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On a question of Pisier

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1. Introduction.

It is well-known that if X_0 and X_1 are Banach spaces with X_1 of (Rademacher) type 2 then a complex interpolation space $[X_0, X_1]_\theta$ is of type p where $1/p = 1 - \theta/2$. In [3] Pisier asks if there is a converse to this observation. Precisely he asks whether given a Banach space X of (Rademacher) type p where $1 < p < 2$ one can find a pair of Banach spaces X_0, X_1 with X_1 of type 2 so that X is isomorphic to the complex interpolation space $[X_0, X_1]_\theta$ where $1/p = 1 - \theta/2$.

In this note we will consider the natural finite-dimensional version of this question. It is natural to ask whether given $1 < p < 2$ and $1 < K < \infty$ there exist a constant $C = C(p, K)$ so that if X is an n-dimensional Banach space with type p constant $T_p(X) \leq K$ then one can find $N \geq n$, two N-dimensional Banach spaces X_0, X_1 so that $T_2(X_1) \leq C$, and an n-dimensional subspace F of $[X_0, X_1]_\theta$ (where $1/p = 1 - \theta/2$) such that $d(X, F) \leq C$. (Notice that we only ask if X can be well-embedded into $[X_0, X_1]_\theta$ whereas Pisier asks for X to be isomorphic to $[X_0, X_1]_\theta$.)

We show that the answer to this question is negative. In fact we show the following result about the Lorentz spaces $\ell_{p,s}$ where $1 < p < 2 < s < \infty$ and $1/p + 1/s < 1$. Given any constant K and any $\epsilon > 0$ there exists a constant $c = c(K, \epsilon, p, s) > 0$ so that if X_0, X_1 are N-dimensional Banach spaces with $T_2(X_1) \leq K$ and if F is an n-dimensional subspace of $[X_0, X_1]_\theta$ where $1/p = 1 - \theta/2$ then $d(F, \ell_{p,s}^n) \geq c(\log n)^{1-1/p-1/s-\epsilon}$. The Lorentz spaces $\ell_{p,s}$ are of type p (see [2], Theorem 1.f.10) so that the spaces $\ell_{p,s}^n$ have uniformly bounded type p constants.

We now explain our notation. Let us note first that we regard any N-dimensional Banach space as being the same underlying vector space $\mathbf{C}^N$ equipped with a particular norm. On $\mathbf{C}^N$ we have the natural bilinear pairing defined simply by

$$\langle x, y \rangle = \sum_{k=1}^{N} x_k y_k.$$

Hence if X is an N-dimensional Banach space we can define X^* by

$$\|x\|_{X^*} = \sup_{\|y\|_X \leq 1} |\langle x, y \rangle|.$$

For any $n \in \mathbf{N}$ we let $D_n = \{-1, +1\}^n$ be the dyadic group with 2^n elements. For $t \in D_n$ and $1 \leq k \leq n$ we define $\epsilon_k(t) = t_k$ where $t = (t_j)_{j=1}^n$. We use dt to denote normalized Haar measure on the finite group D_n. Similarly we let S_n be the group of

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permutations of $\{1,2,\ldots,n\}$ and use $d\sigma$ to denote Haar measure on it. We may then consider the finite-dimensional spaces $L_2(D_n;X)$ and here the bilinear pairing is defined by

$$\langle f,g\rangle = \int \langle f(t),g(t)\rangle dt$$

and similarly for $L_2(S_n \times D_n;X)$.

We recall that the type p constant of X, $T_p(X)$ is the least constant T such that for any $n \in \mathbf{N}$, and $x_1,\ldots,x_n \in X$ we have

$$\left(\int_{D_n} \|\sum_{k=1}^{n} \epsilon_k(t)x_k\|^p dt\right)^{1/p} \leq T\left(\sum_{k=1}^{n} \|x_k\|^p\right)^{1/p}.$$

We shall also need the K-convexity constant $K(X)$ which is the least constant K such that for any n and any $f \in L_2(D_n;X)$ we have

$$\|\sum_{k=1}^{n}\left(\int_{D_n} \epsilon_k(t)f(t)dt\right)\epsilon_k\|_{L_2(X)} \leq K\|f\|_{L_2(X)}.$$

Pisier [4] shows that if $p > 1$ then $K(X)$ is bounded by a constant depending only on p and $T_p(X)$.

