The Endomorphisms of L_p $(0 \leq p \leq 1)$

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1. Introduction. In [12] Kwapień gives a representation theorem for the endomorphisms of the F-space $L_0(= L_0(0, 1))$ of all real (or complex) measurable functions on $(0, 1)$ with the topology of convergence in measure; this followed an earlier paper of Berg, Peck and Porta [1] on the complemented subspaces of L_0. We first state Kwapień's theorem for future reference. The version given below is a slight modification of Kwapień's original statement.

Theorem 1.1. (Kwapień) *Let K be a Polish space and let λ be a probability measure defined on the Borel subsets $\mathcal{B}$ of K. Let (X, Σ, μ) be a measure space with $\mu(X) = 1$. Let $T : L_0(K, \mathcal{B}, \lambda) \to L_0(X, \Sigma, \mu)$ be a continuous linear operator; then T takes the form*

$$(1.1.1) \qquad Tf(x) = \sum_{n=1}^{\infty} a_n(x) f(\sigma_n x) \qquad \mu\text{-a.e.} \qquad f \in L_0(K)$$

where

(i) $\{a_n\}$ is a sequence of elements of $L_0(X, \Sigma, \mu)$ such that $\mu\{x : a_n(x) \neq 0$ for infinitely many $n\} = 0$.

(ii) $\sigma_n : X \to K$ is a sequence of $\Sigma - \mathcal{B}$ measurable mappings satisfying the condition that if $\lambda(A) = 0$ then $\mu(\sigma_n^{-1}(A) \cap \{x : a_n(x) \neq 0\}) = 0$.

Conversely if $\{a_n\}$ and $\{\sigma_n\}$ satisfy (i) and (ii), then (1.1.1) *defines an operator from $L_0(K)$ into $L_0(X)$.*

We have modified Kwapién's statement to the extent that we assume each σ_n Σ-measurable and not merely measurable with respect to the completion of Σ. This is permissible since K is Polish.

Let us now state Theorem 1.1 in a different form.

Theorem 1.2. *Let K be a compact metric space and let λ be a probability measure on K. Let (X, Σ, μ) be a measure space with $\mu(X) = 1$. Let $T: L_0(K, \mathcal{B}, \lambda) \to L_0(X, \Sigma, \mu)$ be a continuous linear operator; then T takes the form*

$$(1.2.1) \qquad Tf(x) = \int_K f(t)\,d\nu_x(t) \qquad \mu\text{-a.e.}, \qquad f \in L_0(K)$$

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where $x \mapsto \nu_x$ is a mapping of X into the space $M(K)$ of all regular Borel measures on K which is Σ-measurable for the weak-Borel sets of $M(K)$, and satisfies the conditions:*

(i) *If $\lambda(A) = 0$ then $|\nu_x|(A) = 0$* μ-a.e.

(ii) *ν_x is of finite support* μ-a.e.

Conversely if $x \mapsto \nu_x$ satisfies these conditions then (1.2.1) *defines an operator from $L_0(K)$ into $L_0(X)$.*

If T is a continuous linear operator then ν_x is obtained by letting

$$\nu_x = \sum_{n=1}^{\infty} a_n(x)\delta(\sigma_n x)$$

(where $\delta(t)$ is the Dirac measure at the point $t \in k$). The converse is easy to verify.

The purpose of this paper is to investigate similar representation theorems for the spaces L_p $(0 < p < 1)$. These spaces are locally bounded and our techniques are rather different from those of Kwapień. We prove a theorem of the type of 1.2 and deduce the analogue to 1.1, rather than conversely. At least some of the argument can be applied to the case $p = 1$, where we obtain a representation theorem for the endomorphisms of L_1 which seems to be essentially different from the known representation theorems to be found in Dunford and Schwartz ([3] p. 505) and Grothendieck ([5] p. 62).

In Section 2, we collect together all the measurability results that we require. In Section 3 we prove our main representation theorems. Sections 4, 5 and 6 are devoted to applications of these theorems. Thus in Section 4, we consider the problem: if $0 \le p < 1$, and $L_p(X, \Sigma, \mu)$ is separable, what conditions on a sub-σ-algebra Σ_0 of Σ ensure that $L_p(X, \Sigma_0, \mu)$ is complemented in $L_p(X, \Sigma, \mu)$? If $p = 0$, then Berg, Peck and Porta [1] showed that the subspace of $L_0([0, 1] \times [0, 1])$ of functions constant on horizontal lines is not complemented. Here we give necessary and sufficient conditions on Σ_0 for $L_p(X, \Sigma_0, \mu)$ to be complemented and extend the Berg-Peck-Porta theorem to all p, $0 < p < 1$. Of course if $p = 1$, every $L_p(X, \Sigma_0, \mu)$ is complemented by a conditional expectation operator.

In Section 5, we show that every non-zero endomorphism of $L_p(0, 1)$ $(0 \le p < 1)$ is an isomorphism on some $L_p(B)$ where $B \subset (0, 1)$ is a subset of positive measure. We also identify the unique maximal two-sided ideal in the algebra $\mathscr{L}(L_p)$ of all endomorphisms of L_p. In Section 6, we show that every complemented subspace of $L_0(0, 1)$ is isomorphic to L_0, and that if (X_n) is an unconditional Schauder decomposition of L_p $(0 < p \le 1)$ then at least one X_n is isomorphic to L_p. For $p = 1$, this result has been obtained independently by T. Starbird, and extends a result of Enflo ([4], [16], [22] and [23]) that L_1 is primary (*i.e.* if $L_1 \cong X \oplus Y$ then either $X \cong L_1$ or $Y \cong L_1$).

Finally in Section 7 we construct for each p, $0 \leq p < 1$ a quotient of L_p which admits no linear operators into L_p.

Throughout this paper, we concentrate on the case of real scalars, although for complex scalars only minor modifications are necessary. In Section 7, we do have occasion to consider the case of complex scalars. We also note that if K is a Polish space and λ is any continuous probability measure on K, then $L_p(K, \lambda)$ is isomorphic to $L_p(0, 1)$. Thus most of our results could be stated for the special case $K = (0, 1)$ without much loss of generality. However there seems no great advantage in doing this.

We also note that operators on $L_p(G)$ (G a locally compact group) which commute with translation have been characterized recently by Oberlin [17], [18]. For the metric case, these results follow from our main representation theorems.

2. Preliminary measurability results. We recall that a Polish space is a topological space which is homeomorphic to a separable complete metric space and that a continuous image of a Polish space is called a Souslin space. For any topological space K we denote the σ-algebra of Borel sets (*i.e.* the smallest σ-algebra generated by the open sets) by $\mathcal{B}(K)$, or simply $\mathcal{B}$ when no confusion can arise. If λ is a (finite) Borel measure on K, then $\mathcal{B}_\lambda$ is the λ-completion of $\mathcal{B}$. We denote by $\mathcal{U} = \mathcal{U}(K)$, the σ-algebra of universally measurable sets, i.e. the intersection of all $\mathcal{B}_\lambda$.

In this section we gather together a number of known or essentially known results on measurable mappings which we require later.

Theorem 2.1. (Kuratowski [10]) *Suppose K_1 and K_2 are two Polish spaces of the same cardinality. Then there is a Borel isomorphism $\varphi : K_1 \to K_2$ (i.e. φ is a bijection and φ and φ^{-1} are Borel maps).*

Theorem 2.2. *Let K be a Souslin space and let M be a metric space. Suppose $\varphi : K \to M$ is a continuous surjection (so that M is Souslin space). Then there is a map $\psi : M \to K$ which is $\mathcal{U}(M) - \mathcal{B}(K)$ measurable (i.e. $\psi^{-1}(B) \in \mathcal{U}(M)$ for $B \in \mathcal{B}(K)$) and $\varphi(\psi(x)) = x$ for $x \in M$.*

Proof. This is a consequence of the Kuratowski-Ryll-Nardzewski selection theorem (see [11], [2] p. 268). Suppose P is a Polish space and $\theta : P \to K$ is a continuous surjection. By Corollary 2, p. 270 of [2], there is a $\mathcal{U}(M) - \mathcal{B}(P)$ measurable map $\pi : M \to P$ such that $\varphi\theta\pi(x) = x$ ($x \in M$). Let $\psi = \theta\pi$.

If (X, Σ) is a measurable space and μ is a positive measure on Σ, we denote by Σ_μ the completion of Σ.

The following theorem follows easily from the fact that a Polish space has a countable base of open sets.

Theorem 2.3. *Let (X, Σ, μ) be a measure space and let K be a Polish space. Suppose $\sigma : X \to K$ is $\Sigma_\mu - \mathcal{B}$ measurable. Then there exists a $\Sigma - \mathcal{B}$ measurable map $\bar{\sigma} : X \to K$ such that $\bar{\sigma}(x) = \sigma(x)$* μ-a.e.

Theorem 2.4. (Lusin) *Suppose K_1 and K_2 are compact metric spaces and that $\sigma : K_1 \to K_2$ is a Borel measurable map. Then for any positive Borel measure λ on K_1 and $\epsilon > 0$ there is a closed subset E of K_1 such that $\sigma \mid E$ is continuous and $\lambda(K_1 \setminus E) < \epsilon$.*

For the remainder of this section we fix K as an infinite compact metric space. We denote by $C(K)$ the separable Banach space of all real-valued (or complex-valued) continuous functions on K. The dual of $C(K)$, which we denote by $\mathcal{M}(K)$, is the space of signed Borel measures on K with norm

$$\|\lambda\| = |\lambda|(K) \qquad \lambda \in \mathcal{M}(K)$$

where $|\lambda|$ is the total variation of λ; *i.e.*

$$|\lambda|(B) = \sup \sum_{i=1}^{\infty} |\lambda(B_i)| \qquad B \in \mathcal{B}$$

where the supremum is taken over all disjoint sequences (B_i) in $\mathcal{B}$ with $\cup B_i = B$.

