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BANACH SPACES EMBEDDING INTO L_0

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ABSTRACT
Our main result in this paper is that a Banach space X embeds into L_1 if and only if $l_1(X)$ embeds into L_0; more generally if $1 \leqq p < 2$, X embeds into L_p if and only if $l_p(X)$ embeds into L_0.

1. Introduction

It is still unknown whether every Banach space which embeds into $L_0 = L_0(0,1)$ is isomorphic to a subspace of L_1. This problem was suggested by Kwapien [9]. Our main result in this paper is that a Banach space X embeds into L_1 if and only if $l_1(X)$ embeds into L_0; more generally if $1 \leqq p < 2$, X embeds into L_p if and only if $l_p(X)$ embeds into L_0.

Before discussing the problem and the contents of the paper, we introduce some notation. Throughout the paper Ω will denote a compact metric space, Σ the σ-algebra of Borel subsets of Ω and P a nonatomic probability measure on Σ. Of course there is no loss of generality in taking $\Omega = [0,1]$ and P Lebesgue measure on $[0,1]$. For $0 \leqq p < \infty$, $L_p(\Omega, \Sigma, P)$ will be abbreviated to L_p. We also denote by $L(p,\infty)$ the Lorentz space, weak L_p, of all $f \in L_0(\Omega,\Sigma,P)$ so that

$$\|f\|_{p,\infty} = \sup_{0<x<\infty} x(P(|f|>x))^{1/p} < \infty.$$

Let us say that a linear operator $V: X \to L_p$ (or $V: X \to L(p,\infty)$) is a *strong embedding* if V is an isomorphism onto its range, and the topology of L_p (or $L(p,\infty)$) on its range coincides with the L_0-topology (convergence in measure).

For $1 \leqq p \leqq 2$, a Banach space X is said to be of type p (Rademacher) if there

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is a constant C so that

$$\operatorname{Avg}\left\|\sum_{i=1}^{n}\varepsilon_i x_i\right\|^p \leqq C\sum_{i=1}\|x_i\|^p$$

where the average is taken over all choices of sign $\varepsilon_i = \pm 1$. Every Banach space is of type one.

The first progress on Kwapien's problem was made by Nikishin [14] who showed that a Banach subspace of L_0 is in fact isomorphic and is a subspace of L_p for every $p<1$. Later [15] (cf. expositions in [13] and [5]) he refined this result to establish the following factorization theorem:

THEOREM 1.1. (*Nikishin*). *Let X be a Banach space of type p $(1\leqq p<2)$ and let $V: X\to L_0$ be any continuous linear operator. Then given $\varepsilon>0$, there exists a set E with $P(E)\geqq 1-\varepsilon$ so that if*

$$Wx = 1_E Vx$$

then W is a bounded linear operator from X into $L(p,\infty)$.

COROLLARY 1.2. *Every Banach subspace of L_0 can be strongly embedded in $L(1,\infty)$.*

COROLLARY 1.3. *Every Banach subspace of L_0 of type p can be strongly embedded in $L(p,\infty)$.*

We obtain the results announced in the introduction by a close analysis of the spaces $L(p,\infty)$. Our methods hinge on the existence of non-trivial continuous linear functionals on the non-locally convex quasi-Banach space $L(1,\infty)$. This fact was first observed by Cwikel and Sagher [2] and recently the dual of $L(1,\infty)$ has been studied by Cwikel and Fefferman [1] and by Kupka and Peck [8]. Our methods are quite similar to techniques in [8] but were obtained independently.

In fact for convenience, in Section 2, we study not $L(p,\infty)$ but instead the l_∞-product $l_\infty(L(p,\infty))$ which we abbreviate to $\mathcal{Y}_p$. Thus $\mathcal{Y}_p$ consists of all sequences $F=(f_n)$ where $f_n\in L(p,\infty)$ and

$$\|F\| = \sup_n \|f_n\|_{p,\infty} < \infty.$$

We do, however, give an application of these ideas to $L(p,\infty)$ showing the standard embedding of L_p into $L(p,\infty)$ using p-stable processes yields a complemented subspace.

In Section 3 we give our applications to Banach subspaces of L_0 and in Section 4 we make our closing remarks.

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2. An explicit linear functional on $\mathcal{Y}_1$

Let $\mathcal{U}$ be any ultrafilter on $(2,\infty)$ which includes each of the sets (x,∞) for $x>2$. Let $\mathcal{V}$ be any ultrafilter on $\mathbf{N}$ containing each of the sets $\{m\in\mathbf{N}: m\geqq n\}$ for $n\in\mathbf{N}$.

