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COMPACT AND STRICTLY SINGULAR OPERATORS ON ORLICZ SPACES

BY
N. J. KALTON

ABSTRACT
If $0 < p < 1$ and $T: L_p(0,1) \to E$ is a continuous linear operator into a topological vector space, there is an infinite-dimensional subspace X of L_p on which T is an isomorphism; thus there are no compact operators on L_p. Results of this type are proved for general non-locally convex Orlicz spaces.

1. Introduction

Let $\phi: [0,\infty) \to [0,\infty)$ be an Orlicz function, i.e. an increasing function continuous at 0, such that $\phi(0) = 0$ and $\phi(x) > 0$ for some $x > 0$. The Orlicz space $L_\phi(0,1)$ is the space of all measurable functions (identify functions equal almost everywhere) such that for some $\varepsilon > 0$

$$\int_0^1 \phi(\varepsilon^{-1}|x(t)|)\,dt < \infty.$$

$L_\phi(0,1)$ is an F-space (complete metric linear space) if we equip it with the topology (the ϕ-topology) whose base of neighbourhoods of zero consists of all sets of the form $rB_\phi(\varepsilon)$ where $r > 0$ and

$$B_\phi(\varepsilon) = \left\{x : \int_0^1 \phi(|x(t)|)\,dt \leqq \varepsilon\right\}$$

for $\varepsilon > 0$. If ϕ is convex then $L_\phi(0,1)$ is a Banach space. (ϕ is convex if $\phi(\frac{1}{2}(x+y)) \leqq \frac{1}{2}(\phi(x) + \phi(y))$, $x \geqq 0$, $y \geqq 0$.)

An Orlicz function ϕ is said to satisfy the Δ_2-*condition at* ∞ if

$$\sup_{x \geqq 1} \frac{\phi(2x)}{\phi(x)} < \infty.$$

If ϕ satisfies this condition then $L_\infty(0,1)$ is dense in $L_\phi(0,1)$. In general we denote the closure of $L_\infty(0,1)$ in $L_\phi(0,1)$ by $\tilde{L}_\phi(0,1)$. $\tilde{L}_\phi$ consists of all x such that

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$$\int_0^1 \phi(\lambda |x(t)|)\,dt < \infty \quad \text{for all} \quad \lambda > 0.$$

An operator $T: X \to Y$ between two topological vector spaces is *compact* if it maps a neighbourhood of zero in X to a relatively compact set. Recently Turpin [6] (see also Pallaschke [5]) has shown that if

$$\lim_{x \to \infty} x^{-1}\phi(x) = 0$$

then $L_\phi(0,1)$ admits no non-zero compact endomorphisms. In response to a question of Turpin ([6] (and [7] 3.4.9)) we shall show that this assumption and the Δ_2-condition at ∞ imply that there is no non-zero compact operator $T: L_\phi(0,1) \to X$ where X is any topological vector space. In fact there is no such non-zero operator which is not an isomorphism on some infinite-dimensional Hilbertian subspace of L_ϕ. For the special case $\phi(x) = x^p$, $0 < p < 1$, this also improves a result of the author [2] that L_p admits no Hausdorff vector topology for which the unit ball is relatively compact.

Before going on to the proofs we give a sketch of the technique to be employed. Let $\mathscr{A}$ be a collection of continuous operators with domain $\tilde{L}_\phi$. We form on $\tilde{L}_\phi$ the finest topology, γ, such that each $T \in \mathscr{A}$ is continuous. We then construct on $\tilde{L}_\phi$ a topology β stronger than γ but weaker than the original ϕ-topology; this topology has for its base of neighbourhoods of zero the set of γ-closed ϕ-neighbourhoods. Under certain circumstances, it is shown that β can be identified as the topology induced by a convex Orlicz function ψ such that $\tilde{L}_\phi \subset L_\psi$. Thus each $T \in \mathscr{A}$ factors through the inclusion map $\tilde{L}_\phi \hookrightarrow \tilde{L}_\psi$.

In §2, we apply this technique to those operators which map $\{x: \text{ess.sup}\,|x(t)| \leqq 1\}$ into a relatively compact set. In §3, we apply it to those operators which are strictly singular on every Hilbertian subspace of $\tilde{L}_\phi$.