For $0 < p < 1$, we define, similarly

$$|\lambda|_p(B) = \sup \sum_{i=1}^{\infty} |\lambda(B_i)|^p \qquad B \in \mathcal{B}.$$

Then $|\lambda|_p$ is a positive, but not necessarily finite Borel measure on $\mathcal{B}$. We define

$$\|\lambda\|_p = \{|\lambda|_p(K)\}^{1/p}.$$

Proposition 2.5. ([8], [17]) *If $0 < p < 1$, then $\|\lambda\|_p < \infty$ if and only if*

$$\lambda = \sum_{i=1}^{\infty} \alpha_i \delta(t_i)$$

where $t_i \in K$ are distinct and

$$\|\lambda\|_p = \left(\sum_{i=1}^{\infty} |\alpha_i|^p \right)^{1/p} < \infty.$$

We denote the weak*-Borel sets of $\mathcal{M}(K)$ by $\mathcal{B}^*$. Since $C(K)$ is separable, there is a metrizable topology on $\mathcal{M}(K)$, γ say, which agrees with the weak*-topology on bounded sets and is weaker than the weak*-topology. Then γ defines the same Borel sets as the weak*-topology. Using this observation the following lemma is easy.

Lemma 2.6. *Let (X, Σ) be a measurable space and let $\varphi : X \to \mathcal{M}(K)$ be a map such that for $f \in C(K)$, the map $x \mapsto \int f\, d\varphi(x)$ is Σ-measurable. Then φ is $\Sigma - \mathcal{B}^*$ measurable.*

Now the next proposition follows from Lemma 2.6.

Proposition 2.7. *Let (X, Σ) be a measurable space and suppose $\varphi_n : X \to \mathcal{M}(K)$ is a sequence of $\Sigma - \mathcal{B}^*$ measurable maps. Suppose* $\lim \varphi_n(x) = \varphi(x)$ *$(x \in X)$. Then φ is $\Sigma - \mathcal{B}^*$ measurable.*

Lemma 2.8. *Suppose* $0 < p \leq 1$.

(i) Suppose $U \subset K$ *is open; then the map* $\lambda \mapsto |\lambda|_p(U)$ *is weak*-lower-semi-continuous.*

(ii) Suppose $B \in \mathscr{B}$; *then the map* $\lambda \mapsto |\lambda|_p(B)$ *is* $\mathscr{B}^*$*-measurable.*

(iii) The map $\lambda \mapsto |\lambda|$ *is* $\mathscr{B}^* - \mathscr{B}^*$ *measurable.*

Proof. (i) For $p = 1$ this is well-known. For $p < 1$, we use Lemma 2.1 (i) of [8] which implies that the map $\lambda \mapsto |\lambda|_p(K)$ is weak*-lower-semi-continuous. Suppose λ_α is a net such that $\lambda_\alpha \to \lambda$, weak*. Let $\mu_\alpha = \lambda_\alpha \mid U$ and $\nu_\alpha = \lambda_\alpha \mid K \setminus U$. If

$$\limsup_\alpha |\lambda_\alpha|_p(U) < \infty$$

then

$$\limsup_\alpha \|\mu_\alpha\| \leq \limsup_\alpha \|\mu_\alpha\|_p < \infty.$$

Hence there is a subnet μ_β of (μ_α) which converges to some μ weak*. Hence $\nu_\beta \to \lambda - \mu$ and $\text{supp}(\lambda - \mu) \subset K \setminus U$.
Thus

$$|\lambda - \mu|_p(U) = 0$$

and so

$$\begin{aligned}
|\lambda|_p(U) &\leq |\mu|_p(U) + |\lambda - \mu|_p(U) \\
&= |\mu|_p(U) \\
&\leq |\mu|_p(K) \\
&\leq \limsup_\alpha |\mu_\alpha|_p(K) \\
&= \limsup_\alpha |\lambda_\alpha|_p(U).
\end{aligned}$$

This proves (i). Parts (ii) and (iii) follow trivially.

Lemma 2.9. *The set* $\mathscr{M}_c(K)$ *of continuous Borel measures on* K *is a weak*-Borel subset of* $\mathscr{M}(K)$.

Proof. For $n \in \mathbb{N}$, let $U_{n,1}, \cdots U_{n,m(n)}$ be a covering of K by open sets of diameter less than n^{-1}. Then $\lambda \in \mathscr{M}_c(K)$ if and only if

$$\lim_{n \to \infty} \sup_{1 \leq k \leq m(n)} |\lambda|(U_{n,k}) = 0,$$

and so $\mathscr{M}_c(K) \in \mathscr{B}^*$.

Our next result is the main one of the section.

Theorem 2.10. *There exist maps*

$$b_n : \mathcal{M}(K) \to R \qquad n = 1, 2 \cdots$$

$$h_n : \mathcal{M}(K) \to K \qquad n = 1, 2 \cdots$$

$$\varphi : \mathcal{M}(K) \to \mathcal{M}_c(K)$$

such that

(2.10.1) b_n is $\mathcal{U}^*(=\mathcal{U}(\mathcal{M}(K)))$-measurable $(n \in \mathbb{N})$

(2.10.2) h_n is $\mathcal{U}^* - \mathcal{B}$-measurable $(n \in \mathbb{N})$

(2.10.3) φ is $\mathcal{U}^* - \mathcal{B}^*$-measurable

(2.10.4) $|b_n(\lambda)| \geq |b_{n+1}(\lambda)| \qquad (n \in \mathbb{N}),\ \lambda \in \mathcal{M}(K).$

(2.10.5) $h_n(\lambda) \neq h_m(\lambda) \qquad m \neq n,\ \lambda \in \mathcal{M}(K)$

(2.10.6) $$\lambda = \sum_{n=1}^{\infty} b_n(\lambda)\delta(h_n(\lambda)) + \varphi(\lambda) \qquad \lambda \in \mathcal{M}(K).$$

Proof. Consider the space $\ell_1 \times K^{\mathbb{N}}$, which, in its natural topology is Polish. Let $Y \subset \ell_1 \times K^{\mathbb{N}}$ be the subset of $(\xi_1, \xi_2, \cdots; t_1, t_2, \cdots)$ such that $|\xi_n| \geq |\xi_{n+1}|$ $(n \in \mathbb{N})$ and $t_i \neq t_j$ $(i \neq j)$. Then Y is a Borel subset of $\ell_1 \times K^{\mathbb{N}}$, and so $Y \times \mathcal{M}(K)$ is a Souslin space ([21], p. 96). Let Z be the subset of $Y \times \mathcal{M}(K)$ of all $(\xi_1, \xi_2, \cdots; t_1, t_2, \cdots; \lambda)$ such that $\lambda - \Sigma\xi_i\delta(t_i) \in \mathcal{M}_c(K)$. Then Z is also a Borel subset of $Y \times \mathcal{M}(K)$ and hence is Souslin.

Define $\psi : Z \to \mathcal{M}(K)$ by

$$\psi(\xi_1, \xi_2, \cdots; t_1, t_2, \cdots; \lambda) = \lambda.$$

Then ψ is continuous, and so by equipping $\mathcal{M}(K)$ with its metrizable topology γ, introduced before Lemma 2.6, we can deduce from Theorem 2.2 that there is a map $\theta : \mathcal{M}(K) \to Z$ such that $\psi(\theta(\lambda)) = \lambda$ $(\lambda \in \mathcal{M}(K))$ and θ is $\mathcal{U}^* - \mathcal{B}(Z)$ measurable. Now let

$$\theta(\lambda) = (b_1(\lambda), b_2(\lambda), \cdots; h_1(\lambda), h_2(\lambda), \cdots; \lambda)$$

and

$$\varphi(\lambda) = \lambda - \sum_{n=1}^{\infty} b_n(\lambda)\delta(h_n(\lambda)).$$

To conclude the section we quote one further consequence of Lemma 2.6 which will be required later.

Lemma 2.11. *Let* K_1 and K_2 *be compact metric spaces and suppose* $\varphi : K_1 \to K_2$ *is a Borel map. Then the map* $\varphi^* : \mathcal{M}(K_1) \to \mathcal{M}(K_2)$ *defined by*

$$\varphi^*\mu(B) = \mu(\varphi^{-1}(B)) \qquad B \in \mathcal{B}$$

is $\mathcal{B}^* - \mathcal{B}^*$ *measurable.*

3. The main representation theorems. If (X, Σ, μ) is a measure space, then for $0 < p < \infty$, $L_p(X, \Sigma, \mu)$ denotes the (complete, locally bounded) F-space of all real- (or complex-) valued Σ-measurable functions satisfying

$$\|f\|_p = \left\{ \int_X |f(x)|^p d\mu(x) \right\}^{1/p} < \infty.$$

Theorem 3.1. Let K be a compact metric space and let λ be a probability measure on K. Suppose (X, Σ, μ) is a measure space and that $0 < p \leq 1$.

(a) Suppose $x \mapsto \nu_x$ is a $\Sigma - \mathcal{B}^*$ measurable map of X into $\mathcal{M}(K)$ satisfying

$$\sup_{\lambda(B) > 0} \frac{1}{\lambda(B)} \int_X |\nu_x|_p(B) d\mu(x) = M < \infty. \tag{3.1.1}$$

Then, if $f \in L_p(K, \mathcal{B}, \lambda)$, f is $|\nu_x|$-integrable μ-a.e. $(x \in X)$, and there is a bounded linear operator $T : L_p(K, \mathcal{B}, \lambda) \to L_p(X, \Sigma, \mu)$ satisfying

$$Tf(x) = \int_K f(t) d\nu_x(t) \qquad \mu\text{-a.e.} \qquad x \in X \tag{3.1.2}$$

and

$$\|T\| = \sup(\|Tf\| : \|f\| = 1) = M^{1/p}. \tag{3.1.3}$$

(b) Conversely if $T : L_p(K, \mathcal{B}, \lambda) \to L_p(X, \Sigma, \mu)$ is a bounded linear operator, then there is a $\Sigma - \mathcal{B}^*$ measurable map $x \mapsto \nu_x$ of X into $\mathcal{M}(K)$ satisfying (3.1.1), (3.1.2) and (3.1.3) with $M = \|T\|^p$. Furthermore this map is unique up to sets of μ-measure zero.

