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AN F-SPACE WITH TRIVIAL DUAL WHERE THE KREIN–MILMAN THEOREM HOLDS

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ABSTRACT
We show that in certain non-locally convex Orlicz function spaces L_Φ with trivial dual every compact convex set is locally convex and hence the Krein–Milman theorem holds. This complements the example constructed by Roberts of a compact convex set without extreme points in L_p $(0 < p < 1)$ and answers a question raised by Shapiro.

1. Introduction

In [4] Roberts answered a long outstanding question by constructing an example of a compact convex subset of a non-locally convex F-space without extreme points; thus the Krein–Milman theorem fails in general without local convexity. Later in [3], Roberts showed that such examples can be constructed in the spaces L_p $(0 < p < 1)$ (or more generally Orlicz spaces L_Φ where ϕ is sub-additive and $x^{-1}\phi(x) \to 0$ as $x \to \infty$).

The basic ingredient of Roberts's construction is the notion of a *needle point.* If E is an F-space with associated F-norm $|\ \ |$, then $x \in E$ is a needle point if given any $\varepsilon > 0$, there exist $u_1, \cdots, u_n \in E$ such that $|u_i| < \varepsilon$ $(i = 1, 2, \cdots, n)$ and

(i) $x = (1/n)(u_1 + \cdots + u_n)$,

(ii) if $a_1 + \cdots + a_n = 1$ and $a_i \geqq 0$ $(i = 1, 2, \cdots, n)$ then there exists t, $0 \leqq t \leqq 1$ such that

$$\left| tx - \sum_{i=1}^{n} a_i u_i \right| < \varepsilon.$$

Roberts [3] showed that if E contains a non-zero needle point then E contains a compact convex subset which is not locally convex. Also if every element of E is a needle point then E contains a compact convex set with no extreme points; in this case E is called a *needle-point space.*

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Following the work of Roberts, the question was asked (Shapiro [7]) whether every F-space with trivial dual contains a compact convex set without extreme points. We shall show that this is not the case and that there exist F-spaces with trivial dual in which every compact convex set is locally convex. In particular every compact convex set is affinely embeddable in a locally convex space [5] and obeys the Krein–Milman theorem. Our example is an Orlicz function space L_Φ.

2. The construction

We start by defining an element x of an F-space E to be *approachable* if there is a bounded subset B of E such that whenever $\varepsilon > 0$ there exist $u_1, \cdots, u_n \in E$ with $|u_i| < \varepsilon$ $(i = 1, 2, \cdots, n)$ and

(i) $|x - (1/n)(u_1 + \cdots + u_n)| < \varepsilon$

(ii) if $|a_1| + \cdots + |a_n| \leqq 1$ then $\sum_{i=1}^{n} a_i u_i \in B$.

THEOREM 1. *Suppose E is an F-space in which 0 is the only approachable point. Then every compact convex subset of E is affinely embeddable in a locally convex space.*

PROOF. Suppose $K \subset E$ is a compact convex set and let $K_1 = \overline{\text{co}}(K \cup (-K))$. Then K_1 is also compact. We show $0 \in K_1$ has a base of convex neighborhoods in K_1. For $\varepsilon > 0$, let $V_\varepsilon = \{x : |x| < \varepsilon\}$. Suppose $x \in K_1$ and $x \in \overline{\text{co}}(K_1 \cap V_\varepsilon)$ for every $\varepsilon > 0$. Then x is approachable (take $B = K_1$ in the definition) and hence $x = 0$. Now by compactness for any $\delta > 0$ there exists $\varepsilon > 0$ so that

$$\overline{\text{co}}(K_1 \cap V_\varepsilon) \subset V_\delta.$$

Now the finest vector topology on the linear span F of K_1 (i.e. $F = \bigcup(nK_1 : n \in N)$), which agrees with the given topology on K_1 has a base of neighborhoods of 0 of the form

$$\bigcup_{n=1}^{\infty} \sum_{m=1}^{n} (mK_1 \cap V_{\varepsilon_m})$$

where ε_m is a sequence of positive numbers ([9] p. 51). By the above result this is locally convex, and the theorem is proved.

We remark that the second half of this proof was used in [1] in the introduction; an alternative approach would be to show that every point of K_1 has a base of convex neighborhoods (this follows easily from the same fact for 0) and then use Roberts's deeper results in [5].