2. Compact operators

Consider the following classes of operators on the space L_0 of all measurable functions on $(0,1)$

$$(C_1) \qquad S_\theta x(s) = \theta(s)x(s)$$

where $\theta \in L_0$ and $\|\theta\|_\infty = \text{ess.sup}\,|\theta(s)| \leqq 1$. If $\theta = \chi_A$, the characteristic function of A, then we write S_A for S_θ.

$$(C_2) \qquad R_t x(s) = x(K(s,t))$$

where

$$\begin{aligned} K(s,t) &= s+t && 0<s<1-t \\ &= t && s=1-t \\ &= s+t-1 && 1-t<s<1. \end{aligned}$$

Then each operator of classes C_1 and C_2 defines a continuous operator on L_ϕ. In fact if T is in C_1 or C_2

$$\| Tx \|_\phi \leqq \| x \|_\phi \qquad x \in L_\phi$$

(where $\| x \|_\phi = \int_0^1 \phi(|x(t)|)\,dt$).

Note also that L_∞, and hence $\tilde{L}_\phi$, is invariant for each $T \in C_1$ or C_2.

In fact for our applications in this note, we could take for the class (C_2) the larger class of all operators $Rx(s) = x(\alpha(s))$ where $\alpha:(0,1)\to(0,1)$ is any measure preserving map. The operators R_t can be interpreted as follows. Let X be the space of measurable functions on $\mathbf{R}$ of period one. Then $L_0(0,1)$ and X can be naturally identified, and R_t corresponds to operator $x \mapsto x'$ on X where $x'(s) = x(s+t)$. (Of course we are ignoring sets of measure zero in this identification.)

LEMMA 2.1. *Let γ be a vector topology on $\tilde{L}_\phi$, weaker than the ϕ-topology, but not indiscrete. Suppose each endomorphism of $\tilde{L}_\phi$ of classes C_1 and C_2 is γ-continuous. Let β be the vector topology on $\tilde{L}_\phi$ whose base consists of all γ-closed ϕ-neighbourhoods of zero. Then β is the topology induced by an Orlicz function ψ such that $\psi \leqq \phi$ and $\| \cdot \|_\psi$ is lower-semi-continuous for γ.*

PROOF. For $x \in \tilde{L}_\phi$, define

$$F(x) = \inf\{\varepsilon : x \in \overline{B_\phi(\varepsilon)}\}$$

(closure in γ). The sets $\{x : F(r^{-1}x) \leqq \varepsilon\}$ for $r>0$, $\varepsilon>0$ form a base for β, and F is lower-semi-continuous for γ. Clearly we also have $F(x) \leqq \| x \|_\phi$, and it is therefore enough to show that $F(x) = \| x \|_\psi$ for some Orlicz function ψ.

We will show that F satisfies the following conditions:

1) If $x, y \in \tilde{L}_\phi$ and $|x| \leqq |y|$ then $F(x) \leqq F(y)$.

2) If $x_n, x \in \tilde{L}_\phi$, $0 \leqq x_n \leqq x$ and $x_n(t) \to x(t)$ a.e. then $F(x_n) \to F(x)$.

3) If $x, y \in \tilde{L}_\phi$ and $x(t)y(t) = 0$ a.e. then $F(x+y) = F(x) + F(y)$.

4) If $x \in \tilde{L}_\phi$, and $0<t<1$, $F(R_t x) = F(x)$.

First observe that T is of class (C_1) or (C_2), then $T(B_\phi(\varepsilon)) \subset B_\phi(\varepsilon)$. Hence as each T is continuous for γ, $T(\overline{B_\psi(\varepsilon)}) \subset \overline{B_\phi(\varepsilon)}$, and so $F(Tx) \leqq F(x)$ $(x \in \tilde{L}_\psi)$.

For (1), note that $x = S_\theta y$ where $\| \theta \|_\infty \leqq 1$. Hence $F(x) \leqq F(y)$.