The Lorentz sequence spaces $\ell_{p,s}$ where $1 < p,s < \infty$ consist of all sequences $\xi = (\xi_n) \in c_0$ such that if (ξ_n^*) denotes the decreasing rearrangement of $(|\xi_n|)$ then

$$\|\xi\|_{p,s} = \left(\sum_{n=1}^{\infty} n^{1/p-1/s}(\xi_n^*)^s\right)^{1/s} < \infty.$$

Strictly speaking, if $p < s$ then $\| \|_{p,s}$ is a quasi-norm but it is equivalent to a norm. If $1 < p \leq s < \infty$ then $\ell_{p,s}$ is of type p, (see [2], Theorem 1.f.10). We use $\ell_{p,s}^n$ to denote the $n-$dimensional analogue of $\ell_{p,s}$; these spaces have uniformly bounded type p constants.

Finally let us mention interpolation. We will use the ideas of interpolation of families developed in [1]. Suppose, for every $z \in \mathbf{T}$, except on a Borel set of measure zero, we are given a norm $\| \|_{X_z}$ on $\mathbf{C}^N$. Suppose that the map $(z,x) \to \|x\|_{X_z}$ is a Borel function. Suppose that for some fixed norm $\| \|_0$ there exists a Borel function h on $\mathbf{T}$ such that (letting λ denote Haar measure on the circle)

$$\int |\log h| d\lambda < \infty$$

and $h(z)^{-1}\|x\|_0 \leq \|x\|_z \leq h(z)\|x\|_z$. We say that a map $\phi : \Delta = \{z : |z| < 1\}$ is in class $\mathcal{N}^+$ if each co-ordinate is in the Smirnov class N^+; then ϕ extends a.e. to the boundary. We then define for $z \in \Delta$

$$\|x\|_{X_z} = \inf \operatorname{ess} \sup_{z\in\mathbf{T}} \|\phi(z)\|_{X_z}$$

where the infimum is taken over all $\phi \in \mathcal{N}^+$ such that $\phi(z) = x$.

It is then possible to show that the infimum is actually attained. In the special case of interest to us when each X_z is a lattice for $|z| = 1$ then the same is true for $|z| < 1$.

Further if $\|x\|_{X_z} \leq 1$ with $x \geq 0$ then there exists a Borel function $\phi : \mathbf{T} \to R_+^n$ such that for $1 \leq k \leq n$,

$$x_k = \exp\left(\int_{\mathbf{T}} \log(\phi(z))_k d\lambda(z)\right)$$

where the integral can take the value $-\infty$.

2. The main result.

Before stating our first technical result we introduce a definition. We shall say that a finite sequence $\{x_1, \ldots, x_n\}$ in a Banach space X is 1-(real) symmetric if for any $\sigma \in S_n$ and any $t \in D_n$ we have that

$$\|\sum_{k=1}^n a_k x_k\|_X = \|\sum_{k=1}^n \epsilon_k(t) x_{\sigma(k)}\|_X.$$

This does not mean that the sequence is unconditional in the usual complex sense and so we incorporate the word real to remind the reader of this distinction.

THEOREM 2.1. *Suppose* $1 < r < 2, 0 < \theta < 1,$ *and* $1 \leq K < \infty$. *Then there is a constant* $c = c(r, \theta, K) > 0$ *with the following property. Suppose* $N \in \mathbf{N}$ *and that* X_0, X_1 *are two* N*-dimensional Banach spaces with* $T_r(X_0) \leq K$ *and* $T_2(X_1) \leq K$. *Suppose* $X_\theta = [X_0, X_1]_\theta$. *Then for any 1-(real) symmetric basic sequence* $x_1, x_2, \ldots x_n \in X_\theta$ *and any real numbers* $a_1 \geq a_2 \geq \cdots a_n \geq 0$ *we have*

$$\|\sum_{k=1}^n a_k x_k\|_{X_\theta} \geq cn^{-1/p}\left(\sum_{k=1}^n k^{2(1-\theta)/\theta r}|a_k|^{2/\theta}\right)^{\theta/2}\|\sum_{k=1}^n x_k\|_{X_\theta}$$

where $1/p = \theta/2 + (1-\theta)/r$.