Proof. (a) By Lemma 2.8 (iii), the map $x \mapsto |\nu_x|$ is also $\Sigma - \mathcal{B}^*$ measurable and satisfies condition (3.1.1). For any simple function $f \in L_p(K, \mathcal{B}, \lambda)$ we define

$$S_0 f(x) = \int_K f(t) d|\nu_x|(t)$$

$$T_0 f(x) = \int_K f(t) d\nu_x(t).$$

Then if $f = \alpha_1 1_{B_1} + \alpha_2 1_{B_2} + \cdots + \alpha_n 1_{B_n}$ (where 1_A denotes the characteristic function of A and $B_1, \cdots, B_n \in \mathcal{B}$), we have

$$T_0 f(x) = \sum_{i=1}^{n} \alpha_i \nu_x(B_i).$$

Hence

$$\begin{aligned} \|T_0 f\|_p^p &= \int_X |T_0 f(x)|^p d\mu(x) \\ &\leq \int_X \sum_{i=1}^{n} |\alpha_i|^p |\nu_x(B_i)|^p d\mu(x) \end{aligned}$$

$$\leq \sum_{i=1}^{n} |\alpha_i|^p \int_X |\nu_x|_p(B_i) d\mu(x)$$

$$\leq M \sum_{i=1}^{n} |\alpha_i|^p \lambda(B_i)$$

$$= M\|f\|_p^p.$$

Hence T_0, and similarly S_0, extend to bounded linear operators $T, S : L_p(K) \to L_p(X)$ with $\|T\|, \|S\| \leq M^{1/p}$. Now suppose $f \in L_p(K, \mathscr{B}, \lambda)$ and $f \geq 0$. Then we may choose a sequence (f_n) of simple Borel functions such that $0 \leq f_n \leq f$ and $\lim f_n(x) = f(x)$ $(x \in K)$. Thus $\|f - f_n\|_p \to 0$ and hence $\|Sf - Sf_n\|_p \to 0$ and $\|Tf - Tf_n\|_p \to 0$. By selection of a subsequence we may suppose that $Sf_n \to Sf$ (μ-a.e.) and $Tf_n \to Tf$ (μ-a.e.).

$$Sf(x) = \lim_{n \to \infty} \int_K f_n(t) d|\nu_x|(t) \qquad \mu\text{-a.e.}$$

$$= \int_K f(t) d|\nu_x|(t) \qquad \mu\text{-a.e.}$$

Hence for μ-a.e. $x \in X$, f is $|\nu_x|$-integrable. Now by the Lebesgue dominated convergence theorem

$$Tf(x) = \lim_{n \to \infty} \int_K f_n(t) d\nu_x(t) \qquad \mu\text{-a.e.}$$

$$= \int_K f(t) d\nu_x(t) \qquad \mu\text{-a.e.}$$

Clearly this establishes (3.1.2) for any $f \in L_p(K)$. We also have $\|T\| \leq M^{1/p}$. Equation (3.1.3) will follow from (b).

(b) By Theorem 2.1 and Lemma 2.11 it is sufficient to establish (b) when K is totally disconnected. For $n \in \mathbb{N}$, let $\mathscr{A}_n = \{A_{n,1}, \cdots, A_{n,\ell(n)}\}$ be a covering of K of disjoint non-empty clopen sets so that $\mathscr{A}_{n+1}$ refines $\mathscr{A}_n$ and diam $A_{n,k} \leq n^{-1}$ for $1 \leq k \leq \ell(n)$. Let $\chi_k^n = 1_{A_{n,k}}$.

By the Stone-Weierstrass theorem, the functions $\{\chi_k^n : k \leq \ell(n),\ n \in \mathbb{N}\}$ span a dense subspace of $C(K)$. Fix $\tau_k^n \in A_{n,k}$ and let

$$b_k^n(x) = T\chi_k^n(x).$$

For each $x \in K$, define $\nu_x^n \in \mathscr{M}(K)$ by

$$\nu_x^n = \sum_{k=1}^{\ell(n)} b_k^n(x)\delta(\tau_k^n).$$

Then $x \mapsto \nu_x^n$ is $\Sigma - \mathscr{B}^*$ measurable. As

$$b_k^n(x) = \sum_{A_{n+1,i} \subset A_{n,k}} b_i^{n+1}(x) \qquad \mu\text{-a.e.}$$

we have

$$(3.1.4)\qquad |\nu_x^n|_p(K) \le |\nu_x^{n+1}|_p(K) \qquad n \in \mathbb{N} \qquad \mu\text{-a.e.}$$

$$(3.1.5)\qquad \nu_x^{n+1}(A_{n,k}) = \nu_x^n(A_{n,k}) \qquad 1 \le k \le \ell(n), \qquad n \in \mathbb{N}, \qquad \mu\text{-a.e.}$$

However

$$\begin{aligned}\int_X |\nu_x^n|_p(K)d\mu(x) &= \int_X \sum_{k=1}^{\ell(n)} |b_k^n(x)|^p d\mu(x)\\ &\le \sum_{k=1}^{\ell(n)} \|T\|^p\lambda(A_{n,k})\\ &= \|T\|^p.\end{aligned}$$

Thus, by the Monotone Convergence Theorem,

$$\int_X \sup_n |\nu_x^n|_p(K)d\mu(x) \le \|T\|^p$$

and

$$\sup_n \|\nu_x^n\|_p < \infty \qquad \mu\text{-a.e.}$$

As $\|\nu_x^n\| \le \|\nu_x^n\|_p$, we deduce that there exists $X_0 \in \Sigma$ with $\mu(X \setminus X_0) = 0$ and such that (3.1.4), (3.1.5) and (3.1.6) hold for $x \in X_0$ where

$$(3.1.6)\qquad \sup_n \|\nu_x^n\| < \infty.$$

Hence if $x \in X_0$, $\{\nu_x^n\}$ has weak*-cluster point ν_x. This point is unique since if $m \ge n$

$$\nu_x^m(A_{n,k}) = \nu_x^n(A_{n,k}) = b_k^n(x)$$

and hence

$$\nu_x(A_{n,k}) = b_k^n(x).$$

It thus follows that $\nu_x^n \to \nu_x$ weak* $(x \in X_0)$. Let us define $\nu_x = 0$ $(x \notin X_0)$; then the map $x \mapsto \nu_x$ is $\Sigma - \mathcal{B}^*$ measurable by Proposition 2.7.

Now suppose $m \ge n$ and $1 \le k \le \ell(n)$,

$$\begin{aligned}\int_X |\nu_x^m|_p(A_{n,k})d\mu(x) &= \int_X \sum_{A_{m,i} \subset A_{n,k}} |b_i^m(x)|^p d\mu(x)\\ &\le \|T\|^p \sum_{A_{m,i} \subset A_{n,k}} \lambda(A_{m,i})\\ &= \|T\|^p\lambda(A_{n,k}).\end{aligned}$$

Hence

$$\int_X |\nu_x|_p(A_{n,k})d\mu(x) \leq \int_X \liminf_{n\to\infty} |\nu_x^n|_p(A_{n,k})d\mu(x)$$

(by Lemma 2.8 (i))

$$\leq \|T\|^p\lambda(A_{n,k}).$$

It now follows easily that for all Borel sets B

$$\int_X |\nu_x|_p(B)d\mu(x) \leq \|T\|^p\lambda(B).$$

Thus (3.1.1) holds with $M \leq \|T\|^p$. Hence there is a bounded linear operator $S : L_p(K) \to L_p(X)$ defined by

$$Sf(x) = \int_K f(t)d\nu_x(t) \qquad \mu\text{-a.e.}$$

and $\|S\| \leq M^{1/p}$. As $S\chi_k^n = T\chi_k^n$ (μ-a.e.), it is clear that $S = T$ and hence $\|T\| = M^{1/p}$.

Finally, if $x \mapsto \nu_x'$ is any other $\Sigma - \mathscr{B}^*$ measurable map satisfying

$$\int \chi_k^n d\nu_x' = \int \chi_k^n d\nu_x \qquad \mu\text{-a.e.}$$

for $1 \leq k \leq \ell(n)$, $n \in \mathbb{N}$, there is a set $X_0 \in \Sigma$ with $\mu(X\setminus X_0) = 0$ and

$$\int \chi_k^n d\nu_x' = \int \chi_k^n d\nu_x \qquad x \in X_0$$

for all n, k. For $x \in X_0$ $\nu_x' = \nu_x$, and this establishes the uniqueness of the map $x \mapsto \nu_x$.

Remark. $\mathscr{L}(L_p(K), L_p(X))$ is an ordered locally bounded F-space if we define $T \geq 0$ whenever $Tf \geq 0$ for $f \geq 0$. It is then a lattice and if T is defined by (3.1.2) then $|T|$ is defined by

$$|T|f(x) = \int f(t)d|\nu_x|(t)$$

and $\| |T| \| = \|T\|$.

For the case $p = 1$, this is observed by Starbird [21].

Theorem 3.2. *Let K be an infinite Polish space and let λ be a probability measure on K. Let (X, Σ, μ) be a measure space and suppose $0 < p < 1$.*

(a) Let $T : L_p(K, \mathscr{B}, \lambda) \to L_p(X, \Sigma, \mu)$ be a bounded linear operator. Then there is a sequence $\{a_n\}$ of real-valued Borel functions on X and a sequence $\sigma_n : X \to K$ of $\Sigma - \mathscr{B}$ measurable maps such that

(3.2.1) $\quad \sigma_m(x) \neq \sigma_n(x) \qquad m \neq n \qquad x \in X$

(3.2.2) $\sum_{n=1}^{\infty} |a_n(x)|^p < \infty \qquad \mu\text{-a.e.} \qquad x \in X$

(3.2.3) $\sum_{n=1}^{\infty} \int_{\sigma_n^{-1}(B)} |a_n(x)|^p d\mu(x) \leq \|T\|^p \lambda(B) \qquad B \in \mathscr{B}$

(3.2.4) $|a_n(x)| \geq |a_{n+1}(x)| \qquad \mu\text{-a.e.} \qquad x \in X$

(3.2.5) $Tf(x) = \sum_{n=1}^{\infty} a_n(x) f(\sigma_n x) \qquad \mu\text{-a.e.} \qquad x \in X.$

(b) Conversely $\{a_n\}$ *is a sequence of Borel functions on* K *and* $\sigma_n : X \to K$ *is a sequence of* $\Sigma - \mathscr{B}$ *measurable maps satisfying*

(3.2.6) $\sup_{\lambda(B) > 0} \frac{1}{\lambda(B)} \sum_{n=1}^{\infty} \int_{\sigma_n^{-1}(B)} |a_n(x)|^p d\mu(x) = M < \infty$

then (3.2.5) *defines a bounded linear operator* T *with* $\|T\| = M^{1/p}$.