For (2), suppose $\lambda > 0$,

$$\| \lambda (x - x_n) \|_\phi = \int_0^1 \phi(\lambda(x(t) - x_n(t)))\, dt$$

$$\phi(\lambda(x(t) - x_n(t))) \leqq \phi(\lambda x(t)) \qquad \text{a.e.}$$

and

$$\int_0^1 \phi(\lambda x(t))\, dt < \infty \quad \text{since} \quad x \in \tilde{L}_\phi.$$

Hence by the Lebesgue convergence theorem $\| \lambda (x - x_n) \|_\phi \to 0$ for all $\lambda > 0$, and hence $x_n \to x$ in the ϕ-topology. Thus $x_n \to x$ is the γ-topology, and hence as F is γ-lower-semi-continuous, $F(x) \leqq \liminf_{n\to\infty} F(x_n)$. However by (1), $F(x) \geqq F(x_n)$ for all $n \in N$ and hence $F(x_n) \to F(x)$.

For (3), suppose x, y have disjoint supports A and B. Let u_α be a net such that $u_\alpha \to x + y(\gamma)$ and $\| u_\alpha \|_\phi \to F(x + y)$. Then as S_A is continuous for $\gamma, S_A u_\alpha \to x(\gamma)$ and similarly $S_B u_\alpha \to y$. Thus $F(x) \leqq \liminf_\alpha \| S_A u_\alpha \|_\phi$ and $F(y) \leqq \liminf_\alpha \| S_B u_\alpha \|_\phi$. However

$$\| u_\alpha \|_\phi = \| S_A u_\alpha \|_\phi + \| S_B u_\alpha \|_\phi$$

and hence

$$F(x + y) \geqq F(x) + F(y).$$

Conversely if $v_\alpha \to x(\gamma)$, $w_\nu \to y(\gamma)$ and $\| v_\alpha \|_\phi \to F(x)$, $\| w_\nu \|_\phi \to F(y)$, then $S_A v_\alpha \to x(\gamma)$ and $S_B w_\nu \to y(\gamma)$. Hence

$$\begin{aligned} F(x + y) &\leqq \liminf_{\alpha,\nu\to\infty} \| S_A v_\alpha + S_B w_\nu \|_\phi \\ &= \liminf_{\alpha,\nu} (\| S_A v_\alpha \|_\phi + \| S_B w_\nu \|_\phi) \\ &\leqq \liminf_{\alpha,\nu} (\| v_\alpha \|_\phi + \| w_\nu \|_\phi) \\ &= F(x) + F(y). \end{aligned}$$

Hence

$$F(x + y) = F(x) + F(y).$$

Finally for (4) we observe that $R_{1-t} R_t = I$ for $0 < t < 1$, and hence $F(R_t x) \leqq F(x) \leqq F(R_t x)$.

Now we can appeal to 2.4 of [1], with some slight modifications, to deduce that

$$(*) \qquad F(x) = \int_0^1 \psi(|x(t)|)\,dt$$

for some left-continuous increasing function. For completeness we sketch the details here. Define $\psi(t) = F(t\chi_{(0,1)})$; for fixed τ, we may apply the Radon–Nikodym theorem to the absolutely continuous measure $\lambda_\tau(A) = F(\tau\chi_A)$, to deduce the existence of measurable function π_τ such that

$$F(\tau\chi_A) = \int_A \pi_\tau(s)\,ds \qquad A \subset (0,1).$$

Then since $F(R_t\tau\chi_A) = F(\tau\chi_A)$ we have

$$\begin{aligned} F(\tau\chi_A) &= \int_0^1 \int_0^1 \chi_A(K(t,s))\,\pi_\tau(s)\,ds\,dt \\ &= m(A)\psi(\tau) \end{aligned}$$

by Fubini's theorem, since

$$\int_0^1 \chi_A(K(t,s))\,dt = m(A).$$

Thus we obtain (*) for simple functions, and we may use (2) to obtain (*) in general.

Finally we observe that since γ is not indiscrete and $\beta \geqq \gamma$, β is also not indiscrete. Thus $F \not\equiv 0$ and hence $\psi \not\equiv 0$, i.e. ψ is an Orlicz function.