PROOF: We start by observing (see [4]) that since $r > 1$ there is a constant $K_0 = K_0(K, r)$ so that X_0^* and X_1^* are K-convex with K-convexity constant bounded by K_0. For convenience we set $M = \|x_1 + \ldots x_n\|_{X_\theta}$.

It will now be convenient to transform the setting to that of interpolation of families on the unit disk (see [1]). We define a family of norms on $\mathbf{C}^N$ for $z \in \mathbf{T}$ by setting $\|x\|_{Y_{e^{i\tau}}} = \|x\|_{X_1}$ whenever $|\tau| \leq \pi\theta$ and $\|x\|_{Y_z} = \|x\|_{X_0}$ for other z. We can then interpolate to produce a family Y_z for $|z| \leq 1$ and we have $Y_0 = X_\theta$. We note that the family $L_2(S_n \times D_n; Y_z)$ also forms an interpolation family. We introduce $\xi_k \in L_2(Y_z)$ defined by $\xi_k(\sigma, t) = \epsilon_k(t) x_{\sigma(k)}$. Then the sequence (ξ_k) is isometrically equivalent to (x_k). Let $\xi = \sum \xi_k$, so that $\|\xi\|_{L_2(Y_0)} = M$.

Now for $t \in D_n$ we may choose $u(t) \in Y_0^*$ with $\|u(t)\|_{Y_0^*} = 1$ and

$$\langle \sum_{k=1}^n \epsilon_k(t) x_k, u(t) \rangle = M.$$

We can then for each such t find an analytic function $\phi(t) : \Delta \to \mathbf{C}^n$ in class $\mathcal{N}^+$ so that $\|\phi(t, z)\|_{Y_z^*} \leq 1$ a.e. for $z \in \mathbf{T}$ and such that $\phi(t, 0) = u(t)$.

Now define, for $k = 1, 2 \ldots, n$, $|z| \le 1$.

$$\phi_k(z) = \int_{D_n} \epsilon_k(t)\phi(t,z)\,dt.$$

Then for $|z| = 1$, and indeed also for $|z| < 1$, we have

$$\left(\int_{D_n} \|\sum_{k=1}^{n} \epsilon_k \phi_k(z)\|_{Y_z^*}^2 dt\right)^{1/2} \le K_0.$$

We next introduce $\psi_k(z) \in L_2(S_n \times D_n; Y_z^*)$ by setting

$$\psi(z)(\sigma, t) = \epsilon_k(t)\phi_{\sigma(k)}(z).$$

Then for each $z \in \Delta$ and for a.e. $z \in \mathbf{T}$, the sequence $\{\psi_k(z)\}_{k=1}^n$ is 1-(real) symmetric. Furthermore we have

$$\|\sum_{k=1}^{n} \psi_k(z)\|_{L_2(Y_z^*)} \le K_0$$

and

$$\begin{aligned}
\langle \xi_k, \psi_j(0)\rangle &= \int_{S_n}\int_{D_n} \langle \epsilon_k(t)x_{\sigma(k)}, \epsilon_j(t)\phi_{\sigma(j)}(0)\rangle dt\, d\sigma \\
&= \delta_{j,k} \int_{S_n} \langle x_{\sigma(k)}, \phi_{\sigma(k)}(0)\rangle d\sigma \\
&= \delta_{j,k} \frac{1}{n} \sum_{j=1}^{n} \langle x_j, \phi_j(0)\rangle \\
&= \delta_{j,k} \frac{1}{n} \sum_{j=1}^{n} \int_{D_n} \epsilon_j(t) \langle x_j, u(t)\rangle dt \\
&= \delta_{j,k} \frac{1}{n} \int_{D_n} \langle \sum_{j=1}^{n} \epsilon_j(t) x_j, u(t)\rangle dt \\
&= \delta_{j,k} \frac{M}{n}
\end{aligned}$$

where $\delta_{j,k}$ is the Kronecker delta. Hence also:

$$\langle \xi, \sum_{k=1}^{n} \psi_k(0)\rangle = M.$$

It follows that $\| \sum_{k=1}^n \psi_k(0)\|_{L_2(Y_0^*)} \ge 1$.