For $f\in L_0$ and $x\geqq 0$ we define the truncation $\tau_x f$ by

$$\begin{aligned}\tau_x f(\omega)&=f(\omega) &&\text{if } |f(\omega)|\leqq x,\\ &=x &&\text{if } f(\omega)>x,\\ &=-x &&\text{if } f(\omega)<-x.\end{aligned}$$

Now suppose $F=(f_n)\in\mathcal{Y}_1$. We define

$$\phi(F)=\lim_{x\in\mathcal{U}}\frac{1}{\log x}\lim_{n\in\mathcal{V}}\mathcal{E}(\tau_x f_n),$$

where $\mathcal{E}$ denotes expectation.

LEMMA 2.1. *ϕ is a positive linear functional on $\mathcal{Y}_1$ and $\|\phi\|=1$.*

PROOF. First we note that ϕ is well-defined and $\|\phi(F)\|\leqq\|F\|$ for $F\in\mathcal{Y}_1$. In fact if $f\in L(1,\infty)$, with $\|f\|\leqq 1$,

$$|\mathcal{E}(\tau_x f)|\leqq 1+\log x,\qquad x\geqq 1$$

and hence ϕ is well-defined and

$$|\phi(F)|\leqq\|F\|.$$

Now we observe that

$$\phi(F+G)=\phi(F)+\phi(G),\qquad |F|\wedge|G|=0; \tag{1}$$

$$\phi(F)\leqq\phi(G),\qquad \text{whenever } F\leqq G; \tag{2}$$

$$\phi(-F)=-\phi(F),\qquad F\in\mathcal{Y}_1. \tag{3}$$

Now suppose $0<\alpha<1$. If $f\in L(1,\infty)$,

$$\begin{aligned}\mathcal{E}(|\tau_x(\alpha f)-\tau_{\alpha x}(\alpha f)|)&\leqq(x-\alpha x)P(|f|>x)\\ &\leqq(1-\alpha)\|f\|.\end{aligned}$$

Hence it follows easily that

$$\phi(\alpha F)=\lim_{x\in\mathcal{U}}\frac{1}{\log x}\lim_{n\in\mathcal{V}}\mathcal{E}(\tau_{\alpha x}(\alpha f_n))$$
$$=\alpha\phi(F).$$

We conclude that, using (3),

$$\phi(\alpha F)=\alpha\phi(F),\qquad \alpha\in\mathbf{R},\quad F\in\mathcal{Y}_1. \tag{4}$$

Now suppose $F,G\geqq 0$ and let $H=F+G$. Fix $N\in\mathbf{N}$. Then for $k=1,2,\dots,N-1$ and $n\in\mathbf{N}$ we set

$$A_{kn}=\left\{\omega\in\Omega:\frac{k-1}{N}h_n(\omega)\leqq f_n(\omega)<\frac{k}{N}h_n(\omega)\right\}.$$

For $k=N$ we set

$$A_{Nn}=\left\{\omega\in\Omega:\frac{N-1}{N}h_n(\omega)\leqq f_n(\omega)\leqq h_n(\omega)\right\}.$$

For each $n\in\mathbf{N}$, $A_{1n},\dots,A_{Nn}$ partitions Ω into Σ-measurable sets and

$$\frac{k-1}{N}h_n1_{A_{kn}}\leqq f_n1_{A_{kn}}\leqq\frac{k}{N}h_n1_{A_{kn}},$$

$$\frac{N-k}{N}h_n1_{A_{kn}}\leqq g_n1_{A_{kn}}\leqq\frac{N-k+1}{N}h_n1_{A_{kn}}.$$

Now using properties (1), (2) and (4)

$$\frac{N-1}{N}\phi(H)\leqq\phi(F)+\phi(G)\leqq\frac{N+1}{N}\phi(H)$$

so $\phi(F)+\phi(G)=\phi(H)$. It follows easily that ϕ is linear.

By considering any f so that $P(f\geqq x)=x^{-1}$ for $x\geqq 1$ we see that $\|\phi\|=1$.

Note that there is an embedding $J:L(1,\infty)\to\mathcal{Y}_1$ given by $J(f)=(f,f,\dots)$. Let

$$\phi_0(f)=\phi_0J(f)=\lim_{x\in\mathcal{U}}\frac{1}{\log x}\mathcal{E}(\tau_xf)$$

which is a positive linear functional on $L(1,\infty)$ with $\|\phi_0\|=1$ (compare [8]).

