to yield Theorem 1.3. We shall use the proof in [8] p. 145.

Proof of Theorem 1.3. Since X is L-convex there exists $0 < r < p$ and a symmetric Banach function space Y so that $X = \{f \in L_0 : |f|^r \in Y\}$. We may suppose X to be quasi-normed by

$$\|f\| = \||f|^r\|_Y^{1/r}.$$

Let U denote the unit ball of Y' (the Köthe dual of Y). Then

$$\|f\| = \sup_{g \in U} \left\{ \int_0^1 |f|^r |g| dt \right\}^{1/r}.$$

Now the proof in [8] p. 145 goes through with minor modifications. Lemma 2.b.12 is replaced by

$$|Tf^*(2t)|^r \leq M \left\{ \int_0^1 f^*(tu)^r u^{r/p-1} du + \int_1^{\infty} f^*(tu)^r u^{r/q-1} du \right\}$$

and the proof of 2.b.11 is almost identical.

Next we observe:

Lemma 3.1. *If X is a symmetric function space not contained in $L_1[0,1]$ then $p_X \leq 1$.*

Proof. Indeed if $p_X > r > 1$ then

$$\|D_t\| \leq Ct^{-1/r} \qquad 1 \leq t < \infty$$

for some constant $C > 0$. Hence if 1_A denotes the characteristic function of $A \in \mathcal{B}$, and if $1 = 1_{[0,1]}$,

$$\|1\| \leq Ct^{-1/r}\|1_{[0,t]}\|$$

so that

$$\|1_{[0,t]}\| \geq C't^{1/r}$$

for some $C' > 0$.

Thus if $f \in X$ and $A = \{|f| > x\}$

$$x\|1_A\| \leq \|f\|$$

and so

$$\lambda(A)^{1/r} \leq (C')^{-1}x^{-1}\|f\|$$

so that

$$d_f(x) \leq (C')^{-r}\|f\|^r x^{-r}$$

so $f \in L_1$, contradicting our asumption.

Lemma 3.2. *Suppose X is a symmetric function space not contained in L_1. Let X_0 be the closure of L_∞ in X. Then X_0 has trivial dual.*

Proof. Let $\phi \in X_0^*$. Then $B \mapsto \phi(1_B)$ is a λ-continuous measure on $\mathcal{B}$ and hence there exists $h \in L_1$ so that for $f \in L_\infty$

$$\phi(f) = \int_0^1 hf\,dt.$$

If $h \neq 0$ we can suppose there exists $A \in \mathcal{B}$ so that $\lambda(A) > 0$ and $h(s) \geq \varepsilon > 0$ for $s \in A$. If $\operatorname{supp} f \subset A$ then

$$\varepsilon \int_0^1 |f|\,dt \leq \phi(|f|) \leq \|\phi\|\|f\|.$$

It follows, since X_0 is symmetric, that for $f \in L_\infty$

$$\int_0^1 |f|\,dt \leq (1 + \lambda(A)^{-1})\frac{\|\phi\|}{\varepsilon}\|f\|.$$

Hence this inequality holds for $f \in X$ and so $X \subset L_1$ contrary to assumption,

Finally we deduce Theorem 1.2 from Theorem 1.1.

Proof of Theorem 1.2 assuming Theorem 1.1. Assume $T \in \mathcal{L}(X)$. Then for some $r \leq 1$, $T \in \mathcal{L}(L_r)$ and in particular T is controllable. Also X_0 is a minimal symmetric function space with trivial dual. The result now follows from Theorem 4.4 of [7].

4. Local multipliers. For $n \geq 1$ let $\mathcal{H}_n$ be the space of all bounded Borel functions $h:[0,\infty) \to \mathbf{R}$ so that $h(x) = 0$ for $x \geq n$. Let $\mathcal{H} = \bigcup \mathcal{H}_n$. If g, $h \in \mathcal{H}$ we write $g \sim h$ if $d_g = d_h$ where

$$d_h(x) = \lambda(|h| > x)$$

{λ denotes Lebesgue measure on $[0,\infty)$}.

Let X be a fixed symmetric function space. If $h \in \mathcal{H}_n$ and $0 < \varepsilon < 1/n$, we can define a "multiplier" $M(h;\varepsilon):X[0,\varepsilon] \to x([0,\varepsilon] \times [0,n])$ by

$$(M(h;\varepsilon)f)(x,y) = h(y)f(x).$$

$M(h,\varepsilon)$ is a bounded linear operator and we define

$$\|h\|_{\varepsilon,M} = \|M(h,\varepsilon)\|.$$

$\|h\|_{\varepsilon,M}$ is monotone increasing in ε, so we can define

$$\|h\|_M = \lim_{\varepsilon\to 0}\|h\|_{\varepsilon,M}.$$

Now $\| \quad \|_M$ is a pseudo-quasi-norm on $\mathcal{H}$. Note that if $|h_1| \leq |h_2|$ then $\|h_1\|_M \leq \|h\|_M$ and if $h_1 \sim h_2$ then $\|h_1\|_M = \|h_2\|_M$.

For h_1, $h_2 \in \mathcal{H}$ define $h_1 \otimes h_2:[0,\infty) \times [0,\infty) \to \mathbf{R}$ by

$$h_1 \otimes h_2(x,y) = h_1(x)h_2(y).$$

Suppose $h_1 \in \mathcal{H}_m$ and $h_2 \in \mathcal{H}_n$ and $0 < \varepsilon < m^{-1}n^{-1}$. Consider the operator $S:X[0,\varepsilon] \to X([0,\varepsilon] \times [0,m] \times [0,n])$ given by

$$Sf(x,y,z) = h_1(y)h_2(z)f(x).$$

Then

$$\|S\| \leq \|h_2\|_{\varepsilon m,M}\|h_1\|_{\varepsilon,M}.$$

It follows that if $g \in \mathcal{H}$ and $g \sim h_1 \otimes h_2$ then

$$\|g\|_{\varepsilon,M} \leq \|h_2\|_{\varepsilon m,M}\|h_1\|_{\varepsilon,M}.$$

Letting $\varepsilon \to 0$ we obtain

$$\|g\|_M \leq \|h_1\|_M\|h_2\|_M.$$

Thus we can conclude

Proposition 4.1. *If h_1, $h_2 \in \mathcal{H}$ and $g \in \mathcal{H}$ with $g \sim h_1 \otimes h_2$ then*

$$\|g\|_M \leq \|h_1\|_M\|h_2\|_M.$$

Now let $q = q_X$ and $p = p_X$ be the Boyd indices of X.