We now define a family of symmetric lattice norms on $\mathbf{C}^n$ for $z \in \mathbf{T}$. Let

$$\|a\|_{E_z} = \|\sum_{k=1}^{n} |a_k| \psi_k(z)\|_{Y_z^*}.$$

Let us show that the family E_z is an admissible family of norms. Clearly $\| \|_{E_z}$ is Borel measurable on $\mathbf{C}^n \times \mathbf{T}$ when suitably extended to be defined on a set of measure zero. We further note that if $h(z) = \|e_1 + \cdots e_n\|_{E_z}$ then

$$\frac{1}{n}h(z)\|a\|_\infty \leq \|a\|_{E_z} \leq h(z)\|a\|_\infty.$$

Clearly $h(z) \leq K_0$, a.e. However

$$\begin{aligned}\int_{\mathbf{T}} \log h(z)d\lambda(z) &= \int_{\mathbf{T}} \log \|\sum_{k=1}^{n} \psi_k(z)\|_{L_2(Y_z^*)}d\lambda \\ &\geq \log \|\sum_{k=1}^{n} \psi_0(z)\|_{L_2(Y_0^*)} \\ &\geq 0.\end{aligned}$$

Thus we can extend the definition of E_z to $|z| < 1$ by interpolation.

Now define $S_z : \mathbf{C}^n \to L_2(Y_z^*)$ by

$$S_z(a) = \sum_{k=1}^{n} a_k\psi_k(z).$$

It is then clear that

$$\frac{1}{2}\|a\|_{E_z} \leq \|S_z a\|_{L_2(Y_z^*)} \leq 2\|a\|_{E_z}.$$

Thus S_z defines a 4-isomorphism of E_z onto a closed subspace of $L_2(Y_z^*)$. Further the map $z \to S_z$ is in class $\mathcal{N}^+$ in the finite-dimensional spaces of all linear maps from $\mathbf{C}^n$ to $L_2(S_n \times D_n; \mathbf{C}^N)$. In particular if $f : \Delta \to C^n$ is in $\mathcal{N}^+$ then so is $S_z \circ f$ and as $\|S_z\|_{E_z \to L_2(Y_z^*)} \leq 2$, we have the same inequality for $|z| < 1$.

Now for any $a = (a_1, a_2, \ldots, a_n) \in \mathbf{C}^n$ we have

$$S_0^*(\sum_{k=0}^{n} a_k\xi_k) = \frac{M}{n}a$$

so that

$$\|\sum_{k=1}^{n} a_k x_k\|_{X_\theta} \geq \frac{M}{2n}\|a\|_{E_0^*}.$$

We turn to the estimation of $\| \|_{E_0^*}$. To do this we consider E_z^* for $|z| = 1$. Then E_z^* is 4-isomorphic to a quotient of $L_2(Y_z)$. If $z = e^{i\tau}$ where $|\tau| \leq \pi\theta$ then $T_2(E_z^*) \leq 4K$. Hence E_z^* has 2-convexity constant bounded by $4\sqrt{2}K$ ([2], Proposition 1.f.17) and if $a \geq 0$, using the symmetry of E_z^*,

$$\|a\|_2\|e\|_{E_z^*} \leq 4\sqrt{2}K\sqrt{n}\|a\|_{E_z^*}$$

Now $\|e\|_{E_z^*} = n/\|e\|_{E_z} \geq n/K_0$. Thus we conclude that for some $K_1 = K_1(K, r, \theta)$,