LEMMA 2.2. *Suppose* $1<p<\infty$ *and* $1/p+1/q=1$. *If* $F\in\mathcal{Y}_p$ *and* $G\in\mathcal{Y}_q$ *then* $FG\in\mathcal{Y}_1$ *and*

$$|\phi(FG)|\leqq(\phi(|F|^p))^{1/p}(\phi(|G|^q))^{1/q}.$$

PROOF. Simply note that

$$|FG| \leqq \frac{|F|^p}{p} + \frac{|G|^q}{q}$$

so that $FG \in \mathcal{Y}_1$ and

$$\phi(|FG|) \leqq \frac{1}{p}\,\phi(|F|^p) + \frac{1}{q}\,\phi(|G|^q).$$

In particular if $\phi(|F|^p) \leqq 1$, $\phi(|G|^q) \leqq 1$ then $\phi(|FG|) \leqq 1$ and the lemma follows by homogeneity.

LEMMA 2.3. *If* $1 \leqq p < \infty$ *then* $F \to (\phi(|F|_p))^{1/p}$ *is a semi-norm on* $\mathcal{Y}_p$.

PROOF. For $p = 1$ this is clear. For $p > 1$ note that

$$\begin{aligned}\phi(|F+G|^p) &\leqq \phi(|F|\cdot|F+G|^{p-1}) + \phi(|G|\cdot|F+G|^{p-1})\\ &\leqq ((\phi(|F|^p))^{1/p} + \phi(|G|^p)^{1/p})\phi(|F+G|^p)^{1/q},\end{aligned}$$

where $1/p + 1/q = 1$. The lemma follows.

On $\mathcal{Y}_p$ let

$$|||F||| = \phi(|F|^p)^{1/p}.$$

Then $|||\cdot|||$ is a lattice semi-norm on $\mathcal{Y}_p$ satisfying

$$|||F+G|||^p = |||F|||^p + |||G|||^p, \qquad |F| \wedge |G| = 0.$$

Let $M_p = \{F \in \mathcal{Y}_p : |||F|||_p = 0\}$. Then with the quotient norm $\mathcal{Y}_p / M_p$ is a normed lattice whose completion is an AL_p-space (see Lacey [11] Chapter 5). We denote this space by $\mathcal{X}_p$ and denote by K_p the induced map $K_p : \mathcal{Y}_p \to \mathcal{X}_p$.

We conclude this section with an elementary application of these ideas. We denote by J_p the natural map $J_p : L(p,\infty) \to \mathcal{Y}_p$ given by $J_p(f) = (f, f, \dots)$. Note that

$$\|K_pJ_pf\| = (\phi_0(|f|^p))^{1/p}$$

where $\phi_0 = \phi J_1$ as above.

We recall (cf. [7]) that for $1 \leqq p \leqq 2$ there is an embedding of L_p into $L(p,\infty)$ using a p-stable process. Precisely, there is a linear map $V_p : L_p \to L(p,\infty)$ so that

(i) V_pf and V_pg are independent whenever $|f| \wedge |g| = 0$.

(ii) $\mathcal{E}(\exp(itV_pf)) = \exp(-|t|^p\|f\|_p^p)$.

THEOREM 2.5. *The range of* V_p *is complemented in* $L(p,\infty)$ *for* $1 \leqq p < 2$.

PROOF. Note that for $f \in L(p,\infty)$

$$\begin{aligned}\phi_0(|f|^p) &= \lim_{x\in\mathcal{U}} \frac{1}{\log x}\, \mathcal{E}(\tau_x\,|f|^p)\\ &= \lim_{x\in\mathcal{U}} \frac{1}{\log x} \int_0^x P(|f|^p > t)dt\\ &= \lim_{x\to\infty} xP(|f|^p > x)\end{aligned}$$

whenever this limit exists.

Now if $\|f\|_p = 1$, $\lim_{x\to\infty} xP(|V_pf|^p > x) = C_p$ (see [17]) where

$$C_p = \left[\int_0^\infty \frac{\sin x}{x^p}\, dx\right]^{-1}.$$

Thus for $f \in L_p$,

$$\phi_0(|V_pf|^p)^{1/p} = C_p\,\|f\|_p.$$

Hence $K_pJ_pV_p(L_p)$ is an isometric copy of L_p contained in $\mathcal{X}_p$. This implies that there is a projection of norm one of $\mathcal{X}_p$ onto $K_pJ_pV_p(L_p)$ (this result goes back to Pelczynski [16] for the case $p > 1$; see also Dor [3]).

Now let $W : K_pJ_pV_p(L_p) \to V_p(L_p)$ be the inverse of K_pJ_p. Then WPK_pJ_p is the required projection.