LEMMA 2. *Suppose E and F are F-spaces and $T : E \to F$ is a continuous linear operator. If $x \in E$ is approachable, then Tx is approachable in F.*

The proof is immediate.

We now recall that an Orlicz function ϕ is an increasing function defined on $[0,\infty)$ which is continuous at 0, satisfies $\phi(0)=0$ and $\phi(x)>0$ for some $x>0$. The function ϕ is said to satisfy the Δ_2-condition if for some constant K, we have $\phi(2x)\leqq K\phi(x)$ $(0\leqq x<\infty)$. If ϕ satisfies the Δ_2-condition then the Orlicz space $L_\phi(0,1)$ is defined to be the set of measurable functions f such that

$$\int_0^1 \phi(|f(t)|)dt < \infty.$$

L_ϕ is an F-space (after the usual identification of functions differing on a set of measure zero) with a base of neighborhoods $V(\varepsilon)$ where $f \in V(\varepsilon)$ if and only if

$$\int_0^1 \phi(|f(t)|)dt < \varepsilon.$$

THEOREM 3. *Suppose ϕ is an Orlicz function satisfying the Δ_2-condition and*

$$\phi(x) = x, \qquad 0 \leqq x \leqq 1, \tag{3.1}$$

$$\text{there exist } c_n \ (n \in N) \text{ such that } c_n \geqq 0 \text{ for all } n, \ \sum c_n < \infty \tag{3.2}$$

and if

$$G(x) = \sum_{n=1}^{\infty} c_n \frac{n}{x} \phi\left(\frac{x}{n}\right), \qquad (x>0)$$

then $G(x)\to\infty$ as $x\to\infty$.

Then 0 is the only approachable point in $L_\phi(0,1)$.

PROOF. Given any $f \in L_\phi$ with $f \neq 0$, there exists a continuous linear operator $T : L_\phi \to L_\phi$ with $Tf = 1$ (where 1 denotes the constantly one function). Hence it suffices to show that 1 is not approachable.

Suppose on the contrary 1 is approachable. In this case there is a constant M so that whenever $\delta>0$ there exist $n=n(\delta)$ and $u_1,\cdots,u_{2n}$, $h \in L_\phi$ with

$$1 = \frac{1}{2n}(u_1 + \cdots + u_{2n}) + h, \tag{3.3}$$

$$\int_0^1 \phi(|u_i(t)|)dt \leqq \delta, \tag{3.4}$$

$$\int_0^1 \phi(|h(t)|)dt \leqq \delta, \tag{3.5}$$

$$\int_0^1 \phi\left(\left|\sum_{i=1}^{2n} a_i u_i(t)\right|\right) dt \leqq M, \tag{3.6}$$

whenever $|a_1|+|a_2|+\cdots+|a_{2n}|\leqq 1$.

Now let

$$B = \sup_{0<x\leqq 2} \frac{\phi(x)}{x},$$

$$C = \sum_{n=1}^{\infty} c_n,$$

so that both B and C are finite. Now choose $\varepsilon < 1/10$ so that if $x \geqq \varepsilon^{-1}$

$$G(x) \geqq C(8e^2M + B).$$

Then we may choose $u_1,\cdots,u_{2n}, h$ as above with $\delta = \varepsilon^3$. Let $u_1^*,\cdots,u_{2n}^*$ be the pointwise decreasing re-arrangement of $|u_1|,\cdots,|u_{2n}|$. Clearly each u_i^* is measurable and belongs to L_ϕ. Next let

$$w_i(t) = \min\left(u_i^*(t), \frac{2n}{i}\right), \qquad 1 \leqq i \leqq 2n.$$

We shall show first that

$$\frac{1}{2n}\sum_{i=1}^{n} \int_{w_i(t)\geqq \varepsilon^{-1}} w_i(t)dt \geqq \frac{1}{2}. \tag{3.7}$$

Let λ denote Lebesgue measure on $(0,1)$ and let $N(t)$ for each t be the largest k so that $u_k^*(t)\geqq 1$ (and $N(t)=0$ if $u_k^*(t)<1$ for all k). Then

$$\begin{aligned}\int_0^1 N(t)dt &= \sum_{i=1}^{2n} \lambda(|u_i|\geqq 1)\\ &\leqq \sum_{i=1}^{2n}\int_0^1 \phi(|u_i(t)|)dt\\ &\leqq 2n\varepsilon^3.\end{aligned}$$

Hence $\lambda(t: N(t)\geqq 2n\varepsilon) \leqq \varepsilon^2$.

