Basically Scattered Vector Measures

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1. Examples. Throughout this paper Σ is a σ-field of subsets of a point set Ω and X is a real Banach space with dual X^*. If $F: \Sigma \to X$ is a (countably additive) vector measure, a set $E \in \Sigma$ is called F-null if F vanishes on every set $A \in \Sigma$ such that $A \subseteq E$. For economy of exposition we shall agree that a subset E of Ω is non-F-null if $E \in \Sigma$ and E is not F-null. A countably additive vector measure $F: \Sigma \to X$ is called *basically scattered* if for each disjoint sequence (E_n) of non-F-null sets the sequence $(F(E_n))$ is a basic sequence in X. Examples of basically scattered vector measures abound. Let (Ω, Σ, μ) be a finite non-atomic measure space and define $F: \Sigma \to L_1(\mu)$ by $F(E) = \psi_E$ for $E \in \Sigma$. Then F is a basically scattered vector measure of bounded variation. We shall show, in Section 2, that any non-atomic non-zero basically scattered vector measure of bounded variation is locally an isomorphic image of the measure F defined above. If we define $F: \Sigma \to L_p(\mu)$ $(1 < p < \infty)$ by $F(E) = \chi_E$ for $E \in \Sigma$ we find that F is a non-atomic basically scattered vector measure. In the fourth section we shall show that if a basically scattered vector measure G has the property that $(G(E_n))$ is equivalent to the ℓ_p $(1 \leq p < \infty)$ unit vector basis for each disjoint sequence (E_n) of non-F-null sets, then G is locally the isomorphic image of the measure F defined above. In the third section we shall see that if X is a Banach space that does not contain ℓ_∞^n's uniformly and such that there is a non-atomic basically scattered vector measure with range in X, then X contains an infinite-dimensional Hilbertian subspace.

More examples of basically scattered vector measures are furnished by the symmetric non-atomic random measures of Urbanik and Woyczynski [16]. This follows easily from the fact that if f and g are independent symmetric random variables, then $E\|f\| \leq E\|f + g\|$. In fact many of the results of this paper can be proved for symmetric non-atomic random measures of finite expectations by the methods of [16].

Additional examples of basically scattered vector measures arise naturally in the theory of cyclic Banach spaces. If X is a cyclic Banach space then according to a theorem of Bade [1] (see [3, XVII. 3.10]) there is a spectral measure P on a σ-field Σ of subsets of a point set Ω such that for each $x \in X$, $F(E)x$ is a countably additive function of $E \in \Sigma$ such that there is an $x_0 \in X$ with the

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property that X is the closed linear span of the range of the countably additive vector measure $P(\cdot)x_0$. By another theorem of Bade (see [3, XVII. 3.3]) the operator norms $\{\|P(E)\| : E \in \Sigma\}$ are uniformly bounded by some constant K.

Thus if (E_n) is a disjoint sequence of non-$P(\cdot)(x_0)$-null sets in Σ, and (α_n) is any sequence of reals, then

$$\left\|\sum_{n=1}^{m} \alpha_n P(E_n)x_0\right\| = \left\|P\left(\bigcup_{n=1}^{m} E_n\right)\left(\sum_{n=1}^{m+p} \alpha_n P(E_n)(x_0)\right)\right\| \leq K\left\|\sum_{n=1}^{m+p} \alpha_n P(E_n)(x_0)\right\|$$

for all positive integers m and p. Hence $P(\cdot)x_0$ is a basically scattered vector measure. Notice also that K is in a sense a basis constant for the basic sequence $(P(E_n)(x_0))$ that is independent of the disjoint sequence (E_n). We shall begin by trying to find a universal basis constant for a basically scattered vector measure.

2. Bounded basis constant. Let $F: \Sigma \to X$ be a basically scattered vector measure. For each non-F-null set E let $\lambda(E)$ be the smallest number in $[0, +\infty]$ such that

$$\left\|\sum_{n=1}^{m} \alpha_n F(E_n)\right\| \leq \lambda(E)\left\|\sum_{n=1}^{m+p} \alpha_n F(E_n)\right\|$$