REMARK. In the case $p = 1$, $\|W\| = 1$ and so the projection has norm one. In fact if $\|f\| = 1$ then

$$P(|V_1f| > x) = \frac{2}{\pi}\tan^{-1}\frac{1}{x}$$

so that

$$\begin{aligned}\|V_1f\| &= \sup_{x>0} \frac{2}{\pi}\, x \tan^{-1}\frac{1}{x}\\ &= \lim_{x\to\infty} \frac{2}{\pi}\, x\tan^{-1}\frac{1}{x}\\ &= \frac{2}{\pi}\end{aligned}$$

and

$$\phi_0(|V_1f|) = \|K_1J_1V_1f\| = 2\pi^{-1}$$

so that K_1J_1 is an isometry on $V_1(L_1)$.

3. Applications

The following proposition will be useful later on. Although we know of no explicit reference for it, we believe that it is already known and that it is due to Maurey.

PROPOSITION 3.1. *Suppose* $1 < p < 2$, *and* X *is a Banach space so that* $l_p(X)$ *embeds into* L_0. *then* X *(and* $l_p(X)$*) are of type* p.

PROOF. We shall regard $l_p(X)$ as a tensor product of l_p and X, and denote by (e_n) the standard vector basis of l_p.

We may assume that we have a strong embedding $V: l_p(X) \to L_q$ where $0 < q \leqq 1$. It follows that there are constants $M < \infty$ and $\varepsilon > 0$ so that

(1) $\| Vu \|_q \leqq M \| u \|$, $u \in l_p(X)$,

(2) $\| Vu . 1_F \|_q \geqq M^{-1} \| u \|$, $u \in l_p(X)$,

whenever $P(F) \geqq 1 - \varepsilon$.

We shall need two further facts. Suppose $\{\varepsilon_n\}$ is a sequence of independent Bernoulli random variables on some probability space (Ω', Σ', P') where $P'\{\varepsilon_n = 1\} = P'\{\varepsilon_n = -1\} = \frac{1}{2}$. Then there is a constant $\gamma > 0$ so that for $f_1, \dots, f_n \in L_q$

$$\gamma \| (\Sigma |f_i|^2)^{1/2} \|_q \leqq (\mathscr{E}'(\| \Sigma \varepsilon_i f_i \|^q))^{1/q} \leqq \| (\Sigma |f_i|^2)^{1/2} \|_q .$$

This is a simple deduction from Khintchine's inequality (see [12] p. 50).

Secondly we observe that $L(p/2, \infty)$ is a $p/2$-normable; see [5] for a proof, although the result was known earlier [13]. Hence there is a constant C_0 so that for $f_1, \dots, f_n \in L(p, \infty)$

$$\| (\Sigma |f_i|^2)^{1/2} \|_{p,\infty} \leqq C_0 \left(\sum_{i=1}^{n} \| f_i \|_{p,\infty}^p \right)^{1/p} .$$

To prove this simply note $\| \Sigma |f_i|^2 \|_{p/2,\infty} \leqq C_1 (\Sigma \| |f_i|^2 \|_{p/2,\infty}^{p/2})^{2/p}$. Hence as the injection $L(p, \infty) \to L_q$ is continuous we have a constant C_1 so that for $f_1, \dots, f_n \in L(p, \infty)$,

$$\| (\Sigma |f_i|^2)^{1/2} \|_q \leqq C_1 \left(\sum_{i=1}^{n} \| f_i \|_{p,\infty}^p \right)^{1/p} .$$

Combining these remarks we see that there is a constant C so that for $f_1, \dots, f_n \in L(p, \infty)$

$$(\mathscr{E}'(\| \Sigma \varepsilon_i f_i \|_q^q))^{1/q} \leqq C \left(\sum_{i=1}^{n} \| f_i \|_{p,\infty}^p \right)^{1/p} .$$

Fix $A = (2/\varepsilon)^{1/q} M / \gamma$.

Now let us suppose $w_1,\dots,w_n \in X$ with $\|w_i\|=1$. Set

$$f_{ij} = V(e_j \otimes w_i)$$

for $j=1,2,\dots$ and $1 \leqq i \leqq n$.

We now select a sequence of disjoint sets $E_j \in \Sigma$ by induction.

For convenience let $E_0=\emptyset$. If E_k have been selected for $k<j$, we let $F=E_0\cup\cdots\cup E_{j-1}$. For each $\delta>0$ and $1\leqq i\leqq n$, let

$$G_i(\delta)=\{\omega : |f_{ij}(\omega)| \geqq A\delta^{-1/p} \text{ and } \omega \notin F\}.$$

Let $G_i(0)=\emptyset$ for $1\leqq i\leqq n$.

