On a question of Pisier

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1. Introduction.

It is well-known that if \( X_0 \) and \( X_1 \) are Banach spaces with \( X_1 \) of (Rademacher) type 2 then a complex interpolation space \([X_0, X_1]_\theta\) is of type \( p \) where \( 1/p = 1 - \theta/2 \). In [3] Pisier asks if there is a converse to this observation. Precisely he asks whether given a Banach space \( X \) of (Rademacher) type \( p \) where \( 1 < p < 2 \) one can find a pair of Banach spaces \( X_0, X_1 \) with \( X_1 \) of type 2 so that \( X \) is isomorphic to the complex interpolation space \([X_0, X_1]_\theta\) where \( 1/p = 1 - \theta/2 \).

In this note we will consider the natural finite-dimensional version of this question. It is natural to ask whether given \( 1 < p < 2 \) and \( 1 < K < \infty \) there exist a constant \( C = C(p, K) \) so that if \( X \) is an \( n \)-dimensional Banach space with type \( p \) constant \( T_p(X) \leq K \) then one can find \( N \geq n \), two \( N \)-dimensional Banach spaces \( X_0, X_1 \) so that \( T_2(X_1) \leq C \), and an \( n \)-dimensional subspace \( F \) of \([X_0, X_1]_\theta\) (where \( 1/p = 1 - \theta/2 \)) such that \( d(X, F) \leq C \). (Notice that we only ask if \( X \) can be well-embedded into \([X_0, X_1]_\theta\) whereas Pisier asks for \( X \) to be isomorphic to \([X_0, X_1]_\theta\).)

We show that the answer to this question is negative. In fact we show the following result about the Lorentz spaces \( \ell_{p,s} \) where \( 1 < p < 2 < s < \infty \) and \( 1/p + 1/s < 1 \). Given any constant \( K \) and any \( \epsilon > 0 \) there exists a constant \( c = c(K, \epsilon, p, s) > 0 \) so that if \( X_0, X_1 \) are \( N \)-dimensional Banach spaces with \( T_2(X_1) \leq K \) and if \( F \) is an \( n \)-dimensional subspace of \([X_0, X_1]_\theta\) where \( 1/p = 1 - \theta/2 \) then \( d(F, \ell_{p,s}^n) \geq c(\log n)^{1-1/p-1/s-\epsilon} \). The Lorentz spaces \( \ell_{p,s} \) are of type \( p \) (see [2], Theorem 1.f.10) so that the spaces \( \ell_{p,s}^n \) have uniformly bounded type \( p \) constants.

We now explain our notation. Let us note first that we regard any \( N \)-dimensional Banach space as being the same underlying vector space \( C^N \) equipped with a particular norm. On \( C^N \) we have the natural bilinear pairing defined simply by

\[
(x, y) = \sum_{k=1}^{N} x_k y_k.
\]

Hence if \( X \) is an \( N \)-dimensional Banach space we can define \( X^* \) by

\[
\|x\|_{X^*} = \sup_{\|y\|_X \leq 1} |(x, y)|.
\]

For any \( n \in \mathbb{N} \) we let \( D_n = \{-1, +1\}^n \) be the dyadic group with \( 2^n \) elements. For \( t \in D_n \) and \( 1 \leq k \leq n \) we define \( \epsilon_k(t) = t_k \) where \( t = (t_j)_{j=1}^n \). We use \( dt \) to denote normalized Haar measure on the finite group \( D_n \). Similarly we let \( S_n \) be the group of

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permutations of \( \{1, 2, \ldots, n\} \) and use \( \sigma \) to denote Haar measure on it. We may then consider the finite-dimensional spaces \( L_2(D_n; X) \) and here the bilinear pairing is defined by

\[
(f, g) = \int (f(t), g(t)) \, dt
\]

and similarly for \( L_2(S_n \times D_n; X) \).