Proposition 4.2.
(i) For $0 < t \leq 1$, $\|1_{[0,t]}\|_M \geq t^{1/q}$.
(ii) For $1 \leq t < \infty$, $\|1_{[0,t]}\|_M \geq t^{1/p}$.

Proof. For $f \in X[0,1]$ and $0 < s, t < 1$,

$$\|D_{st}f\| \leq \|1_{[0,t]}\|_{s,M}\|D_s\|\|f\|.$$

Hence

$$\|D_{st}\| \leq \|1_{[0,t]}\|_{s,M}\|D_s\|.$$

By induction, since $\|1_{[0,t]}\|_{s,M}$ is monotone in s,

$$\|D_{st^n}\| \leq \|1_{[0,t]}\|_{s,M}^n\|D_s\|.$$

Now

$$\frac{1}{q} = \lim_{n\to\infty} \frac{-\log\|D_{st^n}\|}{-n\log t - \log s} \geq \lim_{n\to\infty} \frac{-\log\|D_s\| - n\log\|1_{[0,t]}\|_{s,M}}{-\log s - n\log t}.$$

Thus

$$\frac{1}{q} \geq \frac{\log\|1_{[0,t]}\|_{s,M}}{\log t}$$

and hence

$$\|1_{[0,t]}\|_{s,M} \geq t^{1/q}.$$

Letting $s \to 0$ we obtain (i).
(ii) If $s, t > 1$,

$$\|D_{st}\| \leq \|1_{[0,t]}\|_{s^{-1}t^{-1},M}\|D_s\|$$
$$\leq \|1_{[0,t]}\|_{s^{-1},M}\|D_s\|.$$

By induction,

$$\|D_{st^n}\| \leq \|1_{[0,t]}\|_{s^{-1},M}^n\|D_s\|.$$

Now

$$\frac{1}{p} = \lim_{n\to\infty} \frac{\log\|D_{st^n}\|}{\log s + n\log t}$$
$$\leq \frac{\log\|1_{[0,t]}\|_{s^{-1},M}}{\log t}.$$

Thus $\|1_{[0,t]}\|_{s^{-1},M} \geq t^{1/p}$ and the result follows.

Theorem 4.3. *If* $h \in \mathcal{H}$ *and* $p \leq r \leq q$ *then*

$$\|h\|_M \geq \|h\|_r$$

where

$$\|h\|_r = \left\{ \int_0^\infty |h(t)|^r dt \right\}^{1/r}.$$

Proof. Let $h \in \mathcal{H}$ and let h^* be the decreasing rearrangement of $|h|$. Then

$$h^*(t)\|1_{[0,t]}\|_M \le \|h\|_M$$

so that

$$\begin{aligned} h^*(t) &\le \|h\|_M\, t^{-1/q} \qquad 0 < t \le 1 \\ &\le \|h\|_M\, t^{-1/p} \qquad 1 \le t < \infty. \end{aligned}$$

Suppose now $h_1 \in \mathcal{H}_1$ and $r < q$. Then

$$\begin{aligned} \|h\|_r^r &= \int_0^1 |h^*(t)|^r dt \\ &\le \left(\int_0^1 t^{-r/q} dt \right) |h|_M^r \\ &= \frac{q}{q-r} \|h\|_M^r. \end{aligned}$$

For $n \in \mathbf{N}$ choose $h^{[n]} \in \mathcal{H}_1$ with $h^{[n]} \sim h \otimes \ldots \otimes h$ (n times). Then $\|h^{[n]}\|_M \le \|h\|_M^n$ and hence

$$\|h\|_r^{nr} = \|h^{[n]}\|_r^r \le \frac{q}{q-r} \|h\|_M^{nr}.$$

Letting $n \to \infty$ we obtain

$$\|h\|_r \le \|h\|_M.$$

Letting $r \to q$ we obtain also $\|h\|_q \le \|h\|_M$. We thus have:

(1) If $h \in \mathcal{H}_1$ and $r \le q$, $\|h\|_r \le \|h\|_M$.

Next for any $h \in \mathcal{H}$ and $r > p$

$$\begin{aligned} \|h\|_r^r &= \int_0^\infty |h^*(t)|^r dt \\ &\le \|h\|_\infty^r + \int_1^\infty t^{-r/p} \|h\|_M^r\, dt \\ &= \|h\|_\infty^r + \frac{p}{r-p} \|h\|_M^r. \end{aligned}$$

Arguing as in (1) we find $h^{[n]} \in \mathcal{H}$ with $h^{[n]} \sim h \otimes \ldots \otimes h$ and then

$$\|h^{[n]}\|_r^r \leq \|h\|_\infty^{nr} + \frac{p}{r-p}\|h\|_M^{nr}$$

so that letting $n \to \infty$,

$$\|h\|_r \leq \max(\|h\|_\infty, \|h\|_M).$$

We conclude:

(2) If $h \in \mathcal{H}$ and $r \geq p$ then

$$\|h\|_r \leq \max(\|h\|_\infty, \|h\|_M).$$

Finally for general h, let $u = h^*1_{[0,1]}$ and $\mathrm{v} = h^*1_{(1,\infty)}$. Suppose $p \leq r \leq q$. Then

$$\|u\|_r \leq \|h\|_M.$$

$$\|\mathrm{v}\|_r \leq \max(h^*(1), \|h\|_M) \leq \|h\|_M.$$

Hence

$$\|h\|_r \leq C_r\|h\|_M$$

where $C_r = 2$ if $r \geq 1$ and $C_r = 2^{1/r}$ if $0 < r \leq 1$.

Now an argument similar to that employed in (1) and (2), using $h^{[n]}$, shows that

$$\|h\|_r \leq \|h\|_M$$

as required.