for all disjoint sequences (E_n) of non-F-null subsets of E, for all sequences (α_n) of real numbers and for all positive integers m and p. There is no claim that $\lambda(E)$ is necessarily finite. We shall say that F has *bounded basis constant on E* if $\lambda(E)$ is finite and we shall say that F has bounded basis constant if $\lambda(\Omega)$ is finite. Two simple remarks are in order. The first is that if $\lambda(E)$ is finite, then, since any permutation of a subsequence of a disjoint sequence (E_n) of non-F-null subsets of E is also a disjoint sequence of non-F-null subsets of E, the constant $\lambda(E)$ is a uniform unconditional constant. This leads directly to the second remark. Suppose $\lambda(E)$ is finite and suppose (α_n) and (β_n) are sequences of real numbers. If (E_n) is a sequence of non-F-null subsets of E, then

$$\left\|\sum_{n=1}^{m} \alpha_n\beta_n F(E_n)\right\| \leq \left(\sup_n |\alpha_n|\right)2\lambda(E)\left\|\sum_{n=1}^{m+p} \beta_n F(E_n)\right\|$$

for all positive integers m amd p.

Theorem 1. *Let $F: \Sigma \to X$ be a non-atomic basically scattered vector measure. If A is a non-F-null set, then there is a non-F-null subset B of A such that $\lambda(B)$ is finite.*

In fact there exist finitely many disjoint non-F-null sets $A_1, \cdots, A_m$ such that $\bigcup_{n=1}^{m} A_n = \Omega$ and $\lambda(A_n)$ is finite for each $n = 1, 2, \cdots, m$.

Proof. To prove the first statement use the non-atomicity of F to find a disjoint sequence (A_n) of non-F-null subsets of A such that $A = \bigcup_{n=1}^{\infty} A_n$. We shall prove that there is a positive integer m such that $\lambda\left(\bigcup_{n=m}^{\infty} A_n\right)$ is finite. If we succeed, then we shall have proved the first statement. To this end suppose $\lambda\left(\bigcup_{n=m}^{\infty} A_n\right)$ is infinite for all m. Let $m_1 = 1$ and find finitely many disjoint non-F-null subsets $E_1^1, \cdots, E_{k_1}^1$ of A and real numbers $\alpha_1^1, \cdots, \alpha_{k_1}^1$ such that

$$\left\|\sum_{n=1}^{p_1} \alpha_n^1 F(E_n^1)\right\| > m_1\left\|\sum_{n=1}^{k_1} \alpha_n^1 F(E_n^1)\right\|$$

for some choice of p_1 with $1 \leq p_1 < k_1$. Now use the countable additivity of F to find a positive integer $\bar{m}_2$ such that $E_n^1 \cap \left(\bigcup_{i=1}^{\bar{m}_2} A_i\right)$ is non-F-null for all n and such that

$$\left\|\sum_{n=1}^{p_1} \alpha_n^1 F\left(E_n^1 \cap \left(\bigcup_{i=1}^{\bar{m}_2} A_i\right)\right)\right\| > m_1\left\|\sum_{n=1}^{K_1} \alpha_n^1 F\left(E_n^1 \cap \left(\bigcup_{i=1}^{\bar{m}_2} A_i\right)\right)\right\|.$$

Put

$$B_n^1 = E_n^1 \cap \left(\bigcup_{i=1}^{\bar{m}_2} A_i\right)$$

and set $m_2 = \bar{m}_2 + 1$. Then $\lambda\left(\bigcup_{n=m_2}^{\infty} A_n\right)$ is infinite. Accordingly there are finitely many disjoint non-F-null subsets $E_1^2, \cdots, E_{K_2}^2$ of $\bigcup_{n=m_2}^{\infty} A_n$ and real numbers $\alpha_1^2, \cdots, \alpha_{K_2}^2$ such that

$$\left\|\sum_{n=1}^{p_2} \alpha_n^2 F(E_n^2)\right\| > m_2\left\|\sum_{n=1}^{K_2} \alpha_n^2 F(E_n^2)\right\|$$

for some choice of p_2 with $1 \leq p_2 \leq K_2$. Choose a positive integer $\bar{m}_3 \geq m_2$ such that if

$$B_n^2 = E_n^2 \cap \left(\bigcup_{i=m_2}^{\bar{m}} A_i\right),$$

then B_n^2 is non-F-null for all n and is such that

$$\left\|\sum_{n=1}^{p_2} \alpha_n^2 F(B_n^2)\right\| > m_2\left\|\sum_{n=1}^{K_2} \alpha_n^2 F(B_n^2)\right\|.$$

Put $m_3 = \bar{m}_3 + 1$. Continue this sliding hump process to find a strictly increasing sequence (m_j) of positive integers and disjoint non-F-null sets $\{B_n^j : n = 1, 2, \cdots, K_j; j = 1, 2, \cdots\}$ such that for each j

$$\left\|\sum_{n=1}^{p_j} \alpha_n^j F(B_n^j)\right\| > m_j\left\|\sum_{n=1}^{K_j} \alpha_n^j F(B_n^j)\right\|$$

for some choice of p_j with $1 \leq p_j < k_j$. It follows immediately that $(F(B_n^j))$ is not a basic sequence, a contradiction which proves the first statement.