We recall that the type \( p \) constant of \( X \), \( T_p(X) \) is the least constant \( T \) such that for any \( n \in \mathbb{N} \), and \( x_1, \ldots, x_n \in X \) we have

\[
(\int_{D_n} \left\| \sum_{k=1}^n \epsilon_k(t) x_k \right\|^p dt)^{1/p} \leq T \left( \sum_{k=1}^n \| x_k \|^p \right)^{1/p}.
\]

We shall also need the \( K \)-convexity constant \( K(X) \) which is the least constant \( K \) such that for any \( n \) and any \( f \in L_2(D_n; X) \) we have

\[
\left\| \sum_{k=1}^n (\int_{D_n} \epsilon_k(t) f(t) dt) \epsilon_k \right\|_{L_2(X)} \leq K \| f \|_{L_2(X)}.
\]

Pisier [4] shows that if \( p > 1 \) then \( K(X) \) is bounded by a constant depending only on \( p \) and \( T_p(X) \).

The Lorentz sequence spaces \( \ell_{p, s} \) where \( 1 < p, s < \infty \) consist of all sequences \( \xi = (\xi_n) \in c_0 \) such that if \( (\xi^*_n) \) denotes the decreasing rearrangement of \( (|\xi_n|) \) then

\[
\| \xi \|_{p, s} = \left( \sum_{n=1}^\infty n^{1/p-1/s} (\xi^*_n)^s \right)^{1/s} < \infty.
\]

Strictly speaking, if \( p < s \) then \( \| \|_{p, s} \) is a quasi-norm but it is equivalent to a norm. If \( 1 < p \leq s < \infty \) then \( \ell_{p, s} \) is of type \( p \), (see [2], Theorem 1.f.10). We use \( \ell^n_{p, s} \) to denote the \( n \)-dimensional analogue of \( \ell_{p, s} \); these spaces have uniformly bounded type \( p \) constants.

Finally let us mention interpolation. We will use the ideas of interpolation of families developed in [1]. Suppose, for every \( z \in \mathbb{T} \), except on a Borel set of measure zero, we are given a norm \( \|\|_{X_z} \) on \( \mathbb{C}^N \). Suppose that the map \( (z, x) \to \| x \|_{X_z} \) is a Borel function. Suppose that for some fixed norm \( \| \|_0 \) there exists a Borel function \( h \) on \( \mathbb{T} \) such that (letting \( \lambda \) denote Haar measure on the circle)

\[
\int |\log h| d\lambda < \infty
\]

and \( h(z)^{-1} \| x \|_0 \leq \| x \|_z \leq h(z) \| x \|_2 \). We say that a map \( \phi : \Delta = \{ z : |z| < 1 \} \) is in class \( \mathcal{N}^+ \) if each co-ordinate is in the Smirnov class \( \mathcal{N}^+ \); then \( \phi \) extends a.e. to the boundary. We then define for \( z \in \Delta \)

\[
\| x \|_{X_z} = \inf \sup \| \phi(z) \|_{X_z}
\]

where the infimum is taken over all \( \phi \in \mathcal{N}^+ \) such that \( \phi(z) = x \).

It is then possible to show that the infimum is actually attained. In the special case of interest to us when each \( X_z \) is a lattice for \( |z| = 1 \) then the same is true for \( |z| < 1 \).
Further if \( \|x\|_{X_\sigma} \leq 1 \) with \( x \geq 0 \) then there exists a Borel function \( \phi : T \to R_+^n \) such that for \( 1 \leq k \leq n \),

\[
x_k = \exp(\int_T \log(\phi(z))_kd\lambda(z))
\]

where the integral can take the value \(-\infty\).

2. The main result.

Before stating our first technical result we introduce a definition. We shall say that a finite sequence \( \{x_1, \ldots, x_n\} \) in a Banach space \( X \) is \( 1 \)-real symmetric if for any \( \sigma \in S_n \) and any \( t \in D_n \) we have that

\[
\|\sum_{k=1}^n a_k x_k\|_X = \|\sum_{k=1}^n \epsilon_k(t) x_{\sigma(k)}\|_X.
\]

This does not mean that the sequence is unconditional in the usual complex sense and so we incorporate the word real to remind the reader of this distinction.