5. Main theorem: first version.

Theorem 5.1. *Let X be a symmetric function space and let $T \in \mathcal{L}(X)$ be an elementary operator. Suppose $p_X \leq r \leq \min(1,q_X)$. Then T induces an endomorphism of L_r and $\|T\|_r \leq \|T\|$.*

Proof. We suppose

$$Tf(t) = a(t)f(\sigma t)$$

where $a \in L_0$ and $\sigma:[0,1] \to [0,1]$ is a Borel map. The kernel of T is given by $\mu_t = a(t)\delta(\sigma t)$. By direct calculation

$$\alpha_x(B) = \lambda\{t: |a(t)| > x, t \in \sigma^{-1}B\}$$

for $B \in \mathcal{B}$. Let F be the local distribution of μ_t. If the theorem is false there is a set B of positive Lebesgue measure such that

$$\int_0^1 F(t,s)^r ds > \|T\|^r$$

for $t \in B$.

Let $\mathcal{G}$ denote the collection of $g \in \mathcal{H}$ of the form

$$g = \sum_{j=1}^{m} c_j 1_{[b_{j-1}, b_j)}$$

where $0 = b_0 < b_1 < \ldots < b_m$ and $c_1 > c_2 \ldots > c_m > 0$ are all rational. Since $\mathcal{G}$ is countable there exist $g \in \mathcal{G}$ and $A \in \mathcal{B}$ with $\lambda(A) > 0$ so that

$$F(t,s) \geq g(s) \qquad 0 < s < \infty, t \in A$$

and

$$\int_0^\infty g(s)^r ds > \|T\|^r.$$

Suppose $f \in X$ with $\operatorname{supp} f \subset A$ is simple, i.e.

$$f = \sum_{i=1}^{n} u_i 1_{E_i}$$

where $u_i \in \mathbf{R}$ and $E_1, \ldots, E_n$ are disjoint Borel subsets of A. We compute

$$\begin{aligned} \lambda(|Tf| > x) &= \lambda(t : |a(t)||f(\sigma t)| > x) \\ &= \sum_{i=1}^{n} \lambda\{t : t \in \sigma^{-1}E_i; |a(t)| > x|u_i|^{-1}\} \\ &= \sum_{i=1}^{n} \alpha_{x|u_i|^{-1}}(E_i). \\ &= \sum_{i=1}^{n} \int_{E_i} w(t, x|u_i|^{-1})dt. \end{aligned}$$

Now for $t \in A$, if $x' < g(s)$,

$$w(t,x') \geq s.$$

For each i, let $c_{\theta(i)}$ be the least of $c_1, \ldots, c_m, \infty$ so that $x|u_i|^{-1} < c_{\theta(i)}$. Then $w(t,x|u_i|^{-1}) \geq b_{\theta(i)}$, where $b_{\theta(i)} = 0$ if $c_{\theta(i)} = \infty$, and

$$\lambda(|Tf| > x) \geq \sum_{i=1}^{n} b_{\theta(i)}\lambda(E_i).$$

Now suppose $\lambda(\operatorname{supp} f) \leq \varepsilon < (b_m + 1)^{-1}$, and consider the multiplier $M(g,\varepsilon) : X[0,\varepsilon] \to X([0,\varepsilon] \times [0,N])$ where $N = [b_m + 1]$. Suppose $\hat{f} \in X[0,\varepsilon]$ and $d_{\hat{f}} = d_f$. Then

$$\lambda\{|M(g,\varepsilon)\hat{f}| > x\} = \sum_{i=1}^{n} b_{\theta(i)}\lambda(E_i).$$

We conclude that

$$\|M(g,\varepsilon)\hat{f}\| \leq \|Tf\|$$

and hence that

$$\|M(g,\varepsilon)\| \leq \|T\|.$$

Thus

$$\|g\|_M \leq \|T\|$$

and so

$$\|g\|_r \leq \|T\|$$

contrary to assumption.

Theorem 5.2. *Let X be a minimal symmetric function space with trivial dual and let $T: X \to X_{\max}$ be a positive operator. Suppose $p_X \leq r \leq \min(1, q_X)$. Then $T \in \mathcal{L}(L_r)$ and $\|T\|_r \leq \|T\|$.*

Proof. As in [4] (Section 2) or [10] Theorem 6.2 there is a sequence $\{S_n\}_{n=1}^{\infty}$ of positive operators $S_n: X \to X$ so that $0 \leq S_n \leq S_{n+1} \leq T$, $S_n f \uparrow Tf$ a.e. for $f \geq 0$ and each S_n is locally elementary. (We remark that the construction of each S_n ensures that $S_n(L_\infty) \subset L_\infty$ so that $S_n(X) \subset X$.)

Now if $\lambda(B) > 0$ and $S_n P_B$ is elementary, $S_n P_B \in \mathcal{L}(L_r)$ and $\|S_n P_B\|_r \leq \|S_n P_B\| \leq \|T\|$. Hence $S_n \in \mathcal{L}(L_r)$ and $\|S_n\|_r \leq \|T\|$. The result now follows.

Theorem 5.3. *Let X be a symmetric L-convex function space not contained in L_1. Suppose $p_X \leq r \leq \min(1, q_X)$. Then there is a constant C depending only on r and X so that if $T \in \mathcal{L}(X)$ is controllable then $T \in \mathcal{L}(L_r)$ and $\|T\|_r \leq C\|T\|$.*

Proof. By Theorem 4.4 of [7] applied to the closure X_0 of the simple functions in X, T has an atomic kernel and so we can write T in series form

$$Tf(t) = \sum_{n=1}^{\infty} a_n(t) f(\sigma_n t) \qquad f \in L_\infty$$

where $\sigma_n t \neq \sigma_m t$ for $m \neq n$. Now by Proposition 6.1 of [7] we can find a positive operator $S \in \mathcal{L}(X_{1/2}, X_{\max,1/2})$ defined by

$$Sf(t) = \sum_{n=1}^{\infty} |a_n(t)|^2 f(\sigma_n t) \qquad f \in X_{1/2}.$$

Furthermore $\|S\| \leq A\|T\|^2$ where A depends only on r and X. Here $X_{1/2} = \{f \in L_0 : |f|_{1/2} \in X\}$ quasi-normed by $\|f\|_{1/2} = \|\,|f|^{1/2}\|^2$.