Proof. (a) This follows almost immediately from Theorems 3.1 and 2.10. Define $\bar{\sigma}_n(x) = h_n(\nu_x)$ and $\bar{a}_n(x) = b_n(\nu_x)$. Since $|\nu_x|_p(K) < \infty$ a.e. we deduce from 2.5 that

$$\nu_x = \sum_{n=1}^{\infty} \bar{a}_n(x)\delta(\bar{\sigma}_n(x)) \qquad \mu\text{-a.e.}$$

If $B \in \mathscr{B}$, $h_n^{-1}(B) \in \mathscr{U}^*$. Define a measure γ on $\mathscr{B}^*$ by

$$\gamma(C) = \mu\{x : \nu_x \in C\} = \mu(\nu^{-1}C).$$

Then $h_n^{-1}(B)$ is γ-measurable; *i.e.* there exist $C_1, C_2 \in \mathscr{B}^*$ with

$$C_1 \subset h_n^{-1}(B) \subset C_2$$

and

$$\gamma(C_2 \setminus C_1) = 0.$$

Hence

$$\nu^{-1}C_1 \subset \bar{\sigma}_n^{-1}(B) \subset \nu^{-1}C_2$$

and

$$\mu(\nu^{-1}C_2 \setminus \nu^{-1}C_1) = 0.$$

Thus $\bar{\sigma}_n^{-1}(B) \in \Sigma_\mu$; *i.e.* $\bar{\sigma}_n$ is $\Sigma_\mu - \mathscr{B}$ measurable. Similarly each $\bar{a}_n$ is Σ_μ-measurable. By Proposition 2.3, we may define $\sigma_n : X \to K$ ($n \in \mathbb{N}$) and $a_n : X \to R$ ($n \in \mathbb{N}$) which are $\Sigma - \mathscr{B}$ and Σ-measurable respectively and such that μ-a.e. $\sigma_n(x) = \bar{\sigma}_n(x)$ and $a_n(x) = \bar{a}_n(x)$ ($n \in \mathbb{N}$). Now (3.2.1), (3.2.2), (3.2.4) and (3.2.5) are trivial. For (3.2.3) we observe that

$$\sum_{n=1}^{\infty} \int_{\sigma_n^{-1}(B)} |a_n(x)|^p d\mu(x) = \int_X |\nu_x|_p(B) d\mu(x).$$

(b) This is quite trivial from (a) of (3.1) if we define

$$\nu_x = \sum_{n=1}^{\infty} a_n(x)\delta(\sigma_n x).$$

Remarks. In the case $p = 1$, we can use the same argument to represent any bounded linear operator $T : L_1(K) \to L_1(X)$ in the form

$$Tf(x) = \sum_{n=1}^{\infty} a_n(x)f(\sigma_n x) + \int_K f(x)d\rho_x(s)$$

where the map $x \mapsto \rho_x$ satisfies the condition of Theorem 3.1 and ρ_x is almost everywhere a continuous measure. We can then separate off the "atomic part" T^a of T

$$T^a f(x) = \sum_{n=1}^{\infty} a_n(x)f(\sigma_n x)$$

and the "continuous part" T^c,

$$T^c f(x) = \int_K f(s)d\rho_x(s)$$

T^a is thus obtained by taking the atomic part of ν_x for each x if

$$Tf(x) = \int_K f(s)d\nu_x(s).$$

4. Projections onto subspaces generated by sub-σ-algebras. Suppose K and M are compact metric spaces and $\varphi : K \to M$ is a continuous surjection. Suppose λ is a probability measure on K and let $\mu = \varphi^*\lambda$ be the induced probability measure on M. The map $S_\varphi : L_p(M, \mu) \to L_p(K, \lambda)$ defined by

$$S_\varphi f(s) = f(\varphi s) \qquad s \in K \qquad f \in L_p(M)$$

is an isometric embedding of $L_p(M)$ into $L_p(K)$ $(0 < p < \infty)$.

Proposition 4.1. *Suppose $0 < p < 1$ and there exists a bounded linear operator $T : L_p(K) \to L_p(M)$ such that TS_φ is the identity on $L_p(M)$. Then there is a Borel map $\theta : M \to K$ and $\epsilon > 0$ such that $\varphi\theta = i_M$ (identity map on M) and*

$$\lambda(B) \geq \epsilon\mu(\theta^{-1}B) \qquad B \in \mathscr{B}(K).$$

Proof. Let

$$Tf(t) = \int_K f(s)d\nu_t(s) \qquad f \in L_p(K) \qquad t \in M$$

where $t \mapsto \nu_t$ satisfies the conditions of Theorem 3.1. As in Theorem 3.2 we may write

$$\nu_t = \sum_{n=1}^{\infty} a_n(t)\delta(\sigma_n t)$$

where a_n and σ_n satisfy the conditions of 3.2. For $t \in K$, let

$$\chi_n(t) = 1 \quad \text{if} \quad \varphi\sigma_n(t) = t$$
$$= 0 \quad \text{if} \quad \varphi\sigma_n(t) \neq t.$$

Then each χ_n is Borel measurable. If

$$\bar{\nu}_t = \sum_{n=1}^{\infty} \chi_n(t)a_n(t)\delta(\sigma_n t)$$

then $t \mapsto \bar{\nu}_t$ defines an operator $V : L_p(K) \to L_p(M)$ (by Theorem 3.1) and $\|V\| \leq \|T\|$. For $f \in C(M)$, $f = TS_\Phi f$ so that almost everywhere,

$$f(t) = \int_K f(\varphi s)d\nu_t(s)$$
$$= \int_M f(s)d\varphi^*\nu_t(s)$$

and by the uniqueness of the representation

$$\delta(t) = \varphi^*\nu_t \qquad \text{a.e.}$$

i.e.

$$\delta(t) = \sum_{n=1}^{\infty} a_n(t)\delta(\varphi\sigma_n t) \qquad \text{a.e.}$$

Hence we also have

$$\delta(t) = \varphi^*\bar{\nu}_t$$

and also $VS_\Phi f = f$ for $f \in L_p(M)$. Now write

$$Vf(t) = \sum_{n=1}^{\infty} \chi_n(t)a_n(t)f(\sigma_n t) = \sum_{n=1}^{\infty} \alpha_n(t)f(\rho_n t)$$

where (α_n) and (ρ_n) satisfy the conditions (3.2.1)–(3.2.5) (of course, this is an alternative representation for $\bar{\nu}_t$-terms which were zero have now been eliminated).

For $C \in \mathscr{B}(M)$, let $B = \varphi^{-1}(C) \in \mathscr{B}(K)$. Then for all n, $\rho_n^{-1}(B) = C$, relative to the set $\{t : \alpha_n(t) \neq 0\}$ and up to sets of λ-measure zero,

and hence

$$\int_C \sum_{n=1}^{\infty} |\alpha_n(t)|^p d\mu(t) \leq \|V\|^p\lambda(B)$$
$$= \|V\|^p\mu(C).$$

Hence almost everywhere

$$\sum_{n=1}^{\infty} |\alpha_n(t)|^p \leq \|V\|^p.$$

As $\varphi^* \bar{\nu}_t = \delta(t)$, we have

$$\sum_{n=1}^{\infty} \alpha_n(t) = 1 \qquad \text{a.e.}$$

Thus as $|\alpha_1(t)| \geq |\alpha_n(t)| \; \forall n$,

$$|\alpha_1(t)|^{1-p} \sum_{n=1}^{\infty} |\alpha_n(t)|^p \geq 1 \qquad \text{a.e.}$$

so that

$$|\alpha_1(t)| \geq \|V\|^{-q} \qquad \text{a.e.}$$

where $q = p(1-p)^{-1}$.

Define $\theta : M \to K$ by $\theta t = \rho_1 t$. Then $\varphi\theta = i_M$, and if $B \in \mathscr{B}(K)$

$$\begin{aligned} \lambda(B) &\geq \|V\|^{-p} \int_{\theta^{-1}(B)} |\alpha_1(t)^p d\mu(t) \\ &\geq \|V\|^{-q} \mu(\theta^{-1}B). \end{aligned}$$

Letting $\epsilon = \|V\|^{-q}$, the proof is complete.

We also observe the analogous result for the case $p = 0$, which has a very similar proof, which we omit.

Proposition 4.2. *Suppose $T : L_0(K) \to L_0(M)$ is such that TS_φ is the identity on $L_0(M)$. Then there is a Borel map $\theta : M \to K$ such that $\varphi\theta = i_M$ and $\lambda(B) = 0$ implies $\mu(\theta^{-1}B) = 0$ for $B \in \mathscr{B}(K)$.*

Lemma 4.3. *Let X be an F-space and Y a closed subspace of X. Suppose $T \in \mathscr{L}(X)$ has the properties (i) $T|Y$ is a homeomorphism and (ii) $T(Y)$ is complemented in X. Then Y is complemented in X.*

Proof. Let $P : X \to T(Y)$ be a projection and let $U : T(Y) \to Y$ be the inverse of $T|Y$. Then UPT is a projection of X onto Y.

We now come to our main theorem of the section. In this we classify exactly, for a separable measure space (X, Σ, μ), those sub-σ-algebras Σ_0 of Σ such that $L_p(X, \Sigma_0, \mu)$ is complemented in $L_p(X, \Sigma, \mu)$ where $0 \leq p < 1$. Of course if $p \geq 1$ there is no restriction on such sub-σ-algebras. For $p = 0$, Berg, Peck and Porta [1] gave an example of a non-atomic sub-σ-algebra Σ_0 such that $L_0(X, \Sigma_0, \mu)$ is uncomplemented.

Theorem 4.4. *Let (X, Σ, λ) be a separable probability measure space and let Σ_0 be a sub-σ-algebra of Σ, which contains the sets of λ-measure zero.*

(a) If $0 < p < 1$, $L_p(X, \Sigma_0, \lambda)$ is complemented in $L_p(X, \Sigma, \lambda)$ if and only if there exists $A \in \Sigma$ and $\epsilon > 0$ such that

(4.4.1) $$\lambda(B \cap A) \geq \epsilon\lambda(B) \qquad B \in \Sigma_0$$

and (4.4.2) *for* $C \subset A$ *and* $C \in \Sigma$, *there exists* $B \in \Sigma_0$ *with* $B \cap A = C$.