Let

$$\rho(\delta)=\max_{1\leqq i\leqq n} P(G_i(\delta)).$$

Pick $\delta_j=\sup\{\delta : \rho(\delta)\geqq\delta\}$. Since ρ is increasing $\rho(\delta_j)\geqq\delta_j$, and so there is a set E_j disjoint from F and $k(j)$ with $1\leqq k(j)\leqq n$ so that

(1) $|f_{k(j),j}(\omega)|\geqq A\delta_j^{-1/p}$, $\omega\in E_j$.

(2) If $H\in\Sigma$ is disjoint from $E_1,\dots,E_{j-1}$ and if $1\leqq i\leqq n$ is such that

$$|f_{ij}(\omega)|\geqq AP(H)^{-1/p}, \qquad \omega\in H,$$

then $P(H)\leqq P(E_j)$.

Note here that if $\delta_j=0$, which is possible, then we are taking $E_j=\emptyset$ and (1) and (2) will still be satisfied ($\delta_j^{-1/p}=\infty$!).

Note that $\Sigma\delta_j\leqq 1$ so that for any $N\in\mathbf{N}$,

$$\mathscr{E}'\left(\left\|\sum_{j=1}^{N}\delta_j^{1/p}\varepsilon_j f_{k(j),j}\right\|_q\right)\leqq M\gamma^{-1}$$

and hence

$$\left\|\left(\sum_{j=1}^{N}\delta_j^{2/p}|f_{k(j),j}|^2\right)^{1/2}\right\|_q\leqq M\gamma^{-1}.$$

As $(\Sigma\delta_j^{2/p}|f_{k(j),j}|^2)^{1/2}\geqq A$ on $E_1\cup\cdots\cup E_N$ we conclude

$$P(E_1\cup\cdots\cup E_N)\leqq\left(\frac{M}{\gamma A}\right)^q\leqq\tfrac{1}{2}\varepsilon.$$

Let $E=\bigcup_{i=1}^{\infty}E_i$. Then $P(E)\leqq\frac{1}{2}\varepsilon$.

Now fix j so that $n\delta_j\leqq\frac{1}{2}\varepsilon$. For each j let

$$G_i=\{\omega : \omega\notin E, |f_{ij}(\omega)|\geqq A\delta_j^{-1/p}\}.$$

Then $P(G_i) \leqq \delta_j$ for $i = 1, 2, \dots, n$. Let $F = \Omega \backslash (E \cup \bigcup_{i=1}^n G_i)$. Then $P(F) \geqq 1 - \varepsilon$ and by construction

$$\| 1_F \cdot f_{ij} \|_{p,\infty} \leqq A, \qquad i = 1, 2, \dots, n.$$

Now for any $\alpha_1, \dots, \alpha_n$

$$\begin{aligned}
\left(\mathscr{E}' \left(\left\| \sum_{i=1}^n \varepsilon_i \alpha_i w_i \right\|_q^q \right) \right)^{1/q} &\leqq M(\mathscr{E}'(\| \Sigma \varepsilon_i \alpha_i V(e_j \otimes w_i) \cdot 1_F \|_q^q))^{1/q} \\
&= M(\mathscr{E}'(\| \Sigma \varepsilon_i \alpha_i f_{ij} \cdot 1_F \|_q^q))^{1/q} \\
&\leqq CM \left(\sum_{i=1}^n | \alpha_i |^p \| f_{ij} \cdot 1_F \|_{p,\infty}^p \right)^{1/p} \\
&\leqq ACM \left(\sum_{i=1}^n | \alpha_i |^p \right)^{1/p}.
\end{aligned}$$

By a well-known result of Kahane [4], this shows that X is of type p.

Lemma 3.2. *Suppose $0 < q < p < 2$ and $1 \leqq M < \infty$ are given. Then there is a constant $C = C(p, q, M)$ so that for any $n \in \mathbf{N}$ and any linear operator $V : l_p^{(n)} \to L_q$ which satisfies*

$$M^{-1} \| u \| \leqq \| Vu \|_q \leqq M \| u \|, \qquad u \in l_p^{(n)},$$

then if $f_i = Ve_i$ we have

$$C^{-1} n^{1/p} \leqq \| \max | f_i | \|_q \leqq C n^{1/p}.$$

Proof. We suppose $\{\varepsilon_i\}_{i=1}^\infty$ and $\{\eta_i\}_{i=1}^\infty$ are two sequences of independent indentically distributed random variables on some probability space. We suppose each ε_i is Bernoulli with $P'\{\varepsilon_i = +1\} = P'\{\varepsilon_i = -1\} = \frac{1}{2}$, while each η_i is r-stable where r is chosen so that $p < r < 2$. Thus

$$\mathscr{E}'(e^{it\eta}) = e^{-|t|^r}$$

for any $\eta = \eta_j$.