Similarly

$$\lambda(t:|h(t)|\geqq \varepsilon)\leqq \varepsilon^2.$$

Now let $A = \{t: |h(t)|<\varepsilon,\ N(t)<2n\varepsilon\}$; then $\lambda(A)\geqq 1-2\varepsilon^2$. For $t\in A$

$$\sum_{i=1}^{2n} |u_i(t)| \geqq 2n(1-\varepsilon)$$

and hence

$$\sum_{i=1}^{2n} |w_i(t)| \geqq 2n(1-\varepsilon).$$

Now

$$\begin{aligned}\int_{|u_i|\leqq\varepsilon^{-1}} |u_i|\,dt &= \int_{|u_i|\leqq\varepsilon} |u_i|\,dt + \int_{\varepsilon<|u_i|\leqq\varepsilon^{-1}} |u_i|\,dt \\ &\leqq \varepsilon + \varepsilon^{-1}\lambda(|u_i|>\varepsilon) \\ &\leqq \varepsilon + \varepsilon^{-2}\int_0^1 \phi(|u_i|)dt \\ &\leqq 2\varepsilon.\end{aligned}$$

Hence

$$\frac{1}{2n}\sum_{i=1}^{2n}\int_{u_i^*\leqq\varepsilon^{-1}} u_i^*(t)dt \leqq 2\varepsilon.$$

If $t \in A$ and $w_i(t) \leqq \varepsilon^{-1}$ then $u_i^*(t) \leqq \varepsilon^{-1}$. For otherwise $2n/i \leqq \varepsilon^{-1}$ so that $i \geqq 2n\varepsilon > N(t)$ and hence $u_i^*(t) < 1 \leqq 2n/i$. Hence

$$\frac{1}{2n}\sum_{i=1}^{2n}\int_{A\cap(w_i\leqq\varepsilon^{-1})} w_i(t)dt \leqq 2\varepsilon.$$

However

$$\frac{1}{2n}\sum_{i=1}^{2n}\int_A w_i(t)dt \geqq (1-\varepsilon)\lambda(A) \geqq 1-3\varepsilon.$$

Thus

$$\frac{1}{2n}\sum_{i=1}^{2n}\int_{A\cap(w_i>\varepsilon^{-1})} w_i(t)dt \geqq 1-5\varepsilon \geqq \frac{1}{2}.$$

Since for $t \in A$, $w_i(t) \leqq 1 \leqq \varepsilon^{-1}$ for $i \geqq n(>N(t))$, we see that (3.7) holds.

We now fix r with $1 \leqq r \leqq n$. We define two sets of random variables $(X_1, \cdots, X_{2n})$, $(Y_1, \cdots, Y_{2n})$ on some probability space (Ω, P) where Ω is a finite set. The random variables $(Y_1, \cdots, Y_{2n})$ are mutually independent and independent of $(X_1, \cdots, X_{2n})$ with common distribution given by $P(Y_i = +1) = P(Y_i = -1) = \frac{1}{2}$. The random variables $(X_1, \cdots, X_{2n})$ are *not* mutually independent. Their distribution may be described as follows: select an r-subset γ at random from the collection of r-subsets of $\{1, 2, \cdots, 2n\}$; then let $X_i = 1$ if $i \in \gamma$ and $X_i = 0$ otherwise.

Then for every $\omega \in \Omega$, $\sum_{i=1}^{2n} |X_i(\omega)Y_i(\omega)| = r$ and hence

$$\int_0^1 \phi\left(\frac{1}{r}\left|\sum_{i=1}^{2n} X_i(\omega)Y_i(\omega)u_i(t)\right|\right) dt \leqq M. \tag{3.8}$$