Now by Theorem 5.2, applied to S restricted to the closure of L_∞ in $X_{1/2}$, $S \in \mathcal{L}(L_{r/2})$ and

$$\|S\|_{r/2} \leq A\|T\|^2.$$

However

$$\|S\|_{r/2}^{r/2} = \sup_{\lambda(B)>0} \lambda(B)^{-1} \sum_{n=1}^{\infty} \int_{\sigma_n^{-1}B} |a_n(t)|^r dt = \|T\|_r^r.$$

Hence $T \in \mathcal{L}(L_r)$ and $\|T\|_r \le A^{1/2}\|T\|$.

At this point, the proof of the main theorem is complete for the case when $q_X < 1$, or more generally if $X \supset L(1,\infty)$. In this case every $T \in \mathcal{L}(X)$ is controllable according to either Theorem 6.5 or Theorem 5.1 of [7]. However in the general case we shall need to use Theorem 5.2 above to establish that $T \in \mathcal{L}(X)$ is controllable and then apply Theorem 5.3.

6. A localization principle. For $1 \le k \le 2^n$ let $D(n,k)$ be the dyadic interval $[(k-1).2^{-n},k.2^{-n})$ for $k < 2^n$ or $[1-2^{-n},1]$ if $k = 2^n$. Let $\mathcal{D}_n$ be the algebra generated by the sets $\{D(n,k) : 1 \le k \le 2^n\}$ and let $\mathcal{D} = \bigcup \mathcal{D}_n$. Let $L_0(\mathcal{D}_n)$ be the space of $\mathcal{D}_n$-measurable functions contained in L_0 and let $L_0(\mathcal{D}) = \bigcup L_0(\mathcal{D}_n)$.

If $f_n, f \in L_0$ we say that $f_n \to f$ in law if for every bounded continuous function $\phi : \mathbf{R} \to \mathbf{R}$ we have

$$\int_0^1 \phi \circ f_n dt \to \int_0^1 \phi \circ f dt.$$

Lemma 6.1. *Let $T_n : L_0(\mathcal{D}_n) \to L_0$ be a sequence of linear maps such that for each $f \in L_0(\mathcal{D}_n)$ the sequence $\{T_n f : n \ge k\}$ is bounded in L_0. Then there is an increasing sequence $m(k)$ and a linear map $S : L_0(\mathcal{D}) \to L_0$ so that if $f \in L_0(\mathcal{D})$ then $T_{m(k)} f \to Sf$ in law.*

Proof. Let $\Gamma \subset \mathbf{N} \times \mathbf{N}$ be the set of (j,k) so that $1 \le j \le 2^k$. Let $\mathbf{R}^* = \mathbf{R} \cup \{\infty\}$ be the one-point compactification of $\mathbf{R}$. Let K be the product space $(\mathbf{R}^*)^\Gamma$; K is a compact metric space. For $(j,k) \in \Gamma$ let $\theta_{jk} : K \to \mathbf{R}^*$ be the coordinate map. For each $n \in \mathbf{N}$ define $\tau_n : [0,1] \to K$ by

$$\begin{aligned} \theta_{jk}\tau_n(t) &= T_n 1_{D(k,j)} \qquad 1 \le k \le n \\ &= 0 \qquad k > n. \end{aligned}$$

Now define probability measure $\rho_n \in \mathcal{M}(K)$ by $\rho_n(B) = \lambda(\tau_n^{-1}B)$ for $B \in \mathcal{B}(K)$. Then we find an increasing sequence $\{m(k) : k = 1,2 \ldots\}$ and a probability measure ρ on K so that $\rho_{m(k)} \to \rho$ weak*.

For fixed $(j,k) \in \Gamma$ the sequence $T_n 1_{D(k,j)}$ is L_0-bounded. Hence given $\varepsilon > 0$ there is a compact subset M of $\mathbf{R}$ so that

$$\lambda(T_n 1_{D(k,j)}(t) \in M) \ge 1 - \varepsilon$$

i.e.

$$\rho_n(\theta_{jk}^{-1}(M)) \ge 1 - \varepsilon.$$

Now $\theta_{jk}^{-1}(M)$ is compact and hence

$$\rho(\theta_{jk}^{-1}(M)) \ge 1 - \varepsilon$$

and thus

$$\rho(\theta_{jk}^{-1}(\mathbf{R})) = 1,$$

i.e. θ_{jk} maps K into $\mathbf{R}$ almost everywhere for ρ.
Also

$$1_{D(k,j)} = 1_{D(k+1,2j-1)} + 1_{D(k+1,2j)}.$$

Let U_ε be the open set of all $v \in K$ so that $\theta_{jk}(v)$, $\theta_{2j-1,k+1}(v)$, $\theta_{2j,k+1}(v) \in \mathbf{R}$ and

$$|\theta_{jk}(v) - \theta_{2j-1,k+1}(v) - \theta_{2j,k+1}(v)| > \varepsilon.$$

If $n \geq k+1$, $\rho_n(U_\varepsilon) = 0$ and hence $\rho(U_\varepsilon) = 0$.
Thus

$$\theta_{jk} = \theta_{2j-1,k+1} + \theta_{2j,k+1} \qquad \rho - \text{a.e.}$$

Now we can induce an operator $S_0 : L_0(\mathcal{D}) \to L_0(K,\rho)$ by setting $S_0 1_{D(k,j)} = \theta_{jk}$. Suppose $f \in L_0(\mathcal{D}_k)$ and $n \geq k$. Let

$$f = \sum_{j=1}^{2^n} a_j 1_{D(n,j)}.$$

Let $g = \Sigma_{j=1}^{2^n} a_j \theta_{jn}$. If $B \subset \mathbf{R}$ is a Borel set

$$\rho_n(g^{-1}(B)) = \lambda((T_n f)^{-1}(B)).$$

Let $\phi : \mathbf{R} \to \mathbf{R}$ be a continuous bounded map with $|\phi(x)| \leq 1$ for all x. For $\varepsilon > 0$ choose a compact subset M of $\mathbf{R}$ so that

$$\lambda((T_n f)^{-1}(M)) \geq 1 - \varepsilon \qquad n \in \mathbf{N}.$$

Let ϕ_0 be a continuous function $\phi_0 : \mathbf{R}^* \to \mathbf{R}$ so that $\phi_0 = \phi$ on M and $|\phi_0(x)| \leq 1$ for all $x \in \mathbf{R}^*$. Then

$$\int_0^1 \phi_0 \circ g \, d\rho_{m(k)} \to \int \phi_0 \circ g \, d\rho$$

i.e.