Now

$$\int \left| \sum_{i=1}^n \varepsilon_i f_i \right|^q dP \geqq M^{-q} n^{q/p}$$

so that

$$\int \mathscr{E}'(| \Sigma \varepsilon_i f_i |^q) dP \geqq M^{-q} n^{q/p}$$

and by Khintchine's inequality

$$\|(\Sigma|f_i|^2)^{1/2}\|_q \geqq M^{-1}n^{1/p}.$$

Conversely

$$\mathscr{E}'\left(\int\left|\sum_{i=1}^n \eta_i f_i\right|^q dP\right) \leqq M^q(\mathscr{E}'(\Sigma|\eta_i|^p))^{q/p}$$

$$= c_1^q M^q n^{q/p}$$

where $c_1 = \|\eta_1\|_p < \infty$.

Hence

$$c_2^q \int (\Sigma|f_i|^r)^{q/r} dP \leqq c_1^q M^q n^{q/p}$$

where $c_2 = \|\eta_1\|_q < \infty$. Thus

$$\left\|\left(\sum_{i=1}^n |f_i|^r\right)^{1/r}\right\|_q \leqq c_1 c_2^{-1} M n^{1/p}.$$

Hence

$$\|\max|f_i|\,\|_q \leqq c_1 c_2^{-1} M n^{1/p}.$$

Also

$$\sum_{i=1}^n |f_i|^2 \leqq \left(\max_{i\leqq n}|f_i|\right)^{2-r}\left(\sum_{i=1}^n |f_i|^r\right)$$

so that by Holder's inequality

$$\left\|\left(\sum_{i=1}^n |f_i|^2\right)^{1/2}\right\|_q \leqq \left\|\left(\sum_{i=1}^N |f_i|^r\right)^{1/r}\right\|_q^{r/2} \left\|\max_{i\leqq n}|f_i|\right\|_q^{1-r/2}$$

or

$$M^{-1}n^{1/p} \leqq (c_1 c_2^{-1} M n^{1/p})^{r/2} \|\max|f_i|\,\|_q^{1-r/2}.$$

Thus

$$\|\max|f_i|\,\|_q \geqq \left(\frac{c_2}{Mc_1}\right)^{r/(2-r)} M^{-2/(2-r)} n^{1/p}$$

and the lemma is proved.

LEMMA 3.3. *Suppose $0 < q < p < 2$ and $0 < M < \infty$. Then there is a constant $\gamma = \gamma(p,q,M)$ so that whenever $V: l_p \to L(p,\infty)$ is an embedding such that*

$$\| Vu \|_{p,\infty} \leqq M \| u \|, \qquad u \in l_p,$$

$$\| Vu \|_q \geqq M^{-1} \| u \|, \qquad u \in l_p,$$

then for any $x \geqq 1$, *if* $Ve_n = f_n$,

$$\liminf P(|f_n|^p > x) \geqq \gamma x^{-1}.$$

PROOF. Since the inclusion $L(p,\infty) \to L_q$ is continuous there is a constant $C = C(p,q,M)$ so that for any finite subset A of $\mathbf{N}$ of cardinality n

$$C^{-1} n^{1/p} \leqq \left\| \max_{i \in A} |f_i| \right\|_q \leqq C n^{1/p}.$$

Choose $\beta_1 < (1/3)^{p/q} C^{-p}$ and β_2 so that

$$\beta_2^{1-q/p} > \frac{3q}{p-q} M^p C^q.$$

Let $A \subset \mathbf{N}$ be any finite subset of cardinality n and let $g_A = \max_{i \in A} |f_i|^p$. Then

$$P(g_A > x) \leqq \sum_{i \in A} P(|f_i|^p > x)$$

$$\leqq n M^p x^{-1}.$$

Now

$$\int g_A^{q/p} dP \leqq (\beta_1 n)^{q/p} + (\beta_2 n)^{q/p} P(g_A > \beta_1 n) + \int_{\beta_2 n}^{\infty} \frac{q}{p} x^{(q/p-1)} P(g_A > x) dx$$

$$\leqq \tfrac{1}{3} C^{-q} n^{q/p} + (\beta_2 n)^{q/p} P(g_A > \beta_1 n) + \frac{qnM^p}{p} \int_{\beta_2 n}^{\infty} x^{q/p-2} dx$$