THEOREM 2.2. *Let ϕ be an Orlicz function and let $T: \tilde{L}_\phi \to X$ be a continuous operator such that $T\,\{x : \|x\|_\infty \leqq 1\}$ is relatively compact. Then*

a) *If* $\liminf_{x\to\infty} x^{-1}\phi(x) = 0$ *then* $T = 0$.

b) *If* $\liminf_{x\to\infty} x^{-1}\phi(x) > 0$, *$T$ may be factorized:*

$\tilde{L}_\phi \xrightarrow{J} \tilde{L}_{\hat{\phi}} \xrightarrow{S} X$ *where $\hat{\phi}$ is the largest convex function smaller than ϕ and J is the natural inclusion map.*

PROOF. Let γ be the topology on $\tilde{L}_\phi$ induced by all continuous operators $T: \tilde{L}_\phi \to Y$ where Y is a topological vector space and $T(B)$ is precompact (where $B = \{x : \|x\|_\infty \leqq 1\}$). If $S \in C_1$ or C_2, $S(B) \subset B$ and hence for all such T, $TS(B) \subset T(B)$ is precompact; therefore S is γ-continuous. Furthermore B is γ-precompact.

If γ is not indiscrète we can apply Lemma 2.1 to obtain the Orlicz function ψ.

Let $\{r_n\}$ denote the sequence of Rademacher functions (i.e. a sequence of independent random variables with mean zero and taking only the values ± 1). Since $r_n \in B$ and B is γ-precompact, we may find a net $r_{m(\alpha)} - r_{n(\alpha)}$ where $m(\alpha) \neq n(\alpha)$ and $r_{m(\alpha)} - r_{n(\alpha)} \to O(\gamma)$. However $\|\cdot\|_\psi$ is γ-lower-semi-continuous; hence

$$\liminf_{\alpha} \| s\chi_{(0,1)} + \tfrac{1}{2} t(r_{m(\alpha)} - r_{n(\alpha)})\|_\psi \geqq \psi(s) \quad \text{for} \quad s, t > 0.$$

However

$$\| s\chi_{(0,1)} + \tfrac{1}{2} t(r_{m(\alpha)} - r_{n(\alpha)})\|_\psi = \tfrac{1}{2}\psi(s)$$

$$+ \tfrac{1}{4}(\psi(s+t) + \psi(s-t)) \quad \text{for} \quad 0 < t \leqq s.$$

Hence

$$\tfrac{1}{2}(\psi(s+t) + \psi(s-t)) \geqq \psi(s) \qquad 0 < t \leqq s$$

i.e. ψ is convex.

Since $\psi \leqq \phi$, (a) follows immediately. For (b), note that ψ is an Orlicz function and that $\psi \leqq \hat{\phi}$ by definition. In fact, although we do not need this, $\psi = \hat{\phi}$.

Clearly T is continuous on $\tilde{L}_\phi$ for the $\hat{\phi}$-topology, for if U is a neighbourhood of 0 in X then $T^{-1}(U)$ is a γ-neighbourhood and hence a $\hat{\phi}$-neighbourhood. Thus the factorization is achieved as required.

Corollary 2.3. *Let ϕ be an Orlicz function satisfying $\liminf_{x\to\infty} x^{-1}\phi(x) = 0$. Then any operator $T: \tilde{L}_\phi \to X$ which maps bounded sets to relatively compact sets is identically zero.*

Corollary 2.4. *An Orlicz space $L_\phi(0,1)$ has a Schauder basis if and only if ϕ is equivalent to a convex Orlicz function and ϕ satisfies the Δ_2-condition at ∞.*

Proof. If ϕ is a convex Orlicz function, satisfying the Δ_2-condition at ∞, then L_ϕ (0, 1) has a basis (the Haar system, see [4] p. 101). Conversely if L_ϕ has a basis then the sequence $\{S_n\}$ of partial sum operators on L_ϕ forms an equicontinuous set of operators and $S_n x \to x$ pointwise. Thus the sets $U^* = \bigcap_{n=1}^{\infty} S_n^{-1}(U)$ form a base of neighbourhoods of zero, as U runs through the set of all closed neighbourhoods of 0. Each S_n has finite-dimensional range and it follows that $S_n^{-1}(U)$ is weakly closed. Hence U^* is weakly closed, and $\tilde{L}_\phi$ has a base of weakly closed neighbourhoods of 0. It follows that ψ constructed in the theorem is equivalent to ϕ.