To prove the second statement say that a non-F-null set B has property P is there exist finitely many non-F-null subsets $B_1, \cdots, B_m$ of A such that $B = \bigcup_{n=1}^{m} B_n$ and $\lambda(B_n)$ is finite for all n. By the first statement, each non-F-null set has a subset that has property P. An appeal to the Exhaustion lemma [2, III.2.4] reveals that there is a disjoint sequence (A_n) of non-F-null sets such that $\bigcup_{n=1}^{\infty} A_n = \Omega$ and $\lambda(A_n)$ is finite for all n. By the proof of the first statement there is a positive integer m such that $\lambda\left(\bigcup_{n=m}^{\infty} A_n\right)$ is finite. Write

$$\Omega = A_1 \cup \cdots \cup A_{m-1} \cup \left(\bigcup_{n=m}^{\infty} A_n\right)$$

to complete the proof.

By now the reader probably expects to see a proof that basically scattered vector measures have bounded basis constants. If so, then the reader is in for a surprise.

Example 2. Let μ be Lebesgue measure on [0, 1]. Let X be any non-zero reflexive (even one-dimensional) subspace of $L_1[0, 1]$ that contains no non-zero simple functions. Let Q: $L_1[0, 1] \to L_1[0, 1]/X$ be the quotient map. For each Borel measurable set $E \subseteq [0, 1]$ let $F(E) = Q(\chi_E)$. Then

a) The F-null Borel sets are precisely the Borel sets of Lebesgue measure zero.

b) If (E_n) is a disjoint sequence of non-F-null sets, then $(F(E_n)/\mu(E_n))$ is equivalent to the unit vector basis of ℓ_1.

c) The basically scattered vector measure F does not have bounded basis constant.

Proof. To prove (a) use the fact that X contains no simple functions. To prove (b) let (E_n) be a disjoint sequence of non-F-null sets. Let (α_n) be in ℓ_1 and consider

$$\left\| \sum_{n=1}^{\infty} \alpha_n F(E_n)/\mu(E_n) \right\| = \left\| \sum_{n=1}^{\infty} \alpha_n Q(\chi_{E_n}/\mu(E_n)) \right\|.$$

Since $\chi_{E_n}/\mu(E_n)$ is isometric to the unit vector basis of ℓ_1 the norm of

$$\sum_{n=1}^{\infty} \alpha_n Q(\chi_{E_n}/\mu(E_n))$$

in $L_1[0, 1]/X$ is computed in a subspace $Y + X$ of $L_1[0, 1]$ where Y is isometric to ℓ_1 and $Y \cap X = \{0\}$. Note that ℓ_1 and X are totally incomparable spaces in the sense of Rosenthal [9]. By a theorem of Rosenthal's [9], the space $Y + X$ is closed in $L_1[0, 1]$. Since $Y \cap X = \{0\}$, it follows that Y is complemented in $Y + X$ by a bounded projection P. Accordingly if $y \in Y$ and $x \in X$, then

$$\|y\|_{L_1} = \|P(x + y)\| \le \|P\| \, \|x + y\|.$$

It follows immediately that $1/\|P\| \, \|y\|_{L_1} \le \|y\|_{L_{1/X}} \le \|y\|_{L_1}$. Hence the quotient map Q: $L_1[0, 1] \to L_1[0, 1]/X$ is an isomorphism on Y. This proves (b).