$$\leqq \left(\tfrac{1}{3} C^{-q} + \frac{qM^p}{p-q} \beta_2^{q/p-1} \right) n^{q/p} + (\beta_2 n)^{q/p} P(g_A > \beta_1 n)$$

$$\leqq \tfrac{2}{3} C^{-q} n^{q/p} + (\beta_2 n)^{q/p} P(g_A > \beta_1 n).$$

However

$$\int_A g_A^{q/p} dP \geqq C^{-q} n^{q/p}.$$

Hence

$$P(g_A > \beta_1 n) \geqq \tfrac{1}{3} C^{-q} \beta_2^{-q/p}.$$

Applying this to every n-set A we see that

$$P(|f_i|^p > \beta_1 n) < \frac{1}{3n} C^{-q} \beta_2^{-q/p}$$

at most $n - 1$ times and so

$$\liminf_{j\to\infty} P(|f_j|^p > \beta_1 n) \geqq \frac{1}{3n} C^{-q} \beta_2^{-q/p}$$

for every $n \in \mathbf{N}$ and the lemma follows for some suitable γ.

We can now state and prove the main theorem.

THEOREM 3.4. *Let X be any Banach space. Suppose $1 \leqq p < 2$. The following conditions on X are equivalent*:

(i) *X embeds into L_p,*

(ii) *$l_p(X)$ embeds into L_0,*

(iii) *$L_p(X)$ embeds into L_0,*

(iv) *There is a tensor product $l_p \otimes_\alpha X$ which is of type p and embeds into L_0.*

REMARK. Note in the case $p = 1$ it is not necessary to suppose the tensor product is of type p in (iv).

PROOF. (i) $\Rightarrow$ (iii) If X embeds into L_p then so does $L_p(X)$.

(iii) $\Rightarrow$ (ii) Immediate.

(ii) $\Rightarrow$ (iv) This follows from Proposition 3.1 if $p > 1$ and is obvious if $p = 1$.

(iv) $\Rightarrow$ (i) We may suppose by Nikishin's theorem that we have a linear operator $V : l_p \otimes_\alpha X \to L(p, \infty)$ so that for some $M < \infty$ and $q < p$ we have

$$\| Vu \|_{p,\infty} \leqq M \| u \|,$$

$$\| Vu \|_q \geqq M^{-1} \| u \|.$$

Now define $W : X \to \mathcal{Y}_p$ by

$$Wg = (V(e_n \otimes g))_{n=1}^{\infty}.$$

Then W is bounded. If $\|g\| = 1$ and $z \in l_p$ we have

$$\| V(z \otimes g) \|_{p,\infty} \leqq M \| z \|,$$

$$\| V(z \otimes g) \|_q \geqq M^{-1} \| z \|.$$

Hence by Lemma 3.3 there exists $\gamma > 0$ so that

$$\liminf_{n\to\infty} (P \,|\, V(e_n \otimes g)|^p > x) \geqq \gamma x^{-1}$$

for $x \geqq 1$.

Let $f_n = V(e_n \otimes g)$ and $F = (f_n) \in \mathcal{Y}_p$. Then for $x \geqq 1$,

$$\lim_{n \in \mathcal{V}} \mathcal{E}(\tau_x |f_n|^p) = \lim_{n \in \mathcal{V}} \int_0^x P(|f_n|^p > t)dt$$

$$\geqq \liminf_{n \to \infty} \int_0^x P(|f_n|^p > t)dt$$

$$\geqq \int_0^x \liminf_{n \to \infty} P(|f_n|^p > t)dt$$

$$\geqq \gamma + \gamma \log x.$$

Hence

$$\phi(|F|^p) \geqq \gamma$$

and in general

$$\phi(|Wg|^p)^{1/p} \geqq \gamma^{1/p} \|g\|.$$

This means that $K_p W$ maps X isomorphically into an abstract L_p-space and the result follows. See the remarks following Lemma 2.3 where K_p is defined.

EXAMPLES. $l_p(l_q)$ $(1 \leqq p,\ q < 2)$ embeds into L_0 if and only if $p \leqq q$.

4. Concluding remarks

We first observe that the proofs of all the main results go through unchanged for quasi-Banach spaces. In particular we have:

THEOREM 4.1. *Let X be a quasi-Banach space. Then $l_1(X)$ embeds into L_0 if and only if X embeds into L_1 (and hence X is locally convex).*

The analogous results for embeddability into L_p for $0 < p < 1$ also hold.