REMARK. A further result which follows from Theorem 2.2 is that the weak closure of the set $B_\phi(\varepsilon)$ is always convex (in $\tilde{L}_\phi$).

3. Strictly singular operators

If X is a Banach space and Y an F-space, an operator $T: X \to Y$ is called *strictly singular* ([3]) if there is no infinite-dimensional subspace E of X such that $T|E$ is an isomorphism.

PROPOSITION 3.1 (cf. [3]). *Suppose X is a Banach space and Y is an F-space.*

i) $T: X \to Y$ is strictly singular if and only if every infinite-dimensional subspace E of X contains an infinite-dimensional subspace F such that $T|F$ is compact.

ii) If $T: X \to Y$ and $S: X \to Y$ are strictly singular then so is $S + T$.

PROOF. i) Pick a basic sequence (x_n) in E; then it is easy to show, using the strict singularity of T, that there is a block basic sequence (u_n) with respect to (x_n) such that $\|u_n\| = 1$ and $\|Tu_n\| \leqq 2^{-n}$. Let F be the closed linear span of (u_n).

ii) follows from (i).

The proof of the following lemma is omitted:

LEMMA 3.2. *Let ϕ be an Orlicz function and define*

$$\phi^*(x) = \int_1^2 \phi(tx)\,dt.$$

Then ϕ^* is equivalent to ϕ (i.e. $L_{\phi^*} = L_\phi$); if ϕ is C^k-function ($k \geqq 0$), then ϕ^* is C^{k+1} on $(0, \infty)$.

PROPOSITION 3.3. *Let ϕ and ψ be Orlicz functions such that $L_\phi \subset L_\psi$. Let γ be a vector topology on $\tilde{L}_\phi$, weaker than the ϕ-topology such that $\|\cdot\|_\psi$ is lower-semi-continuous on $(\tilde{L}_\phi, \gamma)$. Then $\|\cdot\|_{\psi^*}$ is also γ-lower-semi-continuous.*

PROOF. $\tilde{L}_\phi$ is ϕ-separable. Let $\mathcal{U}$ be the collection of all γ-neighbourhoods of 0. For $\varepsilon > 0$, rational

$$B_\psi(\varepsilon) \cap \tilde{L}_\phi = \bigcap_{\mathcal{U}} \overline{(B_\psi(\varepsilon) \cap \tilde{L}_\phi + U)}$$

(closure in γ). Hence by the Lindelöf property of $\tilde{L}_\phi$, there is a sequence $U_n^\varepsilon \in \mathcal{U}$ such that

$$B_\psi(\varepsilon) \cap \tilde{L}_\phi = \bigcap_{n=1}^{\infty} (B_\psi(\varepsilon) \cap \tilde{L}_\phi + U_n^\varepsilon).$$

Let γ_0 be a metrizable vector topology, $\gamma_0 \leqq \gamma$, such that each U_n^ε, $\varepsilon \in \mathbf{Q}_+$, $n \in \mathbf{N}$, is a γ_0-neighbourhood of 0. Then $\|\cdot\|_\psi$ is γ_0-lower-semi-continuous.

If $x_n \to x(\gamma_0)$, then

$$\begin{aligned}
\|x\|_{\psi^*} &= \int_0^1 \psi^*(|x(t)|)\,dt \\
&= \int_0^1 \int_1^2 \psi(s|x(t)|)\,ds\,dt \\
&= \int_1^2 \|sx\|_\psi\,ds \\
&\leqq \int_1^2 \liminf_{n\to\infty} \|sx_n\|_\psi\,ds \\
&\leqq \liminf_{n\to\infty} \int_1^2 \|sx_n\|_\psi\,ds \\
&= \liminf_{n\to\infty} \|x_n\|_{\psi^*}
\end{aligned}$$

so that $\|\cdot\|_{\psi^*}$ is γ_0-lower-semi-continuous, and hence γ-lower-semi-continuous.