To prove (c), suppose F has bounded basis constant. Choose a non-zero function $f \in X$ and a sequence of simple functions (f_n) such that $\lim_n f_n = f$ in $L_1[0, 1]$ norm. Then $\lim_n Q(f_n) = 0$. Since $Q(f_n) = \int_{[0,1]} f_n dF$ (see [2, Chapter III]) we see that

$$\lim_n \left\| \int_{[0,1]} f_n dF \right\| = 0.$$

It will be shown that $\lim_n f_n = 0$ in μ-measure. To this end let $\epsilon > 0$ and put $E_n = \{\omega \in \Omega: |f_n(\omega)| \ge \epsilon\}$. Recall that by the remarks immediately preceeding the statement of Lemma 1

$$\left\| \sum_{n=1}^{m} \alpha_n \beta_n F(E_n) \right\| \le \left(\sup_n |\alpha_n| \right) 2\lambda(\Omega) \left\| \sum_{n=1}^{m+p} \beta_n F(E_n) \right\|$$

for any disjoint sequence (E_n) of non-F-null sets for any two sequences (α_n) and (β_n) of real numbers and for all positive integers m and p. From this it follows if A_n is a non-F-null subset of E_n, then

$$0 \le \lim_n \|\epsilon F(A_n)\| \le 2\lambda(\Omega) \left\| \int_{[0,1]} f_n dF \right\|.$$

Hence

$$\lim_n \sup_{A \subseteq B_n} \|F(A)\| = 0.$$

Since μ and F have the same null sets, it follows that $\lim_n \mu(B_n) = 0$. This proves $\lim_n f_n = 0$ in μ-measure. Hence $f = 0$, a contradiction which completes the proof.

The following fact which is embedded in the proof of Example 2 will be isolated for future use. The integral is that of Dunford and Schwartz [3, IV.10].

Corollary 3. *Let F: $\Sigma \to X$ be a non-atomic basically scattered vector measure with bounded basis constant. Suppose μ is a non-negative finite measure on Σ having the same null sets as F. If (f_n) is a sequence of F-integrable functions such that $\lim_n \int_\Omega f_n dF = 0$, then $\lim_n f_n = 0$ in μ-measure.*

3. Basically scattered vector measures of bounded variation. Let (Ω, Σ, μ) be a non-atomic finite measure space and let $1 \leq p < \infty$. Define a vector measure $F: \Sigma \to L_p(\mu)$ by $F(E) = \chi_E$ for $E \in \Sigma$ (here χ_E is the characteristic or indicator function of E). Evidently F is a basically scattered vector measure. Now if $p = 1$, a simple calculation shows that F is of bounded variation. On the other hand for $1 < p < \infty$ simple computations show that F is not of bounded variation [2, I.1.16]. The purpose of this section is to prove that if G is a non-atomic basically scattered measure of bounded variation, then G is locally an isomorphic image of the vector measure $E \to \chi_E \in L_1(\mu)$ for some measure μ. As consequences we shall obtain a vector measure-theoretic characterization of Banach spaces that contain $L_1[0, 1]$ and we shall obtain exceedingly simple proofs of extensions of related theorems of Masani [8] and Sundaresan and Woyczynski [13].

Throughout this section Σ is a σ-field of subsets of a point set Ω, and X is a Banach space. If $F: \Sigma \to X$ is a vector measure, and $E \in \Sigma$, then $|F|(E)$ stands for the total variation of F on E and $|F|_E$ stands for the measure $|F|$ restricted to the sets in Σ that are subsets of E. Recall tht if F is of bounded variation, then $|F|$ is countably additive [2, I.1.9].

Theorem 4. *Let $F: \Sigma \to X$ be a non-atomic basically scattered vector measure of bounded variation. Then there is a non-F-null set A and an isomorphism $T: L_1(|F|_A) \to X$ such that $F(E) = T(\chi_E)$ for all $E \in \Sigma$ such that $E \subseteq A$.*

Consequently a Banach space X contains an isomorphic copy of $L_1[0, 1]$ if and only if there is a non-atomic basically scattered X-valued vector measure of bounded variation.

Proof. Let $F: \Sigma \to X$ be a non-atomic basically scattered of bounded variation. According to Rybakov's theorem [2, IX.2.2], there is an $x_0^* \in X^*$ with $\|x_0^*\| \leq 1$ such that F and $|x_0^*F|$ have the same null sets. By the Hahn decomposition theorem (for scalar measures) matters may be arranged so that there is a non-F-null set A_1 such that

$$|x_0^*F|(E \cap A_1) = x_0^*F(E \cap A_1)$$

for all $E \in \Sigma$. Accordingly $x_0^*F|_{A_1}$ and $|F|_{A_1}$ are mutually absolutely continuous positive measures. Hence there exists a $\beta > 0$ and a non-F-null subset A_2 of A_1 such that

$$x_0^*F(E \cap A_2) \geq \beta|F|(E \cap A_2)$$

for all E in Σ. Next, appeal to Theorem 1 to find a non-F-null subset A of A_2 such that $\lambda(A)$ is finite.