(b) $L_0(X, \Sigma_0, \lambda)$ *is complemented in* $L_0(X, \Sigma, \lambda)$ *if and only if there exists* $A \in \Sigma$ *satisfying* (4.4.2) *and:*

(4.4.3) *If* $B \in \Sigma_0$ *and* $\lambda(B \cap A) = 0$, *then* $\lambda(B) = 0$.

Proof. (a) Suppose first that $L_p(X, \Sigma_0, \lambda)$ is complemented in $L_p(X, \Sigma, \lambda)$. Let $\mathscr{F}_0$ be a countable algebra dense in Σ_0 and suppose $\mathscr{F} \supset \mathscr{F}_0$ is a countable dense algebra in Σ. Let K and M be the Stone spaces of $\mathscr{F}$ and $\mathscr{F}_0$ respectively; thus K and M are compact metric spaces. There is then a naturally induced continuous surjection $\varphi : K \to M$. Let $\mathscr{F}_0^\sigma$ and $\mathscr{F}^\sigma$ be the σ-algebras generated by $\mathscr{F}_0$ and $\mathscr{F}$. Then $L_p(X, \mathscr{F}^\sigma, \lambda)$ identifies naturally with $L_p(K, \mathscr{B}, \bar{\lambda})$ (where $\bar{\lambda}$ is the Borel measure induced on K by λ) and its subspace $L_p(X, \mathscr{F}_0^\sigma, \lambda)$ is then identified with $S_\varphi(L_p(M, \mathscr{B}, \mu))$ where $\mu = \varphi^*\bar{\lambda}$. It is easily seen that it is only necessary to establish (4.4.1) and (4.4.2) for the sub-σ-algebra $\mathscr{B}_0 = \{\varphi^{-1}(B) : B \in \mathscr{B}(M)\}$ of $\mathscr{B} = \mathscr{B}(K)$.

By Proposition 4.1, there exists a Borel map $\theta : M \to K$ and $\epsilon > 0$ such that $\varphi\theta = i_M$ and $\epsilon\mu(\theta^{-1}B) \leq \lambda(B)$ $(B \in \mathscr{B}(K))$. Suppose $(E_n : n \in \mathbb{N})$ is a sequence of closed subsets of M such that $\mu(\cup E_n) = 1$ and $\theta|E_n$ is continuous for each n. Let $\theta(\cup E_n) = A$; A is an F_σ-set and so $A \in \mathscr{B}$. If $B \in \mathscr{B}_0$

$$\begin{aligned}\lambda(B \cap A) &\geq \epsilon\mu(\theta^{-1}(B \cap A)) \\ &= \epsilon\mu(\theta^{-1}B) \\ &= \epsilon\lambda(B).\end{aligned}$$

If $C \subset A$ and $C \in \mathscr{B}$, then $\varphi^{-1}\theta^{-1}(C) \in \mathscr{B}$ and $\varphi^{-1}\theta^{-1}(C) \cap A = C$.

Conversely, suppose the conditions are satisfied and let P_A be the natural projection of $L_p(X, \Sigma, \lambda)$ onto $L_p(A, \Sigma, \lambda)$. From (4.4.1) if $f \in L_p(X, \Sigma_0, \lambda)$

$$\|P_A f\|_p \geq \epsilon^{1/p}\|f\|_p$$

and from (4.4.2) P_A maps $L_p(X, \Sigma_0, \lambda)$ onto $L_p(A)$. Hence by Lemma 4.3, $L_p(X, \Sigma_0, \lambda)$ is complemented in $L_p(X, \Sigma, \lambda)$.

The case $p = 0$ is similar.

Corollary 4.5. *If there is a non-atomic* σ*-algebra* $\Lambda \subset \Sigma$, *independent of* Σ_0, *then there is no projection of* $L_p(X, \Sigma, \lambda)$ *onto* $L_p(X, \Sigma_0, \lambda)$ *for any* p, $0 \leq p < 1$.

Proof. Suppose (4.4.2) and (4.4.3) are satisfied (or the stronger condition (4.4.1)). We can define a measure ν on Σ by

$$\nu(C) = \lambda(B) \qquad C \in \Sigma$$

where $B \in \Sigma_0$ and $B \cap A = C \cap A$. In fact up to sets of λ-measure zero, B is uniquely determined. For if $B' \cap A = C \cap A$ then $(B' \triangle B) \cap A = \varnothing$ and hence $\lambda(B' \triangle B) = 0$. Similar checks show that ν is indeed a measure, and is λ-continuous. Hence given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $\lambda(C) < \delta(\epsilon)$ implies $\nu(C) < \epsilon$.

For $\epsilon > 0$, let $X = E_1 \cup \cdots \cup E_n$ where $E_i \cap E_j = \emptyset (i \neq j)$, $E_i \in \Lambda$ and $\lambda(E_i) \leq \delta(\epsilon)$. For $i = 1, 2, \cdots, n$, $E_i = B_i \cap A$ where $B_i \in \Sigma_0$. As $E_i \cap B_i \cap A = E_i \cap A$ we have

$$\nu(E_i \cap B_i) = \nu(E_i) \qquad 1 \leq i \leq n$$

and hence

$$\nu\left[\bigcup_{i=1}^{n} (E_i \cap B_i)\right] = \nu(X) = 1.$$

However

$$\lambda\left[\bigcup_{i=1}^{n} E_i \cap B_i\right] = \sum_{i=1}^{n} \lambda(E_i)\lambda(B_i) \leq \delta(\epsilon).$$

Letting $\epsilon \to 0$ we obtain a contradiction.

In particular there is no projection onto $L_p\,[[0, 1] \times [0, 1]]$ onto its subspace of all functions constant on horizontal lines if $0 \leq p < 1$, generalizing the result of Berg, Peck and Porta [1].

5. The endomorphisms of L_p. Suppose now that K is a fixed compact metric space and that λ is a nonatomic probability measure on K; let $L_p = L_p(K, \mathcal{B}, \lambda)$. Suppose $T \in \mathcal{L}(L_p)$ $(0 \leq p < 1)$; then T may be represented in the form of Theorem 3.2 and hence is decomposed into operators of the form

$$Sf(x) = a(x)f(\sigma x) \qquad \text{a.e.}$$

where $a : K \to R$ and $\sigma : K \to K$ are Borel maps. Our first lemma summarizes the obvious properties of such an operator.

Lemma 5.1. (a) Suppose $S \in \mathcal{L}(L_p)$ $(0 \leq p \leq 1)$ takes the form

$$Sf(x) = a(x)f(\sigma x) \qquad \text{a.e.}$$

If $S \neq 0$, there is a Borel subset B of K such that $\lambda(B) > 0$ and $S|L_p(B)$ is an isomorphism.

(b) If $p = 1$, then B may be chosen in (a) so that $S(L_p(B))$ is complemented in L_p.

(c) If $0 \leq p < 1$ and there exists a Borel subset C of K such that $\lambda(C) > 0$ and

(5.1.1) $$a(x) \neq 0 \qquad x \in C$$

(5.1.2) $$\sigma|C \text{ is an injection,}$$

then B may be chosen in (a) such that $S(L_p(B))$ is complemented in L_p.

Proof. (a) Since $S \neq 0$, we may choose a closed subset E of K such that $\lambda(E) > 0$, $\sigma|E$ is continuous and $0 < \alpha \leq |a(x)|$ $(x \in E)$. Let $F = \sigma(E)$ so that F is closed, and, since $S1_F = P_E a$, $\lambda(F) > 0$.

Let $\nu = \sigma^*\lambda$ be the measure on K induced by λ and σ; *i.e.*

$$\nu(A) = \lambda(\sigma^{-1}A).$$

If $A \subset F$, and $p > 0$,

$$\begin{aligned}\|S\|^p \lambda(A) &\geq \|S1_A\|_p^p \\ &= \int_{\sigma^{-1}(A)} |a(x)|^p d\lambda(x) \\ &\geq \alpha^p \nu(A).\end{aligned}$$

Hence $\nu|F$ is λ-continuous, and has Radon-Nikodym derivative u, say. Since $\nu(F) = \lambda(E) > 0$, $u \neq 0$ and so there exists $\epsilon > 0$ and a Borel subset B of F such that $\lambda(B) > 0$ and $u(x) \geq \epsilon$ $(x \in B)$.

If $f \in L_p(B)$,

$$\begin{aligned}\|Sf\|_p^p &\geq \|P_E Sf\|_p^p \\ &= \int_E |a(x)|^p |f(\sigma x)|^p d\lambda(x) \\ &\geq \alpha^p \int_E |f(\sigma x)|^p d\lambda(x) \\ &= \alpha^p \int_E |f(x)|^p d\nu(x) \\ &= \alpha^p \int_B |f(x)|^p u(x) d\lambda(x) \\ &\geq \epsilon\alpha^p \|f\|_p^p.\end{aligned}$$

This establishes (a) if $p > 0$. The case $p = 0$ is proved by a similar technique.

(b) Define $V \in \mathscr{L}(L_p)$ by

$$\begin{aligned}Vf(x) &= a(x)^{-1} f(x) \qquad & x \in E \\ &= 0 & x \notin E.\end{aligned}$$

Then V maps $S(L_p(B))$ isomorphically onto the subset of $L_p(\sigma^{-1}B)$ for all functions measurable with respect to the sub-σ-algebra of $\mathscr{B}$ of sets of the form $(\sigma^{-1}A : A \in \mathscr{B})$. If $p = 1$, $VS(L_1(B))$ is complemented in L_1 (by using the composition of $P_{\sigma^{-1}B}$ and a conditional expectation operator). Part (b) follows now from Lemma 4.3.

(c) If (5.1.1) and (5.1.2) hold we may suppose that E in (a) is a subset of C. Then $VS(L_p(B)) = L_p(\sigma^{-1}B)$ is complemented in L_p.