THEOREM 3.4. *Let ϕ be an Orlicz function satisfying*

$$\int_0^\infty x^2\phi(\lambda x)\,e^{-\frac{1}{2}x^2}dx < \infty \qquad \lambda > 0.$$

Suppose X is an F-space and $T: \tilde{L}_\phi \to X$ is a non-zero continuous linear operator. Then at least one of the following conditions holds:

a) *There is a subspace H of $\tilde{L}_\phi$ isomorphic to an infinite-dimensional Hilbert space such that $T|H$ is an isomorphism.*

b) $\liminf_{x\to\infty} x^{-1}\phi(x) > 0$ *and T factorizes*

$$\tilde{L}_\phi \xrightarrow{J} \tilde{L}_{\hat{\phi}} \xrightarrow{\hat{T}} X$$

where $\hat{\phi}$ is the convex minorant of ϕ and $J: \tilde{L}_\phi \to \tilde{L}_{\hat{\phi}}$ is the inclusion map.

PROOF. Let γ be the topology on $\tilde{L}_\phi$ induced by the class $\mathscr{A}$ of all operators $S: \tilde{L}_\phi \to Y$ (where Y is any F-space) such that if $R: l_2 \to \tilde{L}_\phi$ is continuous then SR is strictly singular. Let H be any subspace of $\tilde{L}_\phi$ of infinite dimension isomorphic to a Hilbert space. Suppose γ coincides on H with the ϕ-topology; then there exist maps $S_i: \tilde{L}_\phi \to Y_i$ $(1 \leqq i \leqq n)$ in $\mathscr{A}$, and $\varepsilon > 0$ such that the set $\{x \in H: \max\|S_ix\| \leqq \varepsilon\}$ is bounded. Let $Y = Y_1 \oplus \cdots \oplus Y_n$ under the F-norm

$\|(y_1 \cdots y_n)\| = \max \|y_i\|$ and consider $S = S_1 \oplus \cdots \oplus S_n : \tilde{L}_\phi \to Y$. By Proposition 3.1, S is strictly singular on H, contradicting our assumptions. Hence γ is strictly weaker than ϕ on H. Next suppose $R : \tilde{L}_\phi \to \tilde{L}_\phi$ is continuous; then if $S : \tilde{L}_\phi \to Y$ is in $\mathscr{A}$, and $Q : l_2 \to \tilde{L}_\phi$, SRQ is strictly singular. Thus $SR \in \mathscr{A}$, and R is γ-continuous.

Now suppose γ is not indiscrete and apply Lemma 2.1. We obtain an Orlicz function ψ whose base of neighbourhoods consists of all γ-closed ϕ-neighbourhoods of 0. Then apply Lemma 3.2 three times to obtain a C^2-function θ equivalent to ψ; by Proposition 3.3, $\|\cdot\|_\theta$ is still γ-lower-semi-continuous.

Let (x_n) be a sequence of functions in L_0, which, considered as random variables, are independent, normally distributed, with mean zero and variance one. Then for $\lambda > 0$

$$\int_0^1 \phi(\lambda |x_n(t)|)\, dt = \sqrt{\frac{2}{\pi}} \int_0^\infty \phi(\lambda x) e^{-\frac{1}{2}x^2} dx$$

$$\leqq \sqrt{\frac{2}{\pi}} \left\{ \int_0^1 \phi(\lambda x) e^{-\frac{1}{2}x^2} dx + \int_1^\infty x^2 \phi(\lambda x) e^{-\frac{1}{2}x^2} dx \right\} < \infty$$

so that $x_n \in \tilde{L}_\phi$. For $a_1 \cdots a_n \in \mathbf{R}$, $a_1 x_1 + \cdots + a_n x_n$ is again normally distributed with mean zero and variance, $a_1^2 + \cdots + a_n^2$. Hence