To complete the proof let $f = \sum_{n=1}^{m} \alpha_n \chi_{E_n}$ be a simple function in $L_1(|F|_A)$ such that the E_n's are disjoint non-F-null subsets of A. Define $T(f)$ by

$$T(f) = \sum_{n=1}^{m} \alpha_n F(E_n).$$

Then

$$\|T(f)\| \leq \sum_{n=1}^{m} |\alpha_n| \, \|F(E_n)\|$$
$$\leq \sum_{n=1}^{m} |\alpha_n| \, |F|(E_n) = \|f\|_{L_1(|F|_A)}.$$

On the other hand,

$$\|T(f)\| = \left\| \sum_{n=1}^{m} \alpha_n F(E_n) \right\|$$
$$\geq \frac{1}{2\lambda(A)} \left\| \sum_{n=1}^{m} |\alpha_n| F(E_n) \right\|$$
$$\geq \frac{1}{2\lambda(A)} x_0^* \sum_{n=1}^{m} |\alpha_n| F(E_n)$$
$$= \frac{1}{2\lambda(A)} \sum_{n=1}^{m} |\alpha_n| \, |F|(E_n) \frac{x_0^* F(E_n)}{|F|(E_n)}$$
$$\geq \frac{\beta}{2\lambda(A)} \sum_{n=1}^{m} |\alpha_n| \, |F|(E_n)$$
$$= \frac{\beta}{2\lambda(A)} \|f\|_{L_1}(|F|_A).$$

Hence

$$\frac{\beta}{2\lambda(A)} \|f\|_{L_1(|F|_A)} \leq \|T(f)\| \leq \|f\|_{L_1(|F|_A)}.$$

It follows that T extends to a topological isomorphism of $L_1(|F|_A)$ into X. Moreover

$$F(E) = T(\chi_E)$$

for all $E \in \Sigma$ such that $E \subseteq A$. This proves the first statement.

To prove the second statement use the non-atomicity of F to see that $L_1(|F|_A)$ contains isometric copies of $L_1([0, 1])$. Conversely if T: $L_1[0, 1] \to X$ is a topological isomorphism, then the vector measure F defined for each Borel Set $E \subseteq (0,1)$ by $F(E) = T(\chi_E)$ is an X-valued non-atomic basically scattered vector measure of bounded variation. This completes the proof.

Some suspicious readers may not be happy with the local nature of Theorem 4. It is natural to hope that in the statement of Theorem 4, the set A can be replaced by the whole space Ω. That this is, in general, false can be quickly seen *via* Example 2. If Theorem 4 holds with $A = \Omega$, then it follows that the quotient map Q: $L_1[0, 1] \to L_1[0, 1]/X$ is an isomorphism for any reflexive subspace of $L_1[0, 1]$ that contains no simple functions. That this is false is an immediate consequence of the proof of Example 2.

Some corollaries follow:

Corollary 5. *Let $F: \Sigma \to X$ be a non-atomic basically scattered vector measure. Let μ be a finite non-negative (scalar) measure on Σ such that F is absolutely continuous with respect to μ. If A is a non-F-null set, then there exists no μ-Bochner integrable function $f: A \to X$ such that*

$$F(E \cap A) = \int_{E \cap A} f d\mu$$

for all $E \in \Sigma$.

Proof. On the contrary suppose there is a non-F-null set A and a μ-Bochner integrable function f such that

$$F(E \cap A) = \int_{E \cap A} f d\mu$$

for all $E \in \Sigma$. Then $|F|(A)$ is finite. Consequently there exists an $|F|$-Bochner integrable function $g: \Omega \to X$ such that

$$F(E \cap A) = \int_{E \cap A} g d|F|.$$

An appeal to Theorem 1 produces a non-F-null subset B of A and a topological isomorphism $T: L_1(|F|_B) \to X$ such that $F(E \cap B) = T(\chi_{E \cap B})$ for all $E \in \Sigma$. Thus

$$\chi_{E \cap B} = \int_{E \cap B} T^{-1}(g) d|F|$$

for all $E \in \Sigma$. Since $|F|$ is non-atomic this is impossible (see [2, III.1.2]). This completes the proof.