We now introduce some notation. If $T \in \mathscr{L}(L_p)$ $(0 < p \leq 1)$ is represented by the map $x \mapsto \nu_x$ as in Theorem 3.1, then we define

$$\theta_p(T) = \int_K \|\nu_x\|_p^p d\lambda(x)$$

$$\left(= \int_K |\nu_x|_p(K) d\lambda(x)\right).$$

If B is a Borel subset of K with $\lambda(B) > 0$ then

$$\|T\|_B = \sup(\|Tf\| : \|f\| \leq 1, \qquad f \in L_p(B)).$$

Lemma 5.2. *Suppose $0 < p \leq 1$ and $T_n \in \mathscr{L}(L_p)$. Suppose that $\theta_p(T_n) \to 0$. Then given any $\epsilon > 0$ and $C \in \mathscr{B}$ with $\lambda(C) > 0$, there exists $B \in \mathscr{B}$ with $B \subset C$ and $\lambda(B) > 0$, and $n \in \mathbb{N}$ such that $\|T_n\|_B \leq \epsilon$.*

Proof. For $A \in \mathscr{B}$, let

$$\rho_n(A) = \int_K |\nu_x^n|_p(A)d\lambda(x)$$

where $x \mapsto \nu_x^n$ represents T_n. Then ρ_n is a positive Borel measure on K and $\rho_n(A) \leq \|T_n\|^p\lambda(A)$. Hence ρ_n has Radon-Nikodym derivative ω_n and

$$\theta_p(T_n) = \rho_n(K) = \int_K \omega_n(t)d\lambda(t) \to 0.$$

In particular $\omega_n \to 0$ in λ-measure, and so there exists $B \in \mathscr{B}$, $B \subset C$, $\lambda(B) > 0$ and $n \in \mathbb{N}$ such that

$$\omega_n(t) \leq \epsilon^p \qquad t \in B.$$

Then

$$\begin{aligned}\|T_n\|_B^p &= \|T_nP_B\|^p \\ &= \sup_{\substack{A \in \mathscr{B} \\ \lambda(A) > 0}} \lambda(A)^{-1} \int_K |\nu_x^n|(A \cap B)d\lambda(x) \\ &= \operatorname*{ess\,sup}_{t \in B} \omega_n(t) \leq \epsilon^p.\end{aligned}$$

Lemma 5.3. *(a) Suppose $0 < p \leq 1$ and $T_n, S \in \mathscr{L}(L_p)$. Suppose $\theta_p(T_n - S) \to 0$. Suppose there exists $C \in \mathscr{B}$ with $\lambda(C) > 0$ such that $S|L_p(C)$ is an isomorphism onto its image. Then there is a subset B of C, with $B \in \mathscr{B}$, $\lambda(B) > 0$, and $n \in \mathbb{N}$ such that $T_n|L_p(B)$ is an isomorphism onto its image.*

(b) Suppose, in addition, $S(L_p(C))$ is complemented in $L_p(K)$. Then n and B may be chosen so that $T_n(L_p(B)$ is complemented in $L_p(K)$.

Proof. We prove only (b), as part (a) is essentially proved en route. We may suppose that

$$\|Sf\|_p \geq \delta\|f\|_p \qquad f \in L_p(C)$$

where $\delta > 0$. Let Q be a projection onto $S(L_p(C))$ and let $\epsilon = \frac{1}{2}\delta^2\|Q\|^{-1}\|S\|^{-1}$. Choose B and $n \in \mathbb{N}$ as in Lemma 5.2 so that $\lambda(B) > 0$ and

$$\|T_n - S\|_B \leq \epsilon.$$

For $f \in L_p(B)$

$$\|T_nf\|_p \geq (\delta^p - \epsilon^p)^{1/p}\|f\|_p.$$

Since $\delta \leq \|S\|$ and $1 \leq \|Q\|$, T_n maps $L_p(B)$ isomorphically onto its image. Let $V|S(L_p(C))$ be the inverse of $S|L_p(C)$, so that $\|V\| \leq \delta^{-1}$. Then let $W = SP_BVQ$; W is a projection of L_p onto $S(L_p(B))$.

For $f \in S(L_p(B))$ consider

$$\begin{aligned}\|f - WT_nVf\|_p &= \|(S - WT_n)Vf\|_p \\ &= \|W(S - T_n)Vf\|_p \\ &\leq \epsilon\|W\|\,\|V\|\,\|f\|_p \\ &\leq \delta^{-2}\epsilon\|S\|\,\|Q\|\,\|f\|_p \\ &\leq \frac{1}{2}\,\|f\|_p.\end{aligned}$$

Hence WT_nV maps $S(L_p(B))$ onto itself isomorphically. Thus W maps $T_n(L_p(B))$ onto $S(L_p(B))$ isomorphically and hence $T_n(L_p(B))$ is complemented in L_p by Lemma 4.3.

We now come to the first main theorem of the section:

Theorem 5.4. *Suppose $0 \leq p < 1$ and $T \in \mathcal{L}(L_p)$ is non-zero. Then there is a Borel subset B of K with $\lambda(B) > 0$ and such that $T|L_p(B)$ is an isomorphism.*

Proof. Consider the case $p > 0$. Suppose T has a representation as in Theorem 3.2

$$Tf(x) = \sum_{n=1}^{\infty} a_n(x)f(\sigma_n x) \qquad \text{a.e.}$$

Suppose $C \in \mathscr{B}$ and $n \in \mathbb{N}$ are such that $\lambda(C) > 0$ and $|a_n(x)| > 0$ $(x \in C)$. For each $m \in \mathbb{N}$, let $B_{m,1}, \cdots, B_{m,k(m)}$ be a covering of K by disjoint Borel sets of diameter less than m^{-1}. Let $C_{m,i} = \sigma_n^{-1}(B_{m,i}) \cap C$ $(1 \leq i \leq k(m))$. Define

$$T_m = \sum_{i=1}^{k(m)} P_{C_{m,i}}TP_{B_{m,i}}.$$

If $x \mapsto \nu_x$ represents T, then $x \mapsto \nu_x^m$ represents T_m where

$$\begin{aligned}\nu_x^m(B) &= \nu_x(B \cap B_{m,i}) \qquad & x \in C_{m,i} \\ &= 0 & x \notin C.\end{aligned}$$

Let S denote the map

$$\begin{aligned}Sf(x) &= a_n(x)f(\sigma_n x) \qquad & x \in C \\ &= 0 & x \notin C.\end{aligned}$$

Then S is represented by $x \mapsto \mu_x$ where

$$\begin{aligned}\mu_x &= a_n(x)\delta(\sigma_n x) \qquad & x \in C \\ &= 0 & x \notin C.\end{aligned}$$

For $x \in C_{m,i}$

$$|\nu_x^m - \mu_x|_p(K) = |\nu_x - \mu_x|_p(B_{m,i}).$$

Hence for all $x \in C$, where $x \in C_{m,i(m)}$, $m \in \mathbb{N}$,

$$\begin{aligned}\lim_{m\to\infty} |\nu_x^m - \mu_x|_p(K) &= \lim_{m\to\infty} |\nu_x - \mu_x|_p(B_{m,i(m)}) \\ &\leq |\nu_x - \mu_x|_p(\limsup B_{m,i(m)}) \\ &= |\nu_x - \mu_x|_p\{\sigma_n x\} \\ &= 0.\end{aligned}$$

As $|\nu_x^m - \mu_x|_p(K) \leq |\nu_x|_p(K)$ we conclude from the Dominated Convergence Theorem that $\theta_p(T_m - S) \to 0$.

Now by Lemmas 5.1 and 5.3, there exist m and a Borel set B of positive measure such that $T_m|L_p(B)$ is an isomorphism. For some i, $1 \leq i \leq k(m)$, $\lambda(B_{m,i} \cap B) > 0$, and the $T|L_p(B_{m,i} \cap B)$ is an isomorphism.

For the case $p = 0$, the argument is similar. We define T_m as before and observe that, almost everywhere, ν_x has finite support. Hence $\nu_x^m = \mu_x$ eventually, almost everywhere. We may thus produce B, $\lambda(B) > 0$ such that $T_m|L_0(B) = S|L_0(B)$.

For the case $p = 1$, we obtain the following theorem, which is very closely related to a result of Enflo and Starbird ([4] and [22]). In fact, after suitable translation, it is essentially a special case of their result. It follows in exactly the same way as Theorem 5.4, but appealing to Lemma 5.1(b).

Theorem 5.5. *Suppose $T \in \mathcal{L}(L_1)$ has non-zero atomic part T^a. Then there is a Borel subset B of K with $\lambda(B) > 0$ such that $T|L_1(B)$ is an isomorphism and $T(L_1(B))$ is complemented in L_1.*

For $0 \leq p < 1$ we cannot guarantee that $T(L_p(B))$ is complemented in general, by the results of Section 4. In fact consider the map $S : L_p([0, 1] \times [0, 1]) \to L_p([0, 1] \times [0, 1])$.

$$Sf(s, t) = f(\sigma s)$$

where $\sigma : [0, 1] \to [0, 1] \times [0, 1]$ is a measure preserving Borel isomorphism. If $S(L_p(B))$ is complemented for $B \subset [0, 1] \times [0, 1]$, then it is also complemented in $L_p(\sigma^{-1}B \times [0, 1])$, and this contradicts Corollary 4.5.

We shall say that $T \in \mathcal{L}(L_p)$ is *large* if there exist $U, V \in \mathcal{L}(L_p)$ with $UTV = I$. Otherwise T is *small*.

Theorem 5.6. *Suppose $T \in \mathcal{L}(L_p)$, where $0 \leq p < 1$, is given in the form of Theorems 1.1 and 3.2*

$$Tf(x) = \sum_{n=1}^{\infty} a_n(x) f(\sigma_n x) \qquad \text{a.e.}$$

Then the following conditions are equivalent:

(a) T is large.

(b) There exists $B \in \mathcal{B}$ with $\lambda(B) > 0$ and such that T maps $L_p(B)$ isomorphically onto a complemented subspace of L_p.

(c) There exists $B \in \mathcal{B}$ with $\lambda(B) > 0$, and $n \in \mathbb{N}$ such that $|a_n(x)| > 0$ $(x \in B)$ and $\sigma_n|B$ is injective.