$$\|a\|_2 \leq \frac{K_1}{\sqrt{n}}\|a\|_{E_z^*}.$$

For other values of z we deduce that $T_r(E_z^*) \leq CK$, for some C depending only on r. We have immediately by symmetry

$$\begin{aligned}\|\sum_{k=1}^{l} e_k\|_{E_z^*}^r &\geq (CK)^{-r}\frac{l}{2n}\|\sum_{k=1}^{n} e_k\|_{E_z^*}^r \\ &\geq K_2^{-r} l n^{r-1},\end{aligned}$$

where K_2 depends only on K, r and θ. From this we have that if $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, then

$$\max_{1\leq k\leq n} k^{1/r} a_k \leq K_2 n^{1/r-1}\|a\|_{E_z^*}.$$

Now suppose $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Then there exists a Borel measurable map $v : \mathbf{T} \to R_+^n$ so that $\|v(z)\|_{E_z^*} \leq \|a\|_{E_0^*} = A$, say, and

$$\log a_k = \int_{\mathbf{T}} \log v_k(z) d\lambda.$$

Now let $w(z)$ be the vector obtained from $v(z)$ by rearranging the co-ordinates in decreasing order. Then let $b_k \geq 0$ be defined by

$$\log b_k = \int_{\mathbf{T}} \log w_k(z) d\lambda.$$

Then $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$, and clearly

$$\sum_{j=1}^{k} \log a_j \leq \sum_{j=1}^{k} \log b_j$$

for $k = 1, 2, \ldots, n$. For convenience let us assume that $a_n > 0$; some minor modifications must be made in the other case. In this case we have equality in the above equation when $k = n$. It follows from a well-known lemma of Hardy, Littlewood and Polya (cf [2] Proposition 2.a.5) that $(\log a_k)_{k=1}^n$ is in the convex hull of all permutations of $(\log b_k)_{k=1}^n$. Thus since the Lorentz space $\ell_{p,2/\theta}$ is a Banach space under a suitable renorming we have the existence of a constant K_3 depending only on r, θ such that

$$\sum_{k=1}^{n} k^{2(1-\theta)/\theta r} a_k^{2/\theta} \leq K_3 \sum_{k=1}^{n} k^{2(1-\theta)/\theta r} b_k^{2/\theta}.$$

Now

$$\begin{aligned}\log(k^{(1-\theta)/r}|b_k|) &= \frac{(1-\theta)}{r}\log k + \int_{\mathbf{T}} \log w_k(z)\, d\lambda(z) \\ &= \frac{1}{2\pi}\int_{-\pi\theta}^{\pi\theta} \log w(e^{i\tau}) d\tau + \frac{1}{2\pi}\int_{\pi\theta<|\tau|<\pi} \log(k^{1/r} w_k(e^{i\tau})) d\tau.\end{aligned}$$

Now we may use the estimate if $|\tau| > \pi\theta$ that $k^{1/r}w_k(e^{i\tau}) \leq K_2 n^{1/r-1}A$. We conclude that

$$k^{(1-\theta)/r}b_k \leq (K_2 n^{1/r-1}A)^{1-\theta}\exp\left(\int_{-\pi\theta}^{\pi\theta}\log w(e^{i\tau})\frac{d\tau}{2\pi}\right).$$

From this and the geometric-arithmetic mean inequality we deduce that

$$k^{2(1-\theta)/\theta r}b_k^{2/\theta} \leq (K_2 n^{1/r-1}A)^{2(1-\theta)/\theta}\int_{-\pi\theta}^{\pi\theta} w(e^{i\tau})^2\frac{d\tau}{2\pi\theta}.$$

Now summing over k,

$$\sum_{k=1}^{n} k^{2(1-\theta)/\theta r}b_k^{2/\theta} \leq (K_2 n^{1/r-1}A)^{2(1-\theta)/\theta)}\int_{-\pi\theta}^{\pi\theta}\|w(e^{i\tau})\|_2^2\frac{d\tau}{2\pi\theta}.$$