**Theorem 2.1.** Suppose \( 1 < r < 2, 0 < \theta < 1 \), and \( 1 \leq K < \infty \). Then there is a constant \( c = c(r, \theta, K) > 0 \) with the following property. Suppose \( N \in N \) and that \( X_0, X_1 \) are two \( N \)-dimensional Banach spaces with \( T_r(X_0) \leq K \) and \( T_2(X_1) \leq K \). Suppose \( X_\theta = [X_0, X_1]_\theta \). Then for any \( 1 \)-real symmetric basic sequence \( x_1, x_2, \ldots, x_n \in X_\theta \) and any real numbers \( a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \) we have

\[
\|\sum_{k=1}^n a_k x_k\|_{X_\theta} \geq cn^{-1/p}(\sum_{k=1}^n k^{2(1-\theta)/\theta r}|a_k|^{2/\theta})^{\theta/2}\|\sum_{k=1}^n x_k\|_{X_\theta}
\]

where \( 1/p = \theta/2 + (1 - \theta)/r \).

**Proof:** We start by observing (see [4]) that since \( r > 1 \) there is a constant \( K_0 = K_0(K, r) \) so that \( X_0^*_r \) and \( X_1^* \) are \( K \)-convex with \( K \)-convexity constant bounded by \( K_0 \). For convenience we set \( M = \|x_1 + \ldots + x_n\|_{X_\theta} \).

It will now be convenient to transform the setting to that of interpolation of families on the unit disk (see [1]). We define a family of norms on \( C^N \) for \( z \in T \) by setting \( \|x\|_{Y_z} = \|x\|_{X_1} \) whenever \( |\tau| \leq \pi \theta \) and \( \|x\|_{Y_z} = \|x\|_{X_0} \) for other \( z \). We can then interpolate to produce a family \( Y_z \) for \( |z| \leq 1 \) and we have \( Y_0 = X_\theta \). We note that the family \( L_2(S_n \times D_n; Y_z) \) also forms an interpolation family. We introduce \( \xi_k \in L_2(Y_z) \) defined by \( \xi_k(\sigma, t) = \epsilon_k(t) x_{\sigma(k)} \). Then the sequence \( (\xi_k) \) is isometrically equivalent to \( (x_k) \). Let \( \xi = \sum \xi_k \), so that \( \|\xi\|_{L_2(Y_0)} = M \).

Now for \( t \in D_n \) we may choose \( u(t) \in Y_0^* \) with \( \|u(t)\|_{Y_0^*} = 1 \) and

\[
(\sum_{k=1}^n \epsilon_k(t)x_k, u(t)) = M.
\]

We can then for each such \( t \) find an analytic function \( \phi(t) : \Delta \to C^n \) in class \( \mathcal{N}^+ \) so that \( \|\phi(t, z)\|_{Y_z^*} \leq 1 \) a.e. for \( z \in T \) and such that \( \phi(t, 0) = u(t) \).
Now define, for $k = 1, 2, \ldots, n$, $|z| \leq 1$.

$$
\phi_k(z) = \int_{D_n} \epsilon_k(t) \phi(t, z) \, dt.
$$

Then for $|z| = 1$, and indeed also for $|z| < 1$, we have

$$
\left( \int_{D_n} \left\| \sum_{k=1}^{n} \epsilon_k \phi_k(z) \right\|_{Y_x^*}^2 \, dt \right)^{1/2} \leq K_0.
$$