Proof. (a) $\Rightarrow$ (c): Suppose $UTV = I$ where

$$Uf(x) = \sum_{n=1}^{\infty} b_n(x) f(\tau_n x) \qquad \text{a.e.}$$

$$Vf(x) = \sum_{n=1}^{\infty} c_n(x) f(\theta_n x) \qquad \text{a.e.}$$

Then the lattice moduli of T, U, V, are given by

$$|T| f(x) = \sum_{n=1}^{\infty} |a_n(x)| f(\sigma_n x) \qquad \text{a.e.}$$

$$|U| f(x) = \sum_{n=1}^{\infty} |b_n(x)| f(\tau_n x) \qquad \text{a.e.}$$

$$|V| f(x) = \sum_{n=1}^{\infty} |c_n(x)| f(\theta_n x) \qquad \text{a.e.}$$

Hence for $f \geq 0$

$$|T| \cdot |V| f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_m(x)|\, |c_n(\sigma_m x)| f(\theta_n \sigma_m x) \qquad \text{a.e.}$$

and the series converges absolutely. It follows easily that

$$\sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |b_\ell(x)|\, |a_m(\tau_\ell x)|\, |c_n(\sigma_m \tau_\ell x)|\, |f(\theta_n \sigma_m \tau_\ell x)| < \infty \qquad \text{a.e.}$$

for $f \in L_p$. It now follows that, if $f \in L_p$,

$$UTVf(x) = \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_\ell(x) a_m(\tau_\ell x) c_n(\sigma_m \tau_\ell x) f(\theta_n \sigma_m \tau_\ell x) \qquad \text{a.e.}$$

and

$$\sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |b_\ell(x)|\, |a_m(\tau_\ell x)|\, |c_n(\sigma_m \tau_\ell x)| < \infty \qquad \text{a.e.}$$

Hence by the uniqueness of the representation of I,

$$\delta(x) = \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_\ell(x) a_m(\tau_\ell x) c_n(\sigma_m \tau_\ell x) \delta(\theta_n \sigma_m \tau_\ell x) \qquad \text{a.e.}$$

Hence there is a closed set E of positive measure and $\ell, m, n \in \mathbb{N}$ such that $\tau_\ell|E$ is continuous, $x = \theta_n\sigma_m\tau_\ell x$ $(x \in E)$ and

$$|b_\ell(x)a_m(\tau_\ell x)c_n(\sigma_m\tau_\ell x)| > 0 \qquad x \in E.$$

Thus $\sigma_m|\tau_\ell(E)$ is injective and $\lambda(\tau_\ell E) > 0$ since $\lambda(E) > 0$ and $|b_\ell(x)| > 0$ $(x \in E)$.

$(c) \Rightarrow (b)$. Simply repeat the proof of Theorem 5.4, using Lemma 5.1(c) and Lemma 5.2.

$(b) \Rightarrow (a)$. Let Q be a projection onto $T(L_p(B))$, and let $V : T(L_p(B)) \to L_p(B)$ be the inverse of T. Then if $J : L_p \to L_p(B)$ is an isomorphism, $(J^{-1}VQ)TJ = I$.

Corollary 5.7. *The set $\mathcal{J}$ of small operators on $L_p(0 \le p < 1)$ is the unique maximal two-sided ideal in $\mathcal{L}(L_p)$.*

Proof. If $S, T \in \mathcal{J}$ then $S + T \in \mathcal{J}$ by 5.6(c). If $S \in \mathcal{J}$ and $T \in \mathcal{L}(L_p)$ then ST and $TS \in \mathcal{J}$. Hence $\mathcal{J}$ is an ideal, and it is clearly the unique maximal ideal from the definition.

6. Complemented subspaces and unconditional Schauder decompositions. An F-space X is *prime* if it is isomorphic to each of its infinite-dimensional complemented subspaces, and *primary* if $X \cong Y \oplus Z$ implies that either Y or $Z \cong X$. It is well-known that ℓ_p $(0 < p \le \infty)$ and c_0 are prime (Pełczyński [19], Stiles [24] and Lindenstrauss [13]) and it has recently been shown that L_p $(1 \le p < \infty)$ is primary (Enflo, see [4], [16] and [23]).

Our first result extends Enflo's theorem to the case $p < 1$ and provides an alternative proof for the case $p = 1$.

Theorem 6.1. *L_p $(0 < p \le 1)$ is primary.*

Proof. Let P and Q be complementary projections with representations $x \mapsto \nu_x$ and $x \mapsto \mu_x$. Then

$$\nu_x + \mu_x = \delta(x) \qquad \lambda\text{-a.e.}$$

In particular either $\nu_x\{x\} > 0$ or $\mu_x\{x\} > 0$ on a set of positive measure. Suppose the former; then, by Theorem 5.6, or if $p = 1$, Theorem 5.5, $P(L_p)$ contains a complemented subspace isomorphic to L_p. It now follows from the Pełczyński decomposition technique that $P(L_p) \cong L_p$.

In fact, we conjecture that L_p is prime for $0 < p < 1$. We shall now show that L_0 is prime; of course, L_0 has no complemented finite-dimensional subspaces, so we might say that L_0 is "ultra-prime".

Theorem 6.2. *L_0 is prime.*

Proof. Suppose $P \in \mathcal{L}(L_0)$ is a non-zero projection. Then we may represent P in the form

$$Pf(x) = \sum_{n=1}^{\infty} a_n(x)f(\sigma_n x) \qquad \text{a.e.}$$

as in Theorem 1.1. We may additionally suppose, as for the case $p > 0$, that $|a_n(x)| \geq |a_{n+1}(x)|$ a.e. ($n \in \mathbb{N}$) and that $\sigma_n x \neq \sigma_m x$ for $m \neq n$, a.e. We note that it is sufficient, by the Pełczyński decomposition technique and Theorem 5.6 to show that P is large. In this case we use the fact that $L_0 \cong \omega(L_0)$ (*i.e.* the Cartesian product of a countable number of copies of L_0).

Almost everywhere, there exists a least $N = N(x) < \infty$ such that $a_n(x) = 0$ for $n > N$. In fact, for convenience we may alter each a_n on a set of measure zero so that $N(x) < \infty$ everywhere. Choose a closed subset E on K with positive measure such that $N(x)$ is constant and non-zero on E (say $N(x) \equiv N$), and $\sigma_1, \cdots, \sigma_N$ are continuous on E.

Observe that

$$P^2 f(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n(x) a_m(\sigma_n x) f(\sigma_m \sigma_n x) \qquad \text{a.e.}$$

(the sum is everywhere finite). Since $P^2 = P$ we may appeal to the uniqueness of the representation of Theorem 1.2 (which is established exactly as in the case $p > 0$) to deduce

$$\sum_{n=1}^{\infty} a_n(x)\delta(\sigma_n x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n(x) a_m(\sigma_n x) \delta(\sigma_m \sigma_n x) \qquad \text{a.e.}$$

Thus if $1 \leq \ell \leq N$, for almost every $x \in E$ there exists $1 \leq n \leq N$ and $m \geq 1$ so that $\sigma_\ell x = \sigma_m \sigma_n x$ and $a_m(\sigma_n x) \neq 0$. We can then determine a closed subset F of E of positive measure such that for each ℓ, $1 \leq \ell \leq N$ there exist $1 \leq \varphi(\ell) \leq N$ and $\psi(\ell)$ such that $a_{\psi(\ell)}(\sigma_{\varphi(\ell)}(x)) \neq 0$ and

$$\sigma_\ell x = \sigma_{\psi(\ell)} \sigma_{\varphi(\ell)} x \qquad x \in F.$$

For some k and some ℓ, $1 \leq \ell \leq N$, $\varphi^k(\ell) = \ell$. Thus

$$\sigma_\ell x = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} \sigma_\ell x \qquad x \in F$$

where $a_{i_k}(\sigma_\ell x) \neq 0$ ($x \in F$). Clearly $\sigma_\ell F$ is closed, $\lambda(\sigma_\ell F) > 0$ and $\sigma_{i_k}|\sigma_\ell F$ is injective. Thus P is large.

If $0 < p \leq 1$, we extend these results a little bit further by considering unconditional Schauder decompositions. Lindenstrauss and Pełczyński [15] showed that if (X_n) is an unconditional Schauder decomposition of $C[0, 1]$ then at least one $X_n \cong C[0, 1]$. We show that the same result is true for ($0 < p \leq 1$); for $p = 1$, this result has been independently obtained, *via* a different proof by Starbird.

Lemma 6.3. *Suppose* $T_n \in \mathcal{L}(L_p)$ ($0 < p \leq 1$) *and there exists* $M < \infty$ *such that*

$$\left\| \sum_{i=1}^{n} \epsilon_i T_i \right\| \leq M$$

for all n and $\epsilon_1, \cdots, \epsilon_n = \pm 1$. *Then if* $x \mapsto \nu_x^n$ *represents* T_n,

$$\int_K \|\nu_x^n\|_p^p d\lambda(x) \to 0.$$

Proof. Let r_n be the sequence of Rademacher functions on [0, 1]; *i.e.* $r_n(t) = \operatorname{sgn} \sin 2^k\pi t$ $(0 \le t \le 1)$. Since $\mathcal{M}(K)$ is an L_1-space, and hence of cotype 2, there exists $\alpha > 0$ such that if $\mu_1, \cdots, \mu_n \in \mathcal{M}(K)$

$$\int_0^1 \left\| \sum_{i=1}^n r_i(t)\mu_i \right\| dt \ge \alpha\left(\sum_{i=1}^n \|\mu_i\|^2 \right)^{1/2}.$$

If $0 < p \le 1$, we can also show that for some $\alpha = \alpha(p) > 0$,

$$\int_0^1 \left\| \sum_{i=1}^n r_i(t)\mu_i \right\|_p^p dt \ge \alpha^p\left(\sum_{i=1}^n \|\mu_i\|_p^2 \right)^{p/2}.$$

Hence for $n \in \mathbb{N}$,

$$\begin{aligned} \alpha^p \int_K \left(\sum_{i=1}^n \|\nu_x^i\|_p^2 \right)^{p/2} d\lambda(x) &\le \int_0^1 \int_K \left\| \sum_{i=1}^n r_i(t)\nu_x^i \right\|_p^p d\lambda(x)dt \\ &\le \int_0^1 \left\| \sum_{i=1}^n r_i(t)T_i \right\|^p dt \\ &\le M^p. \end{aligned}$$

Thus

$$\sum_{i=1}^\infty \|\nu_x^i\|_p^2 < \infty \qquad \text{a.e.}$$

and $\left(\sum_{i=1}^\infty \|\nu_x^i\|_p^2 \right)^{p/2}$ is integrable. Hence by the Dominated Convergence Theorem

$$\int_K \|\nu_x^n\|_p^p d\lambda(x) \to 0.$$

Theorem 6.4. *Suppose* (X_n) *is an unconditional Schauder decomposition of* L_p $(0 < p \le 1)$. *Then for at least one value of* n, $X_n \cong L_p$.