$$\int_0^1 \phi_0 \circ T_{m(k)} f \, dt \to \int \phi_0 \circ g \, d\rho.$$

Since $\rho(g^{-1}(M)) \geq 1 - \varepsilon$ and $\lambda((T_{m(k)} f)^{-1}(M)) \geq 1 - \varepsilon$ we conclude

$$\limsup_{k \to \infty} \left| \int \phi \circ T_{m(k)} f \, dt - \int \phi \circ g \, d\rho \right| \leq 4\varepsilon.$$

As $\varepsilon > 0$ is arbitrary

$$\int \phi \circ T_{m(k)} f \, dt \to \int \phi \circ g \, d\rho.$$

To conclude the proof define $S_1 : L_0(\mathcal{D}) \to L_0([0,1] \times K, \lambda \times \rho)$ by $S_1 f(s,v) =$

$S_0 f(v)$. Then since $\lambda \times \rho$ is a diffuse measure on $[0,1] \times K$ there is a Borel map $\tau:[0,1] \to [0,1] \times K$ so that if $B \subset [0,1] \times K$ is Borel then $(\lambda \times \rho)(B) = \lambda(\tau^{-1}B)$. Define $S:L_0(\mathfrak{D}) \to L_0$ by

$$Sf(t) = S_1 f(\tau t) \qquad 0 \leq t \leq 1.$$

Now S is the desired map, since if $\phi:\mathbf{R} \to \mathbf{R}$ is continuous and bounded, and $f \in L_0(\mathfrak{D})$,

$$\begin{aligned}\int_0^1 \phi \circ Sf dt &= \int_{[0,1]\times K} \phi \circ S_1 f d(\lambda \times \rho) \\ &= \int_K \phi \circ S_0 f d\rho \\ &= \lim_{k\to\infty} \int \phi \circ T_{m(k)} f dt.\end{aligned}$$

Lemma 6.2. *Let X be an L-convex symmetric function space not contained in L_1 such that $q_X \geq 1$. Then there is a constant C with the following property. If $T \in \mathcal{L}(X)$ and we let*

$$g_n = \sum_{j=1}^{2^n} |T 1_{D(n,j)}|^2 \qquad n \in \mathbf{N}$$

then, for any $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that if $m \geq N$ there exists a Borel set $E_m \subset [0,1]$ with $\lambda(E_m) \geq 1 - \varepsilon$ and

$$\int_{E_m} |g_m|^{1/2} dt \leq C\|T\|.$$

Proof. Define $S_n : L_0(\mathfrak{D}_n) \to L_0$ by setting

$$S_n 1_{D(n,j)} = |T 1_{D(n,j)}|^2.$$

Then $S_n 1 = g_n$.

By Theorem 3.3 of [6] there is a constant A such that if $a_1, \ldots, a_{2^n} \in \mathbf{R}$,

$$\left\| \left(\sum_{j=1}^{2n} a_j^2 |T 1_{D(n,j)}|^2 \right)^{1/2} \right\| \leq A\|T\| \left\| \sum a_j 1_{D(n,j)} \right\|.$$

This implies considering S_n as an operator on $X_{(1/2)}(\mathfrak{D}_n)$ to $X_{(1/2)}$ that

$$\|S_n f\|_{X_{(1/2)}} \leq A^2 \|T\|^2 \|f\|_{X_{(1/2)}}.$$

We shall set $C = 2A$, and assume the theorem fails. In this case there exists an increasing sequence $m(k)$ so that

$$\int_E |g_{m(k)}|^{1/2} dt > C\|T\|$$

whenever $\lambda(E) \geq 1 - \varepsilon$.

Define $V_n : L_0(\mathcal{D}_n) \to L_0$ by setting $V_n = S_{m(k)}$ for $m(k-1)+1 \le n \le m(k)$ (here $m(0) = 0$ for convenience). By Lemma 6.1 we can find $V : L_0(\mathcal{D}) \to L_0$ so that for each $f \in L_0(\mathcal{D})$ Vf is the limit in law of a subsequence of $\{V_n f : n \in \mathbf{N}\}$. It follows that V maps $L_0(\mathcal{D})$ into $X_{\max,1/2}$ and

$$\|Vf\|_{X_{\max,1/2}} \le A^2\|T\|^2\|f\|_{X_{1/2}}$$

for $f \in L_0(\mathcal{D})$. Thus V extends to a positive operator $V : X_{(1/2)} \to X_{\max(1/2)}$ with $\|V\| \le A^2\|T\|^2$. The upper and lower Boyd indices of $X_{(1/2)}$ are $(1/2)q_X$ and $(1/2)p_X$ and so as $p_X \le 1$ and $q_X \ge 1$, $V \in \mathcal{L}(L_{1/2})$ with $\|V\|_{1/2} \le A^2\|T\|^2$. Let $V1 = h$. Then

$$\int_0^1 |h|^{1/2}dt \le A\|T\|.$$

Choose $M > C\|T\|\varepsilon^{-1}$ and $\phi(x) = |x|^{1/2} \wedge M$.
Then

$$\int_0^1 \phi(|h|)dt \le A\|T\|$$

so that for large enough k,

$$\int_0^1 \phi(|g_{m(k)}|)dt \le C\|T\|.$$

Let $F = \{|g_{m(k)}| > M^2\}$. Then $\lambda(F) \le M^{-1}C\|T\| < \varepsilon$. Let $E = [0,1] \setminus F$. Then $\lambda(E) > 1 - \varepsilon$ and

$$\int_E |g_{m(k)}|^{1/2}dt \le C\|T\|$$

contrary to assumption.

7. Proof of the main theorem.

Lemma 7.1. *Let X be a symmetric L-convex function space not contained in L_1. Suppose $q_X \ge 1$. Then there is a constant A such that if $T \in \mathcal{L}(X)$ then $T1 \in L_1$ and $\|T1\|_1 \le A\|T\|$.*

Proof. For $f \in X$ define

$$|||f||| = \inf\{\|T\| : T \in \mathcal{L}(X), T1 = f\}.$$

Here $|||f||| = \infty$ if there is no operator $T \in \mathcal{L}(X)$ with $T1 = f$. We note that $|||f||| \ge |||g|||$ if $|f| \ge |g|$. If f and g are simple functions and $d_f = d_g$ then $|||f||| = |||g|||$. From this remark it follows easily that the lemma will be established if we can show that there exists $c > 0$ so that

$$(*) \qquad |||f||| \ge c\|f\|_1 \qquad f \in L_0(\mathcal{D}).$$

Let us suppose this is false. Then we can find $f \in L_0(\mathcal{D})$ with $f \geq 0$, $\|f\|_1 = 1$ and $|||f||| < 1/2$.