The above corollary represents a considerable generalization of related theorems of Masani [8] (who treats orthogonally scattered vector measures with values in Hilbert space) and Sundaresan and Woyczynski [13] (who treat lattice orthogonally scattered vector measures with values in L_p for $1 \leq p \leq \infty$). It should be noted that the basically scattered vector measures treated by both Masani and Sundaresan-Woyczynski have basis constant one.

The next corollary is a translation of Theorem 4 and an important theorem of Enflo and Starbird [4] into the context of basically scattered vector measures.

Corollary 6. *A bounded linear operator $T: L_1[0, 1] \to L_1[0, 1]$ preserves a copy of $L_1[0, 1]$ if and only if there exists a set A of positive Lebesgue measure and a σ-field Σ_1 of Borel measurable subsets of A such that Lebesgue measure is non-atomic on Σ_1 and such that the vector measure F defined on Σ_1 by $f(E) = T(\chi_E)$ is basically scattered.*

Proof. The "only if" part is a trivial consequence of the main theorem of Enflo and Starbird [4]. To prove the converse choose a set A of positive Lebesgue measure and a non-atomic σ-field Σ_1 of Borel measurable subsets of A such that F is basically scattered on Σ_1. By Theorem 4 there is a non-F-null subset B of A and a topological isomorphism S: $L_1(\Sigma_1, |F|_B) \to L_1[0, 1]$

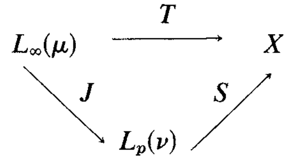
$$S(\chi_{E \cap B}) = F(E \cap B) = T(\psi_{E \cap B})$$

for all $E \in \Sigma_1$. But $L_1(\Sigma_1 \cap B, |F|_B)$, $L_1(\Sigma_1 \cap B)$ and $L_1[0, 1]$ are all isometric and S: $L_1(\Sigma_1 \cap B, \|F\|_B) \to L_1[0, 1]$ is a topological isomorphism. It follows quickly that T: $L_1(\Sigma_1 \cap B) \to L_1[0, 1]$ is a topological isomorphism. This completes the proof.

4. Basically scattered vector measures and the space ℓ_2. In the last section basically scattered vector measures were used to detect the presence of $L_1[0, 1]$ in a Banach space. In this section basically scattered vector measures will be used to detect the presence of ℓ_2 in a Banach space. Again Σ is a σ-field of subsets of Ω and X is a Banach space.

Theorem 7. *Suppose X is a Banach space that does not contain uniformly isomorphic copies of ℓ_∞^n for $n = 1, 2, \cdots$. There is a non-zero non-atomic basically scattered vector measure with values in X if and only if X contains an isomorphic copy of ℓ_2.*

Proof. Let F: $\Sigma \to X$ be a non-atomic basically scattered vector measure. A glance at Theorem 1 shows that without loss of generality we can and do assume that F has bounded basis constant. With the help of a theorem of Bartle, Dunford and Schwartz (see [2, I.2.6]) choose a finite non-negative measure μ on Σ such that F and μ have the same null sets. As is well known [2, I.1.12], integration with respect to F defines a bounded linear operator T: $L_\infty(\mu) \to X$. Since X does not contain uniform copies of ℓ_∞^n for $n = 1, 2, \cdots$, a theorem of Rosenthal and Maurey (see [10, 11]) guarantees that there is a p with $1 \leq p < \infty$ such that T is absolutely p-summing. Combining this with the remarks in Rosenthal [10, p. 352] we find that there exists a non-negative finite measure ν on Σ that is absolutely continuous with respect to μ such that T: $L_\infty(\mu) \to X$ admits a factorization



where J: $L_1(\mu) \to L_p(\nu)$ is the natural embedding and S: $L_p(\nu) \to X$ is a bounded linear operator. Notice that this means that $S(\chi_E) = T(\chi_E) = F(E)$ for all $E \in \Sigma$. It follows that the action of S on $L_p(\nu)$ is given by integration with respect to F.

From this and the fact that F and μ have the same null sets it follows that F and ν have the same null sets. Now let $L_0(\nu)$ be the space of measurable functions and Ω under the topology of convergence in measure. Since $S(f) = \int_\Omega f dF$ for all $f \in L_p(\nu)$, Corollary 3 ensures that there is a continuous linear operator R: $S(L_p(\nu)) \to L_0(\nu)$ such that $R(S(f)) = f$ for all $f \in L_p(\nu)$. To bring home the bacon, recall [6] that since ν is non-atomic, $L_p(\nu)$ contains a Hilbertian subspace H on which the L_p and L_0-topologies agree. Accordingly S evidently preserves a copy of H and now we are home.