Proof. Let T_n be the associated projection, and let $x \mapsto \nu_x^n$ represent T_n. From the preceding lemma, we clearly have that

$$\lim_{m,n\to\infty} \int_K \left\| \sum_{n+1}^m \nu_x^i \right\|_p^p d\lambda(x) = 0.$$

Thus we may find an increasing sequence (n_k) with $n_1 = 0$ such that

$$\sum_{k=1}^\infty \int \left\| \sum_{n_k+1}^{n_{k+1}} \nu_x^i \right\|_p^p d\lambda(x) < \infty.$$

Thus,

$$\sum_{k=1}^{\infty}\left(\sum_{n_k+1}^{n_{k+1}} \nu_x^i\right) \text{ converges in } \mathcal{M}(K) \qquad x\text{-a.e.}$$

Let (f_n) be a dense countable subset of $C(K)$; then

$$\sum_{k=1}^{\infty}\left(\sum_{n_k+1}^{n_{k+1}} \int f_n d\nu_x^i\right) = f_n(x) \qquad \text{a.e.}$$

Hence

$$\sum_{k=1}^{\infty}\left(\sum_{n_k+1}^{n_{k+1}} \nu_x^i\right) = \delta(x) \qquad \text{a.e.}$$

Hence for some n, $\nu_x^n\{x\} > 0$ on a set of positive measure, so that T_n is large. As usual, this implies $T_n(L_p) \cong L_p$. In the case $p = 1$ apply Theorem 5.5.

7. Quotients of L_p. We start by constructing a quotient Y of $L_0 = L_0(0, 1)$ such that the quotient mapping $q : L_0 \to Y$ is not an isomorphism on any subspace $L_0(B)$ (contrast Theorem 5.4).

Theorem 7.1. *There is a proper closed subspace X of L_0 such that for any Borel subset B of $[0, 1]$, the quotient map $q : L_0 \to L_0/X$ is not an isomorphism on $L_0(B)$. Then $\mathcal{L}(L_0/X, L_0) = \{0\}$.*

Proof. Let (θ_n) be a strictly decreasing sequence of real numbers such that $\theta_0 = 2$ and $\theta_n \downarrow 1$. Let $(m_n : n \geq 1)$ be an increasing sequence of integers such that

$$\sum_{n=1}^{\infty} \frac{1}{m_n} \log \frac{\theta_{n-1} + \theta_n}{\theta_{n-1} - \theta_n} < \frac{1}{4}.$$

Let $M_1 = 1$ and $M_n = m_1 \cdots m_{n-1}$. We now define $u_0(t) \equiv 1$ and for $1 \leq n < \infty$, $1 \leq k \leq M_n$

$$u_{n,k}(t) = e^j \quad \frac{(k-1)m_n + j - 1}{M_{n+1}} < t < \frac{(k-1)m_n + j}{M_{n+1}} \quad (1 \leq j \leq m_n)$$

$$= 0 \qquad \text{otherwise.}$$

Let $X_0 = \text{lin}\{u_{n,k}; 1 \leq n < \infty, 1 \leq k \leq M_n\}$. We show that $u_0 \notin \bar{X}_0$. Suppose $g \in X_0$ and

$$\lambda\{x : |2 + g(x)| > 1\} < \frac{1}{2}.$$

Let

$$g = \sum_{n=1}^{N} h_n$$

where $h_n \in \operatorname{lin}\{u_{n,k} : k = 1,2, \cdots, M_n\}$; suppose $g_n = \sum_{i=1}^{n} h_i$, and $g_0 = 0$. Let

$$\lambda\{x : |2 + g_n(x)| < \theta_n\} = \epsilon_n.$$

Clearly $\epsilon_0 = 0$. Suppose $n > 0$, and consider the interval $I = ((k-1)M_n^{-1}, kM_n^{-1})$. Then g_{n-1} is constant on I. Suppose $|2 + g_{n-1}(x)| \geq \theta_{n-1}$. On I, $h_n = \alpha u_{n,k}$ where $\alpha \in R$, and hence $|2 + g_n(x)| < \theta_n$ only when

$$\frac{(k-1)m_m + j - 1}{M_{n+1}} < x < \frac{(k-1)m_n + j}{M_{n+1}}$$

where

$$\theta_{n-1} - \theta_n < |\alpha|e^j < \theta_{n-1} + \theta_n.$$

Thus, if $\alpha \neq 0$,

$$\log(\theta_{n-1} - \theta_n) - \log|\alpha| < j < \log(\theta_{n-1} + \theta_n) - \log|\alpha|.$$

Hence

$$\lambda\{x \in I : |2 + g_n(x)| < \theta_n\} < \frac{2}{M_{n+1}} \log\frac{\theta_{n-1}+\theta_n}{\theta_{n-1}-\theta_n}.$$

Thus

$$\epsilon_n \leq \epsilon_{n-1} + \frac{2}{m_n}\log\frac{\theta_{n-1} + \theta_n}{\theta_{n-1} - \theta_n}.$$

Hence $\epsilon_N < \frac{1}{2}$, and we have reached a contradiction. Thus $u_0 \in \bar{X}_0$.

Let $X = \bar{X}_0$; then X is a proper closed subset of L_0. Now consider the quotient map $q : L_0 \to L_0/X$. Suppose B is a Borel subset of positive measure. For each $n \in \mathbb{N}$, let U_n be an open set $U_n \supset B$ such that $\lambda(U_n) < \left(1 + \frac{1}{n}\right)\lambda(B)$. Then there exists a disjoint sequence $(J_{n,k} : k \in \mathbb{N})$ of intervals of the type $(M_n^{-1}(j-1), M^{-1}j)$ such that $\lambda\left(U_n \setminus \bigcup_k J_{n,k}\right) = 0$. Thus there exists $u_n \in X$ with $u_n(x) \neq 0$ a.e. $(x \in U_n)$. We may suppose

$$\lambda\{x : u_n(x) < 1\} < \frac{1}{n}\lambda(B).$$

Let $v_n = P_B u_n \in L_0(B)$. Then $q(v_n) = q(v_n - u_n) \to 0$ but

$$\lambda\{x : v_n(x) \geq 1\} > \left(1 - \frac{2}{n}\right)\lambda(B).$$

Hence $q|L_0(B)$ is not an isomorphism.

Now if $T \in \mathscr{L}(L_0|X; L_0)$, then $Tq : L_0 \to L_0$ cannot be an isomorphism on $L_0(B)$ for any Borel subset B of positive measure. Hence by Theorem 5.4, $Tq = 0$ *i.e.* $T = 0$.

We do not know whether L_p $(0 < p < 1)$ always has a subspace N such that the quotient map $q : L_p \to L_p/N$ is not an isomorphism on any subspace $L_p(B)$; in particular, must every quotient of L_p contain a subspace isomorphic to L_p? This question is open even for $p = 0$. Note that every quotient of L_p contains a subspace isomorphic to ℓ_2 ([7]) $(0 \leq p < 1)$.

Now let Γ be the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and let λ be normalized Lebesgue measure on Γ. Let $L_p(\Gamma)$ denote the space of *complex-valued* L_p-functions on Γ. For $0 < p \leq 1$, let H_p be the closed subspace of L_p generated by the functions $e_n \in L_p$, $e_n(z) = z^n$ $(n \geq 0)$.

Theorem 7.2. *If* $0 < p < 1$ $\mathscr{L}(L_p(\Gamma)/H_p; L_p(\Gamma)) = \{0\}$.

Proof. Let $T \in \mathscr{L}(L_p(\Gamma)/H_p; L_p(\Gamma))$ and suppose q is the quotient map. Let $x \mapsto \nu_x$ be the representation of Tq. Then $Tq(e_n) = 0$ $(n \geq 0)$ *i.e.*

$$\int_\Gamma z^n d\nu_x(z) = 0 \qquad n \geq 0 \qquad x\text{-a.e.}$$

and so by the F. and M. Riesz theorem ([6] p. 47), ν_x is λ-continuous almost everywhere. As $\|\nu_x\|_p < \infty$ a.e., $\nu_x = 0$ and hence $T = 0$.

We point out two applications of Theorem 7.2. First, we recall ([9]) that an F-space X is *ultratransitive* if any operator of finite rank mapping a closed subspace N of X into X can be extended to an endomorphism of X. It is known that there is, for each $0 < p < 1$, a separable p-Banach X, such that $\ell_p(X)$ is ultratransitive. However ℓ_p is not ultratransitive, and neither is L_p.

Corollary 7.3. $L_p(\Gamma)$ *is not ultratransitive.*

Proof. If $T \in \mathscr{L}(L_p(\Gamma))$ and $T(H_p) = 0$ then $T = 0$.

It is also known that there is a separable p-Banach space which is universal for all separable p-Banach spaces. However no functional representation of such a space is known. The following corollary destroys one possible candidate.

Corollary 7.4. $L_p(\Gamma)/H_p$ *does not embed into* $C(K; L_p(\Gamma))$ *if* K *is a compact metric space.*

(Here $C(K, L_p(\Gamma))$ denote the space of continuous $L_p(\Gamma)$-valued functions on K with the supremum quasi-norm).

Proof. If $T : L_p(\Gamma)/H_p \to C(K, L_p(\Gamma))$, then for each $x \in K$, the map $u \mapsto Tu(x)$ sends $L_p(\Gamma)/H_p$ into $L_p(\Gamma)$. Hence $T = 0$.

For $p = 1$, the argument of Theorem 7.2 can be used to show that H_1 is not complemented in $L_1(\Gamma)$; this is well-known ([14], [6] p. 154). We give here a slightly stronger result, which is due to Pełczyński [20], by a different proof.

Theorem 7.5. L_1/H_1 *does not embed into* $L_1(\Gamma)$.

Proof. Suppose $V : L_1/H_1 \to L_1(\Gamma)$ is such an embedding. Then Vq has a representation $x \mapsto \nu_x$, and the argument of 7.2 shows that ν_x is λ-continuous almost everywhere. This means that Vq is differentiable (we omit the easy verification of this fact) and hence has the Dunford-Pettis property. Consider $e_{-n} \to 0$ weakly, and hence $\|Vq(e_{-n})\| \to 0$. However

$$\|qe_{-n}\| \geq \int_\Gamma e_n(z)e_{-n}(z)d\lambda(z) = 1.$$

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