Now suppose $h \in X$ with $|||h||| < \infty$ and $\varepsilon > 0$. We shall show that there exists $h' \in X$ with $|||h'||| \leq (1/2)|||h|||$ and a Borel set E with $\lambda(E) \geq 1 - \varepsilon$ such that

$$(**) \qquad \int_E |h - h'|dt \leq 2\sqrt{2}C\|f\|_2|||h|||$$

where C is the constant in the statement of Lemma 6.2.

We may suppose $|||h||| = 1$. We may find $T \in \mathcal{L}(X)$ with $T1 = h$, $\|T\| \cdot |||f||| < 1/2$, and $\|T\| < 2$.

Appealing to Lemma 6.2 we find $m \in \mathbf{N}$ so that $f \in L_0(\mathcal{D}_m)$ and if $T1_{D(m,k)} = g_k$ then there is a Borel set E with $\lambda(E) \geq 1 - \varepsilon$ and

$$\int_E \left(\sum_{k=1}^N |g_k(t)|^2\right)^{1/2} dt \leq C\|T\|$$

where $N = 2^m$.

Suppose

$$f = \sum_{k=1}^N a_k 1_{D(m,k)}.$$

Let Π be the group of permutations of $\{1,2,\ldots,N\}$ and for $\pi \in \Pi$ let

$$f_\pi = \sum_{k=1}^N a_{\pi(k)} 1_{D(m,k)}.$$

Now for fixed $t \in [0,1]$

$$\mathcal{E}(|T(f_\pi - 1)(t)|^2) \leq 2\|f - 1\|_2^2 \sum_{k=1}^N |g_k(t)|^2$$

by Lemma 4.2 of [7]. Here $\mathcal{E}$ denotes expectation with respect to natural probability measure on Π_N. Hence

$$\mathcal{E}(|Tf_\pi(t) - h(t)|) \leq \sqrt{2}\,\|f\|_2 \left(\sum_{k=1}^N |g_k(t)|^2\right)^{1/2}.$$

Now

$$\mathcal{E}\left(\int_E |Tf_\pi - h|dt\right) \leq \sqrt{2}\,\|f\|_2 \int_E \left(\sum_{k=1}^N |g_k|^2\right)^{1/2} dt.$$

Thus there exists $\pi \in \Pi$ so that

$$\int_E |Tf_\pi - h|dt \leq \sqrt{2}\,\|f\|_2 C\|T\|.$$

Let $h' = Tf_\pi$. Then since $|||f_\pi||| = |||f|||$ we have $|||h'||| \leq |||T||| \cdot |||f||| < 1/2$, and

$$\int_E |h - h'|dt \leq 2\sqrt{2}C\|f\|_2.$$

This completes the proof of the statement preceding equation (**).

Now apply this in turn with ε replaced by $\varepsilon/2$, $\varepsilon/4$, etc. We can obtain a sequence $h_n \in X$ with $|||h_n||| < 2^{-n}|||h|||$ and a descending sequence of Borel sets E_n with

$$\lambda(E_n) \geq 1 - \varepsilon\left(\frac{1}{2} + \frac{1}{4} \dots + \frac{1}{2^n}\right) \geq 1 - \varepsilon$$

so that

$$\int_{E_n} |h - h_n|dt \leq 2\sqrt{2}C\|f\|_2|||h|||\left(1 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}}\right)$$

$$\leq 4\sqrt{2}C\|f\|_2|||h|||.$$

Let $E = \bigcap E_n$. Observe that $|||h_n||| \to 0$ clearly implies that $h_n \to 0$ in measure. Thus $\lambda(E) \geq 1 - \varepsilon$ and

$$\int_E |h|dt \leq 4\sqrt{2}C\|f\|_2\,|||h|||.$$

However as $\varepsilon > 0$ is arbitrary,

$$\int_0^1 |h|dt \leq 4\sqrt{2}C\|f\|_2|||h|||$$

for h with $|||h||| < \infty$.

This contradicts the failure of equation (*). Hence (*) holds and the lemma is proved.

Lemma 7.1 clearly implies that if $T \in \mathscr{L}(X)$ and $f \in L_\infty$ then $Tf \in L_1$. In fact

$$\|Tf\|_1 \leq A\|T\|\,\|f\|_\infty.$$

Lemma 7.2. *Under the same hypotheses as Lemma 7.1 we have* $T \in \mathscr{L}(L_1)$ *and* $\|T\|_1 \leq \|T\|$.

Proof. We define a lattice norm on L_∞ by

$$|||f||| = \sup(\|Tf\|_1 : \|T\| \leq 1).$$

Note that $|||f||| \leq A\|f\|_\infty$. We shall show that

$$|||f||| \leq \|f\|_1 \qquad f \in L_\infty$$

and the lemma will immediately follow.

Suppose $0 < \eta < 1/2$. Pick $h_n \in L_0(\mathscr{D})$ so that $h_n \geq 0$, $\|h_n\| \to 0$ and $\|h_n\|_1 = 1$. This is possible since X is not contained in L_1. Pick $T_n \in \mathscr{L}(X)$ with $\|T_n\| \leq 1$ so that

$$\|T_nh_n\|_1 \geq \left(1 - \frac{1}{3}\eta\right)|||h_n|||.$$

Clearly $|||h_n||| \geq |||h_n|||_1 = 1$ so that $\|T_nh_n\|_1 \geq 1 - (1/3)\eta$. Also $\|T_nh_n\| \leq \|h_n\| \to 0$, so that $T_nh_n \to 0$ in measure. Thus we can find simple functions g_n with $0 \leq g_n \leq |T_nh_n|$, supp $g_n = F_n$ with $\lambda(F_n) \to 0$ and

$$\int_0^1 g_n(t)dt \geq \left(1 - \frac{\eta}{3}\right) \|T_nh_n\|_1 .$$

We may further suppose that each g_n is of the form

$$g_n = \sum_{k=1}^{\ell} c_k 1_{B_k}$$

where $B_1, \ldots, B_\ell$ are disjoint Borel subsets of F_n with equal measure $\ell^{-1}\lambda(F_n)$.