Let $s = [2n/r]$, and let γ be any fixed s-subset of $\{1, 2, \cdots, 2n\}$. For $j \in \gamma$, let $E_j = \{\omega : X_j(\omega) = 1,\ X_i(\omega) = 0 \text{ if } i \in \gamma \setminus \{j\}\}$. Then if $r > 1$,

$$\begin{aligned}
P(E_j) &= \binom{2n-s}{r-1} \Big/ \binom{2n}{r} \\
&= \frac{r}{2n} \cdot \frac{2n-s}{2n-1} \cdot \frac{2n-s-1}{2n-2} \cdots \frac{2n-r-s+2}{2n-r+1} \\
&\geqq \frac{r}{2n} \exp\left[-(s-1)\left(\frac{1}{2n-s} + \frac{1}{2n-s-1} + \cdots + \frac{1}{2n-r-s+2}\right)\right] \\
&\geqq \frac{r}{2n} \exp\left(-\frac{rs}{n}\right) \\
&\geqq \frac{r}{2ne^2}
\end{aligned}$$

and this also holds for $r = 1$.

Now by symmetry for fixed $t \in (0, 1)$

$$P\left\{E_j \cap \left(\omega : \left|\sum_{i=1}^{2n} X_i(\omega)Y_i(\omega)u_i(t)\right| > |u_j(t)|\right)\right\} \geqq \frac{r}{4ne^2}.$$

Thus

$$\int_{E_j} \phi\left(\frac{1}{r}\left|\sum_{i=1}^{2n} X_i(\omega)Y_i(\omega)u_i(t)\right|\right) dP(\omega) \geqq \frac{r}{4ne^2}\, \phi\left(\frac{|u_j(t)|}{r}\right).$$

As the events $(E_j, j \in \gamma)$ are disjoint, we conclude

$$\int_{\Omega} \phi\left(\frac{1}{r}\left|\sum_{i=1}^{2n} X_iY_iu_i(t)\right|\right) dP(\omega) \geqq \frac{r}{4ne^2} \sum_{j\in\gamma} \phi\left(\frac{|u_j(t)|}{r}\right).$$

Choosing γ to maximize the right-hand side, we have

$$\int_{\Omega} \phi\left(\frac{1}{r}\left|\sum_{i=1}^{2n} X_iY_iu_i(t)\right|\right) dP(\omega) \geqq \frac{r}{4ne^2} \sum_{j=1}^{s} \phi\left(\frac{u_j^*(t)}{r}\right).$$

Thus by (3.8) and Fubini's theorem, we have

$$\int_0^1 \frac{r}{2n} \sum_{j=1}^{s} \phi\left(\frac{u_j^*(t)}{r}\right) dt \leqq 2e^2 M. \tag{3.9}$$

Now summing over $r = 1, 2, \cdots, n$ we have

$$\frac{1}{2n}\int_0^1 \sum_{r=1}^{n}\sum_{j=1}^{[2n/r]} c_r r\phi\left(\frac{u_j^*(t)}{r}\right) dt \leqq 2e^2CM.$$

Interchanging the order of summation and discarding terms with $rj > n$ we have

$$\frac{1}{2n}\int_0^1 \sum_{j=1}^{n}\sum_{r=1}^{[n/j]} c_r r\phi\left(\frac{u_j^*(t)}{r}\right) dt \leqq 2e^2CM. \tag{3.10}$$

If $x \leqq 2n/j$, we have

$$\sum_{r=1}^{[n/j]} c_r r\phi\left(\frac{x}{r}\right) = x\left[G(x) - \sum_{r>[n/j]} c_r \frac{r}{x}\phi\left(\frac{x}{r}\right)\right]$$

$$\geqq x[G(x) - BC].$$

Thus

$$\sum_{r=1}^{[n/j]} c_r r\phi\left(\frac{w_j(t)}{r}\right) \geqq w_j(t)[G(w_j(t)) - BC]. \tag{3.11}$$

From (3.10) since $w_j \leqq u_j^*$ we have

$$\frac{1}{2n}\sum_{j=1}^{n}\int_{w_j \geqq \varepsilon^{-1}} \sum_{r=1}^{[n/j]} c_r r\phi\left(\frac{w_j(t)}{r}\right) dt \leqq 2e^2CM$$

and hence, recalling the choice of ε and (3.11),

$$\frac{1}{2n}\sum_{j=1}^{n}\int_{w_j \geqq \varepsilon^{-1}} 8e^2CMw_j(t)dt \leqq 2e^2CM$$

or

$$\frac{1}{2n}\sum_{j=1}^{n}\int_{w_j \geqq \varepsilon^{-1}} w_j(t)dt \leqq \frac{1}{4}$$

which contradicts (3.7) and completes the proof.