The converse is trivial since ℓ_2 is isomorphic to $L_2[0, 1]$.

The following consequence of Theorem 7 has been independently obtained by H. P. Lotz.

Corollary 8. *A non-atomic Banach lattice with an order continuous norm that does not contain uniform copies of ℓ_∞^n, $n = 1, 2, \cdots$ does contain a copy of ℓ_2.*

Proof. Let X be a non-atomic Banach lattice with an order continuous norm. According to Schaeffer [12, p. 315], the space X has a subspace that is a cyclic Banach space Y. By the remarks in Section 1, there is a non-atomic (since X is non-atomic) basically scattered vector measure whose range is in Y.

We remark here that Lotz has obtained further results in this direction, showing that if X also does not contain uniformly complemented ℓ_1^n's, then an infinite-dimensional Hilbertian subspace of X is complemented in X.

5. Cyclic Banach spaces and a theorem of Tzafriri. Let F: $\Sigma \to X$ be a basically scattered vector measure with bounded basis constant. Let $L_1(F)$ stand for all (equivalence classes of) real-valued measurable functions that are F-integrable in the sense of Dunford and Schwartz [3, IV.10.7]. Glances at Corollary 3 and Dunford and Schwartz [3, IV.10.9] reveal that $L_1(F)$ is isometric to a subspace of X under the norm

$$\|f\| = \left\| \int_\Omega f dF \right\|.$$

Moreover the projections $P(E)$ defined for $E \in \Sigma$ by $P(E)f = f\chi_E$ (note that $\|P(E)\| \leq \lambda(\Omega)$) define a spectral measure such that $P(E)f$ is countably additive for all $f \in L_1(F)$. It follows quickly that $L_1(F)$ is a cyclic Banach space. On the other hand if X is a cyclic Banach space, a glance at the end of Section 1 reveals that $X = L_1(P(\cdot)x_0)$. This observation is due (in different terminology) to Walsh [17] who essentially noted that the approach to the study of cyclic Banach spaces *via* basically scattered vector measures is efficient and rewarding. This approach can be extended to give rather easy proofs of some of Tzafriri's [14, 15] theorems that deal with isomorphic characterizations of $L_p(\mu)$ spaces. Here is an example:

Theorem 9 (Tzafriri). *Let F: $\Sigma \to X$ be a non-atomic basically scattered vector measure with bounded basis constant. Suppose there is a p with $1 \le p < \infty$ such that for each disjoint sequence of non-F-null sets (E_n) there is a constant K such that*

$$K^{-1}\|(\alpha_n)\|_{\ell_p} \le \left\| \sum_{n=1}^{\infty} \alpha_n \frac{F(E_n)}{\|F(E_n)\|} \right\| \le K\|(\alpha_n)\|_{\ell_p}$$

for all finitely non-zero sequences (α_n) of scalars. Then there exists finite non-negative non-atomic measure μ on Σ such that $L_1(F)$ is topologically isomorphic to $L_p(\mu)$.

Proof. By imitating the sliding hump procedure of Lemma 1, one finds disjoint non-F-null sets $A_1, A_2, \cdots, A_m$ and positive constants $K_1, K_2, \cdots, K_m$ such that for each $j = 1, 2, \cdots, m$

$$K_j^{-1}\|(\alpha_n)\|_{\ell_p} \le \left\| \sum_{n=1}^{\infty} \alpha_n \frac{F(E_n)}{\|F(E_n)\|} \right\| \le K_j\|(\alpha_n)\|_{\ell_p}$$

for every disjoint sequence (E_n) of non-F-null subsets of A_j and all finitely non-zero sequences (α_n) of real numbers. Now since the projections defined for f in $L_1(F)$ by $f \to f\chi_{A_j}$ are continuous and disjoint and since

$$\left\| \sum_{j=1}^{m} f\chi_{A_j} \right\|_{L_1(F)} = \|f\|_{L_1(F)},$$

it follows that the norm $|||\cdot|||$ defined by

$$|||f||| = \left(\sum_{j=1}^{m} \left\| \int_{A_j} f dF \right\|^p \right)^{1/p}$$

is equivalent to the original $L_1(F)$ norm. Hence if we can show that for each j the subspace

$$\{f\chi_{A_j} : f \in L_1(F)\}$$

is isomorphic to an $L_p(\nu)$ for some finite measure ν on the sets in Σ that are subset of A_j, then we shall be done.