Now $|||g_n||| \leq |||T_nh_n||| \leq |||h_n|||$. However

$$\begin{aligned} \|g_n\|_1 &\geq (1 - \eta/3)\|T_nh_n\|_1 \\ &\geq (1 - \eta/3)^2|||h_n||| \\ &\geq (1 - \eta/3)^2|||g_n||| \\ &\geq (1 + \eta)^{-1}|||g_n|||. \end{aligned}$$

Thus $|||g_n||| \leq (1 + \eta)\|g_n\|_1$. For any permutation π of $\{1,2,\ldots,\ell\}$

$$\left|\left|\left| \sum_{k=1}^{\ell} c_{\pi(k)} 1_{B_k} \right|\right|\right| \leq (1 + \eta)\|g_n\|_1$$

and hence by averaging ($|||\cdot|||$ is a norm),

$$|||1_{F_n}||| \leq (1 + \eta)\lambda(F_n).$$

It follows that for any Borel set B with $\lambda(B)$ a multiple of $\lambda(F_n)$ we also have

$$|||1_B||| \leq (1 + \eta)\lambda(B).$$

As $\lambda(F_n) \to 0$ we quickly conclude

$$|||1_B||| \leq (1 + \eta)\lambda(B)$$

for all $B \in \mathcal{B}$ and hence as $\eta > 0$ is arbitrary.

$$|||1_B||| \leq \lambda(B).$$

Again since $|||\cdot|||$ is a norm we quickly obtain that for simple f

$$|||f||| \leq \|f\|_1$$

and the lemma follows.

Proof of Theorem 1.1. We have now essentially proved Theorem 1.1; we reiterate the argument for completeness. We consider two cases.

(i) If $q_X < 1$, then by Theorem 6.6 of [7] every $T \in \mathcal{L}(X)$ is controllable. The result now follows from Theorem 5.3.

(ii) If $q_X \geq 1$, then by Lemma 7.2 if $T \in \mathcal{L}(X)$ then $T \in \mathcal{L}(L_1)$. Hence T is controllable, as $\mathcal{L}(L_1)$ is a lattice, and thus Theorem 5.3 again gives the conclusion.

8. An application. In this section we present an illustrative application of Theorem 1.1. Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{B}$. We shall say that $\mathcal{A}$ is *complemented* in $\mathcal{B}$ if there exists $\varepsilon > 0$ and $C \in \mathcal{B}$ such that

(a) For every $A \in \mathcal{A}$, $\lambda(A \cap C) \geq \varepsilon\, \lambda(A)$.

(b) For every $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ with $\lambda((A \cap C)\Delta(B \cap C)) = 0$.

It is shown in [3] that for $0 < p < 1$, $L_p(\mathcal{A})$ is complemented in L_p if and only if $\mathcal{A}$ is complemented in $\mathcal{B}$. We also note that if $p = 1$, $L_1(\mathcal{A})$ is complemented by a projection with atomic kernel in L_1 if and only if $\mathcal{A}$ is complemented in $\mathcal{B}$ by essentially the same proof as in [3].

Proposition 8.1. *Let X be a symmetric quasi-Banach function space and suppose $\mathcal{A}$ is a complemented sub-σ-algebra of $\mathcal{B}$. Then $X(\mathcal{A})$ is complemented in X and either $X/X(\mathcal{A})$ is isomorphic to X or $X(\mathcal{A}) = X$.*

Proof. Consider the projection $P_C : X \to X$. From conditions (a) and (b) P_C maps $X(\mathcal{A})$ isomorphically onto $P_C(X)$. Thus $X(\mathcal{A})$ is complemented by the projection $P_C^{-1} \circ P_C$ where $P_C^{-1} : P_C(X) \to X(\mathcal{A})$ is the inverse map. The kernel of the projection is $P_{[0,1]\setminus C}(X)$ so that either $\lambda(C) = 1$ or $X/X(\mathcal{A}) \cong X$.

Theorem 8.2. *Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{B}$. Suppose X is an L-convex minimal symmetric function space with trivial dual. Then $X(\mathcal{A})$ is complemented in X if and only if $\mathcal{A}$ is complemented in $\mathcal{B}$.*

Proof. This is an immediate deduction from Theorem 1.1 and the remarks preceding Proposition 8.1. Indeed if $P : X \to X(\mathcal{A})$ is a projection then P is a projection of L_r onto $L_r(\mathcal{A})$ for $p_X \leq r \leq \min(1, q_X)$ and has an atomic kernel.

Theorem 8.3. *Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{B}$. Suppose X is an L-convex minimal symmetric function space with trivial dual. Suppose the quotient space $X/X(\mathcal{A})$ is isomorphic to X. Then $\mathcal{A}$ is complemented in $\mathcal{B}$ and $X(\mathcal{A})$ is complemented in X.*

Proof. Let K be the Stone space of a dense countable subalgebra of $\mathcal{A}$. Then K is a compact metric space; let μ be the induced probability measure on K. Then there is a natural linear operator $V : L_0(K,\mu) \to L_0$, which may be represented in the form

$$Vf(s) = f(\tau s)$$

where $\tau : [0,1] \to K$ is a Borel map which is measure-preserving in the sense that $\lambda(\tau^{-1}(B)) = \mu(B)$ for $B \subset K$ a Borel set. We note that V maps $X(K,\mu)$ isomorphically onto $X(\mathcal{A})$.

Define $\phi : [0,1]^2 \to [0,1]$ by $\phi(s,t) = s$ and consider the map $\tau \circ [0,1] \to K$. This map is anti-injective in the sense of [6] Section 2 and hence by Proposition

2.2 of [6] there is a compact metric space M, a diffuse probability measure π on M and Borel maps $\theta : [0,1]^2 \to M$ and $\rho : K \times \mathcal{M} \to [0,1]^2$ so that

(i) $\rho(\tau\phi \times \theta)(\omega) = \omega \quad$ a.e. $\omega \in [0,1]^2$

(ii) $\tau\phi \times \theta$ is measure-preserving for $\lambda \times \lambda$ and $\mu \times \pi$.