We are now in a position to construct the example.

EXAMPLE 4. *There exists a locally bounded Orlicz space $L_\phi(0,1)$ with trivial dual in which the only approachable point is* $\{0\}$.

We shall construct ϕ to satisfy (3.1), (3.2), the Δ_2-condition and

$$\liminf_{x\to\infty} \frac{\phi(x)}{x} = 0, \tag{4.1}$$

(4.2) for some $\beta > 0$, $x^{-\beta}\phi(x)$ is non-decreasing.

Then (4.1) will imply that $L_\phi^* = \{0\}$ (Rolewicz [6], Turpin [8] p. 95) and (4.2) will imply that L_ϕ is locally bounded (Rolewicz [6], Turpin [8] p. 77).

Let $(t_n : n = 0, 1, 2, \cdots)$ be an increasing sequence of positive numbers such that $t_{n+1} > t_n + 4n + 2$ $(n \geqq 0)$. Define a function $\sigma : R \to R$ by

$$\begin{aligned} \sigma(t) &= 0, && t \leqq t_0; \\ \sigma(t) &= (1-\beta)(n - (t - t_n)), && t_n \leqq t \leqq t_n + 2n; \\ \sigma(t) &= (1-\beta)(t - t_n - 3n), && t_n + 2n \leqq t \leqq t_n + 4n + 1; \\ \sigma(t) &= (1-\beta)(n+1), && t_n + 4n + 1 \leqq t \leqq t_{n+1}. \end{aligned}$$

Suppose $0 < \alpha < \frac{1}{4}(1-\beta)$ and define

$$\theta(t) = \max_{n=0,1,2,\cdots} (\sigma(t - n \log 2) - \alpha n \log 2).$$

Then if $t_n \leqq t \leqq t_n + 4n + 1$, there exists m with $m \log 2 < 4n + 2$ and

$$\sigma(t - m \log 2) = n(1-\beta).$$

Hence

$$\sigma(t) \geqq n(1-\beta) - \alpha(4n+2).$$

If $t_n + 4n + 1 \leqq t \leqq t_{n+1}$, $\theta(t) \geqq \sigma(t) = (1-\beta)(n+1)$, so that $\lim_{t\to\infty} \theta(t) = \infty$. Now we define

$$\begin{aligned} \phi(x) &= x \exp(\sigma(\log x)), \qquad 0 < x < \infty, \\ \phi(0) &= 0. \end{aligned}$$

Then $\phi(x) = x$ for $0 \leqq x \leqq 1$, and satisfies the Δ_2-condition. Also $\log x^{-\beta}\phi(x) = \sigma(\log x) + (1-\beta)\log x$ is non-decreasing, so that (4.2) holds. For (4.1) observe that $\log(\phi(x)/x) = \sigma(\log x)$ and $\sigma(t_n + 2n) = -n(1-\beta)$.

Finally we show that (3.2) holds:

$$\begin{aligned} \sum_{n=0}^{\infty} 2^{-n\alpha} \frac{2^n}{x} \phi\left(\frac{x}{2^n}\right) &= \sum_{n=0}^{\infty} 2^{-n\alpha} \exp(\sigma(\log x - n \log 2)) \\ &\geqq \exp \theta(\log x) \\ &\to \infty \qquad \text{as } x \to \infty. \end{aligned}$$

Of course by Theorem 1 the space L_ϕ we have constructed has the property that every compact convex subset is locally convex.

3. Concluding remarks

There are a number of obvious questions arising from this example. We do not know if a condition like (3.2) is necessary for the conclusion of Theorem 3. In particular if we simply have

$$\liminf_{x\to\infty} x^{-1}\phi(x) = 0 \quad \text{and} \quad \limsup_{x\to\infty} x^{-1}\phi(x) = \infty,$$

then can L_ϕ contain a non-zero needle point? In [7] Shapiro asks whether the Krein–Milman theorem holds in certain quotients of H_p $(0 < p < 1)$. This example perhaps suggests that the failure of the Krein–Milman theorem and the existence of needle points is a rarer phenomenon than previously suspected.

In [2], the author and N. T. Peck plan to investigate further the relationship between approachable points and needle points.

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