To this end fix j and let $\Sigma \cap A_j = \{E \in \Sigma: E \subseteq A_j\}$. Following Tzafriri [16, p. 22] define set functions μ_1 and μ on $\Sigma \cap A_j$ by

$$\mu_1(E) = \sup \sum_{n=1}^{\infty} \|F(E_n)\|^p$$

and

$$\mu(E) = \inf \sum_{n=1}^{\infty} \mu_1(E_n)$$

where the respective supremum and infimum are taken over all disjoint sequences (E_n) in $\Sigma \cap A_j$ such that $\bigcup_{n=1}^{\infty} E_n = E$. Note that μ_1 is super additive and μ is countably subadditive. Now if $E \in \Sigma \cap A_j$ is not μ-null and (E_n) is a sequence of non-F-null sets whose union is E, then

$$F(E) = \sum_{n=1}^{\infty} \|F(E_n)\|(F(E_n)/\|F(E_n)\|).$$

Hence

$$K_j^{-p} \sum_{n=1}^{\infty} \|F(E_n)\|^p \leq \|F(E)\|^p$$
$$\leq K_j^p \sum_{n=1}^{\infty} \|F(E_n)\|^p.$$

In other words

$$K_j^{-p}\mu_1(E) \leq \|F(E)\|^p \leq K_j^p\mu_1(E).$$

From these two chains of inequalities it follows directly that

$$(*) \qquad K_j^{-2p}\mu(E) \leq \|F(E)\|^p \leq K_j^{2p}\mu(E)$$

for all $E \in \Sigma \cap A_j$. Hence μ is a finite countably subadditive set function on $\Sigma \cap A_j$ which has the same null sets as F. Use this fact, the super additivity of μ_1 and the standard computation in Halmos [5, p. 49] to see that μ is finitely additive. Thus μ is a countably additive finite measure on $\Sigma \cap A_j$.

Finally let $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ where the α_i's are scalars and the E_i's are disjoint non-F-null (= non-μ-null) subsets of A_j. We have

$$\left\|\int_{A_j} f dF\right\| = \left\|\sum_{i=1}^{n} \alpha_i F(E_i)\right\|$$
$$= \left\|\sum_{i=1}^{n} \alpha_i \Big\| F(E_i)\|(F(E_i)/\|F(E_i)\|)\right\|.$$

Hence

$$K_j^{-1}\left(\sum_{i=1}^{n} |\alpha_i|^p\|F(E_i)\|^p\right)^{1/p} \leq \|f\|_{L_1(F)}$$
$$\leq K_j\left(\sum_{i=1}^{n} |\alpha_i|^p\|F(E_i)\|^p\right)^{1/p}.$$

Combining this with (*) above yields

$$K_j^{-3}\left(\sum_{i=1}^{n} |\alpha_i|^p\mu(E_i)\right)^{1/p} \leq \|f\|_{L_1(F)}$$
$$\leq K_j^3\left(\sum_{i=1}^{n} |\alpha_i|^p\mu(E_i)\right)^{1/p}.$$

In other words

$$K_j^{-3}\|f\|_{L_p(\mu)} \leq \|f\|_{L_1(F)} \leq K_j^3\|f\|_{L_p(\mu)}.$$

Since simple functions are dense in both $L_p(\mu)$ and $L_1(F)$, this completes the proof.

Corollary 10. *Let $1 \leq p < \infty$. A Banach space X contains a copy of $L_p[0, 1]$ if and only if there is a σ-field Σ of subsets of a point set and a non-atomic basically scattered vector measure F: $\Sigma \to X$ such that for each disjoint sequence (E_n) of non-F-null sets there is a constant K such that*

$$K^{-1}\left(\sum_{n=1}^{\infty} \alpha_n^p\right)^{1/p} \leq \left\|\sum_{n=1}^{\infty} \alpha_n(F(E_n)/\|F(E_n)\|)\right\| \leq K\left(\sum_{n=1}^{\infty} \alpha_n^p\right)^{1/p}$$

for all finitely non-zero sequences (α_n) of real numbers.

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