Thus there is an isomorphism $U : X(K \times M) \to X[0,1]^2$ given by

$$Uf(s,t) = f(\tau s, \theta(s,t))$$

with inverse

$$U^{-1}f(k,m) = f(\rho(k,m)).$$

Now we define a map $Q : X(K \times M) \to X(K \times M \times M)$ by

$$Qf(k,m_1,m_2) = f(k,m_1) - f(k,m_2).$$

Let $W : X[0,1] \to X[0,1]^2$ be defined by

$$Wf(s,t) = f(s).$$

We consider the map $T : X[0,1] \to X[K \times M \times M]$ given by $T = QU^{-1}W$. It is easily calculated that

$$Tf(k,m_1,m_2) = f(\phi\rho(k,m_1)) - f(\phi\rho(k,m_2)).$$

It is clear that $f \in \ker T$ if and only if there exists $g \in X(K)$ with

$$f(\phi\rho(k,m)) = g(k) \qquad \text{a.e.}$$

Then $f = \mathrm{V}g$ i.e. $f = g \circ \tau$ a.e. Hence $\ker T = \mathcal{R}(\mathrm{V})$ (the range of V) i.e. $X(\mathcal{A})$.

Next if $f \in X[0,1]$ define $g \in L_0(K)$ by

$$g(k) = \inf\left\{ t : \pi(U^{-1}Wf(k,m) \geq t) \leq \frac{1}{2} \right\}.$$

Thus for each $k \in K$, $g(k)$ is a median for $U^{-1}Wf(k,m)$.

In fact $\mu(|g| > t) \leq 2(\mu \times \pi)\,(|U^{-1}Wf| > t)$ so that $g \in X(K,\mu)$. Furthermore

$$\begin{aligned}\lambda(|f(s) - g\tau(s)| > t) &= (\mu \times \pi)(|U^{-1}Wf(k,m) - g(k)| > t) \\ &\leq 2(\mu \times \pi \times \pi)(|Tf(k,m_1,m_2)| > t).\end{aligned}$$

It thus follows that if $Tf_n \to 0$ then $d(f_n, X(\mathcal{A})) \to 0$ i.e. T maps X onto a closed subspace of $X(K \times M \times M)$ isomorphic to $X/X(\mathcal{A})$.

Let $S : X \to X(K \times M \times M)$ be an isomorphism onto $\mathcal{R}(T)$, and suppose first that $p = p_X < 1$. We observe that T naturally extends to an operator $T : L_p \to L_p(K \times M \times M)$ with closed range and kernel $L_p(\mathcal{A})$. Since $S^{-1}T \in \mathcal{L}(X,X)$ we conclude that $S^{-1}T \in \mathcal{L}(L_p, L_p)$. Similarly $S \in \mathcal{L}(L_p, L_p(K \times M \times M))$. Combining these statements we see that S leads to an isomorphism of L_p onto $T(L_p) \cong L_p/L_p(\mathcal{A})$. Now by results in [5] (Theorem 7.3) $\mathcal{A}$ is complemented in $\mathcal{B}$ (or $L_p(\mathcal{A})$ is complemented in L_p).

Next consider the case $p_X = 1$. In this case $L_1(\mathcal{A})$ is automatically complemented in L_1 so that there is an L_1-continuous operator $E : T(L_1) \to L_1$ so that TE

is the identity. The map S extends to an L_1-continuous operator $S : L_1 \to T(L_1)$ and so $ES : L_1 \to L_1$ is continuous. Thus [3] we can write

$$ESf(t) = \int f d\nu_t \qquad \text{a.e.}$$

where $t \to \nu_t$ is a weak*-Borel map of [0,1] into $\mathcal{M}[0,1]$. Now

$$TESf(k,m_1,m_2) = \int f d(\nu_{\phi\rho(k,m_1)} - \nu_{\phi\rho(k,m_2)}).$$

If $f \in L_\infty$, $TESf = Sf$ and S has an atomic representation (Theorem 1.2). Hence $\nu_{\phi\rho(k,m_1)} - \nu_{\phi\rho(k,m_2)}$ is almost everywhere atomic. Let ν_t^a be the atomic part of ν_t. Then we can define an operator $D : L_1 \to L_1$ by

$$Df(t) = \int d\nu_t^a$$

and by the above remarks $TD = S$ also. Consider the map $DS^{-1}T : L_1 \to L_1$. This is a projection with $T(DS^{-1}T) = T$ and hence has kernel $L_1(\mathcal{A})$ (= ker T). Thus $L_1(\mathcal{A})$ is the range of a projection $I - DS^{-1}T$. We now observe that $S^{-1}T : X \to X$ has an atomic representation and hence so does $I - DS^{-1}T$. As $L_1(\mathcal{A})$ is the range of an atomic projection, $\mathcal{A}$ is complemented in $\mathcal{B}$.

REFERENCES

1. D. W. BOYD, *Indices of function spaces and their relationship to interpolation,* Canad. J. Math. **21** (1969), 1245–1254.
2. M. CWIKEL & Y. SAGHAR, *$L(p,\infty)$,* Indiana Univ. Math. J. **21** (1972), 781–786.
3. N. J. KALTON, *The endomorphisms of L_p, $0 \le p \le 1$,* Indiana Univ. Math. J. **27** (1978), 353–381.
4. N. J. KALTON, *Linear operators on L_p for $0 < p < 1$,* Trans. Amer. Math. Soc. **259** (1980), 319–355.
5. N. J. KALTON, *Isomorphisms between L_p-function spaces when $p < 1$,* J. Funct. Anal. **42** (1981), 299–337.
6. N. J. KALTON, *Convexity conditions for non-locally convex lattices,* Glasgow Math. J. **25** (1984), 141–152.
7. N. J. KALTON, *Representation of operators between function spaces,* Indiana Univ. Math. J. **33** (1984), 639–665.
8. J. LINDENSTRAUSS & L. TZAFRIRI, *Classical Banach spaces II, function spaces,* Ergeb. Math. Grenzgeb. **97,** Springer-Verlag, Berlin, 1979.
9. A. R. SOUROUR, *Pseudo-integral operators,* Trans. Amer. Math. Soc. **253** (1979), 339–363.
10. L. WEIS, *On the representation of order continuous operators by random measures,* Trans. Amer. Math. Soc. **285** (1984), 535–564.

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