For $|\tau| \leq \pi\theta$, we recall $\|w(e^{i\tau})\|_2 \leq K_1 n^{-1/2}A$. Hence

$$\sum_{k=1}^{n} k^{2(1-\theta)/\theta r}b_k^{2/\theta} \leq (K_2 n^{1/r-1}A)^{2(1-\theta)/\theta}(K_1 n^{-1/2}A)^2 \leq (K_4 n^{1/p-1}A)^{2/\theta}$$

for some K_4 depending only on K, θ and r. Thus

$$\begin{aligned}(\sum_{k=1}^{n} k^{2(1-\theta)/\theta r}b_k^{2/\theta})^{\theta/2} &\leq K_4 n^{1/p-1}\|a\|_{E_0^*}\\ &\leq \frac{2K_4}{M}n^{1/p}\|\sum_{k=1}^{n} a_k x_k\|_{X_\theta}.\end{aligned}$$

and the theorem follows.■

THEOREM 2.2. *Suppose $1 < p < 2$ and $q < s < \infty$ where $\frac{1}{p}+\frac{1}{q} = 1$. Then given $\epsilon > 0$, $K < \infty$ there is a constant $c = c(K, \epsilon, p, s) > 0$ so that whenever X_0 and X_1 are $N-$dimensional Banach spaces with $T_2(X_1) \leq K$, and F is an $n-$dimensional subspace of $[X_0, X_1]_\theta$ where $1/p = 1 - \theta/2$ then $d(F, \ell^n(p,s)) \geq c(\log n)^{1/q-1/s-\epsilon}$.*

PROOF: Pick any $0 < \alpha < \theta$ and consider$[X_0, X_1]_\alpha = X_\alpha$. Then Y is of type r with $T_r(Y_0) \leq K^\alpha \leq K$. Further $\theta = (1-\beta)\alpha + \beta$ where $0 < \beta < 1$ and by the re-iteration theorem $X_\theta = [X_\alpha, X_1]_\beta$.

Let $S : \ell^n_{p,s} \to F$ be any isomorphism, and let $Se_k = f_k$ for $1 \leq k \leq n$. We will argue that we may suppose that $(f_k)_{k=1}^n$ is 1-(real) symmetric. We can define $\tilde{S} : \ell^n_{p,s} \to L_2(S_n \times D_n; [X_0, X_1]_\theta)$ by $\tilde{S}e_k(\sigma, t) = \epsilon_k(t)f_{\sigma(k)}$ and $\tilde{S}$ is an isomorphism onto a subspace $\tilde{F}$ of $[L_2(X_0), L_2(X_1)]_\theta$ such that $\|\tilde{S}\|\|\tilde{S}^{-1}\| \leq \|S\|\|S^{-1}\|$. This reasoning allows us to suppose that f_k is already 1-(real) symmetric.

Now there is a constant $c = c(K, \alpha, p) > 0$ so that for any $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ we have

$$\|\sum_{k=1}^{n} a_k f_k\|_{X_\theta} \geq cn^{-1/p}(\sum_{k=1}^{n} k^{2(1-\beta)/\beta r}a_k^{2/\beta})^{\beta/2}\|\sum_{k=1}^{n} f_k\|_{X_\theta}.$$

Let us choose $a_k = k^{-1/p}$. Then we have

$$\|\sum_{k=1}^{n} a_k f_k\|_{X_\theta} \geq cn^{-1/p}(\log n)^{\beta/2}\|\sum_{k=1}^{n} f_k\|_{X_\theta}.$$

Clearly

$$\|\sum_{k=1}^{n} f_k\|_{X_\theta} \geq c_1\|S^{-1}\|^{-1}n^{1/p}$$

for a suitable $c_1 = c_1(p,s)$. Also

$$\|\sum_{k=1}^{n} a_k f_k\| \leq C\|S\|(\log n)^{1/s}$$

for some suitable $C = C(p,s)$. If we combine these we have

$$\|S\|\|S^{-1}\| \geq c_2(\log n)^{\beta/2-1/s}$$

where $c_2 > 0$ depends on α, p, K. As $\alpha \to 0$, we have $\beta \to \theta$ and $\beta/2 \to 1/q$ whence the result.∎

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