An obvious question which arises is whether a quasi-Banach subspace of L_0 of type p $(0 < p \leqq 2)$ necessarily embeds into L_p. For $p = 2$ this is the case, and is a simple consequence of a theorem of Kwapien [10] since every subspace of L_0 has cotype 2. However for $p = 1$ it is false; there is a non-locally convex subspace of L_0 which is of type one. This is the Ribe space (see [6], [8]). The l_1-product of this space cannot be embedded in L_0. For $0 < p < 1$ or $1 < p < 2$ the problem is open.

A second question is whether Theorem 3.4 holds when $p = 2$, i.e. if $l_2(X)$ embeds into L_0 is X a Hilbert space? The methods of Proposition 3.1 only show that X is of type p for every $p < 2$ in this case.

We shall however settle the case $p = 2$ in the special case when X has local unconditional structure. The key is the following lemma.

LEMMA 4.1. *Let (u_n) be an unconditional basic sequence in L_p where $0<p<2$. If $0<q<p$ there is an unconditional basic sequence (v_n) in L_q and a constant C so that*

$$C^{-1}\left\|\sum_{n=1}^{N}|a_n|^{q/p}u_n\right\|_p^{p/q}\leqq\left\|\sum_{n=1}^{N}a_nv_n\right\|_q\leqq C\left\|\sum_{n=1}^{N}|a_n|^{q/p}u_n\right\|_p^{p/q}.$$

PROOF. Let η_n be any sequence of independent identically distributed $2q/p$-stable random variables on some probability space (Ω',Σ',P') with

$$\mathscr{E}'(\exp(it\eta_n))=\exp(-|t|^{2q/p}).$$

Consider $g_n(\omega,\omega')=|u_n(\omega)|^{p/q}\eta_n(\omega')$ in $L_p(\Omega\times\Omega')$.

It is easy to see that for some constant C_0

$$\left\|\sum_{n=1}^{N}a_ng_n\right\|_q=C_0\left\|\left(\sum_{n=1}^{N}|a_n|^{2q/p}|u_n|^2\right)^{1/2}\right\|_p^{p/q}$$

and the result follows.

REMARK. The above lemma implies that the q/p-concavification of an unconditional basic sequence in L_p embeds into L_q (we thank the referee for this remark). A similar result holds for Banach lattices embedding in L_p.

We say a Banach space X has local unconditional structure if there is a constant K so that if $E\subset X$ and $\dim E<\infty$ there is a finite-dimensional subspace F of X containing E with a K-unconditional basis.

THEOREM 4.2. *Let X be a Banach space with local unconditional structure. Suppose $l_2(X)$ embeds into L_0. Then X is a Hilbert space.*

PROOF. First note that we can reduce this theorem in the case when X has an unconditional basis. Indeed for the general case one need only take any sequence F_n of finite-dimensional subspaces each with a K-unconditional basis and observe if $Y=l_2(F_n)$ then Y has an unconditional basis and $l_2(Y)$ embeds into L_0.

Now suppose X has an unconditional basis (f_n), and $l_2(X)$ embeds into L_p where $0<p<2$. Then since X has cotype 2, if $\Sigma\, a_nf_n$ converges then $\Sigma|a_n|^2<\infty$.

Conversely, note that there is an unconditional basic sequence (u_{mn}) in L_p so that $\Sigma\, a_{mn}u_{mn}$ converges if and only if

$$\sum_{m=1}^{\infty}\left\|\sum_{n=1}^{\infty}a_{mn}f_n\right\|^2<\infty.$$

Fix any $q<p$ and determine an unconditional basic sequence (v_{mn}) in L_q by

Lemma 4.1. Then $\Sigma\, a_{mn}v_{mn}$ converges if and only if

$$\sum_{m=1}^{\infty}\left\|\sum_{n=1}^{\infty}|a_{mn}|^{q/p}f_n\right\|^2<\infty.$$

Let Y be $[v_{1n}]$ in L_q. Then Y has an unconditional basis (v_{1n}) and

$$C^{-1}\left\|\sum|a_n|^{q/p}f_n\right\|^{p/q}\leqq\left\|\sum a_nv_{1n}\right\|\leqq C\left\|\sum|a_n|^{q/p}f_n\right\|^{p/q}.$$

Hence $\Sigma\, a_{mn}v_{mn}$ converges if and only if

$$\sum_{m=1}^{\infty}\left\|\sum_{n}a_{mn}v_{1n}\right\|_Y^{2q/p}<\infty$$

and if $r=2q/p$, $l_r(Y)$ embeds into L_q. Hence Y embeds into L_r and is of type r. Thus if $\Sigma|a_n|^2<\infty$ then $\Sigma|a_n|^{2/r}v_{n1}$ converges and hence $\Sigma|a_n|f_n$ converges.

We conclude that (f_n) is equivalent to the standard basis of l_2.

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