$$\| a_1 x_1 + \cdots + a_n x_n \|_\phi = \| \sqrt{\Sigma a_i^2}\, x_1 \|_\phi$$

and hence that there is an embedding $W : l_2 \to \tilde{L}_\phi$ with $We_i = x_i$ (where e_i is the unit vector basis of l_2). Hence there exists a net

$$u_\alpha = \sum_{i=1}^{N(\alpha)} c_i^{(\alpha)} x_i$$

such that $u_\alpha \to O(\gamma)$ but $\Sigma_{i=1}^{N(\alpha)} | c_i^\alpha |^2 = 1$. Thus for $s > 0$ and $\varepsilon > 0$,

$$\| s\chi_{(0,1)} \|_\theta = \theta(s)$$

$$\leqq \liminf_\alpha \| s\chi_{(0,1)} + \varepsilon u_\alpha \|_\theta.$$

However

$$\| s\chi_{(0,1)} + \varepsilon u_\alpha \|_\theta = \int_0^1 \theta(| s + \varepsilon u_\alpha(t)|)\, dt$$

$$= \int_0^1 \theta(| s + \varepsilon u(t)|)\, dt$$

where $u = u(t)$ is any fixed normally distributed random variable with mean zero and variance 1.

If $s > 0$, since θ is a C^2-function

$$\theta(|s+\tau|) = \theta(s) + \tau\theta'(s) + \tfrac{1}{2}\tau^2\theta''(s) + \tau^2\rho(\tau)$$

where $\rho(0) = 0$ and ρ is continuous. Clearly ρ satisfies

$$|\rho(\tau)| \leqq A + \theta(2|\tau|) \leqq A + \phi(16|\tau|)$$

for some constant A.

Hence

$$\int_0^1 \theta(|s + \varepsilon u(t)|)\,dt = \theta(s) + \frac{\varepsilon^2}{2}\theta''(s) + \varepsilon^2 \int_0^1 u^2(t)\rho(\varepsilon u(t))\,dt$$

and so

$$\theta''(s) + 2\int_0^1 u^2(t)\rho(\varepsilon u(t))\,dt \geqq 0.$$

For $\varepsilon \leqq 1$

$$u^2(t)|\rho(\varepsilon u(t))| \leqq u^2(t)(A + \phi(16\varepsilon|u(t)|))$$

$$\leqq u^2(t)(A + \phi(16|u(t)|))$$

and

$$\int_0^1 u^2(t)(A + \phi(16|u(t)|))\,dt = \frac{\sqrt{2}}{\sqrt{\pi}}\int_0^\infty x^2(A + \phi(16x))e^{-\frac{1}{2}x^2}\,dx$$

$$< \infty.$$

Hence, by the Lebesgue Convergence Theorem,

$$\lim_{\varepsilon \to 0} \int_0^1 u^2(t)\rho(\varepsilon u(t))\,dt = 0.$$

Thus $\theta''(s) \geqq 0$ and θ is a convex function.

As $\theta(x) \leqq \psi(8x) \leqq \phi(8x)$, we clearly have $\theta(x) \leqq \hat{\phi}(8x)$. Hence if T fails to satisfy (a), then γ is not indiscrete, and so $\theta \not\equiv 0$ and $\hat{\phi} \not\equiv 0$. Then

$$\liminf_{x \to \infty} x^{-1}\phi(x) > 0$$

and T factorizes through $\tilde{L}_\theta$ and hence through $\tilde{L}_{\hat{\phi}}$.

COROLLARY 3.5. *If ϕ is an Orlicz function satisfying the Δ_2-condition at ∞ and such that*

$$\liminf_{x \to \infty} x^{-1}\phi(x) = 0,$$

and X is an F-space containing no subspace isomorphic to l_2, then there is no non-zero continuous operator $T: L_\phi \to X$.

PROOF. The Δ_2-condition at ∞ implies $\phi(x) \leqq A + Bx^p$ for $x \geqq 0$ and for some p. Hence

$$\int_0^\infty x^2\phi(\lambda x)e^{-\frac{1}{2}x^2}dx < \infty \qquad \lambda > 0$$

and the result follows by the theorem.

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DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY COLLEGE OF SWANSEA
SINGLETON PARK, SWANSEA, SA2 8PP, U.K.