We next introduce $\psi_k(z) \in L_2(S_n \times D_n; Y_x^*)$ by setting

$$
\psi(z)(\sigma, t) = \epsilon_k(t) \phi_{\sigma(k)}(z).
$$

Then for each $z \in \Delta$ and for a.e. $z \in T$, the sequence $\{\psi_k(z)\}_{k=1}^{n}$ is 1-(real) symmetric. Furthermore we have

$$
\left\| \sum_{k=1}^{n} \psi_k(z) \right\|_{L_2(Y_x^*)} \leq K_0
$$

and

$$
\langle \xi_k, \psi_j(0) \rangle = \int_{S_n} \int_{D_n} \langle \epsilon_k(t)x_{\sigma(k)}, \epsilon_j(t)\phi(0) \rangle \, dt \, d\sigma
$$

$$
= \delta_{j,k} \int_{S_n} \langle x_{\sigma(k)}, \phi(k) \rangle \, d\sigma
$$

$$
= \delta_{j,k} \frac{1}{n} \sum_{j=1}^{n} \langle x_j, \phi_j(0) \rangle
$$

$$
= \delta_{j,k} \frac{1}{n} \sum_{j=1}^{n} \int_{D_n} \epsilon_j(t) \langle x_j, u(t) \rangle \, dt
$$

$$
= \delta_{j,k} \frac{1}{n} \int_{D_n} \left( \sum_{j=1}^{n} \epsilon_j(t)x_j, u(t) \right) \, dt
$$

$$
= \delta_{j,k} \frac{M}{n}
$$

where $\delta_{j,k}$ is the Kronecker delta. Hence also:

$$
\langle \xi, \sum_{k=1}^{n} \psi_k(0) \rangle = M.
$$

It follows that $\| \sum_{k=1}^{n} \psi_k(0) \|_{L_2(Y_x^*)} \geq 1$.

We now define a family of symmetric lattice norms on $C^n$ for $z \in T$. Let

$$
\|a\|_{E_z} = \| \sum_{k=1}^{n} |a_k| \psi_k(z) \|_{Y_x^*}.
$$
Let us show that the family $E_z$ is an admissible family of norms. Clearly $\|\cdot\|_{E_z}$ is Borel measurable on $C^n \times T$ when suitably extended to be defined on a set of measure zero. We further note that if $h(z) = \|e_1 + \cdots + e_n\|_{E_z}$ then

$$\frac{1}{n} h(z) \|a\|_\infty \leq \|a\|_{E_z} \leq h(z) \|a\|_\infty.$$ 

Clearly $h(z) \leq K_0$, a.e. However

$$\int_T \log h(z) d\lambda(z) = \int_T \log \|\sum_{k=1}^n \psi_k(z)\|_{L_2(Y_z^*)} d\lambda(z) \geq \log \|\sum_{k=1}^n \psi_0(z)\|_{L_2(Y_z^*)} \geq 0.$$ 

Thus we can extend the definition of $E_z$ to $|z| < 1$ by interpolation.

Now define $S_z : C^n \to L_2(Y_z^*)$ by

$$S_z(a) = \sum_{k=1}^n a_k \psi_k(z).$$

It is then clear that

$$\frac{1}{2} \|a\|_{E_z} \leq \|S_z a\|_{L_2(Y_z^*)} \leq 2 \|a\|_{E_z}.$$ 

Thus $S_z$ defines a 4-isomorphism of $E_z$ onto a closed subspace of $L_2(Y_z^*)$. Further the map $z \to S_z$ is in class $\mathcal{N}^+$ in the finite-dimensional spaces of all linear maps from $C^n$ to $L_2(S_n \times D_n; C^N)$. In particular if $f : \Delta \to C^n$ is in $\mathcal{N}^+$ then so is $S_z \circ f$ and as $\|S_z\|_{E_z \to L_2(Y_z^*)} \leq 2$, we have the same inequality for $|z| < 1$.

Now for any $a = (a_1, a_2, \ldots, a_n) \in C^n$ we have

$$S_0^*(\sum_{k=0}^n a_k \xi_k) = \frac{M}{n} a$$

so that

$$\|\sum_{k=1}^n a_k x_k\|_{X_\theta} \geq \frac{M}{2n} \|a\|_{E_0^*}.$$ 

We turn to the estimation of $\|\cdot\|_{E_0^*}$. To do this we consider $E_z^*$ for $|z| = 1$. Then $E_z^*$ is 4-isomorphic to a quotient of $L_2(Y_z)$. If $z = e^{i\tau}$ where $|\tau| \leq \pi \theta$ then $T_3(E_z^*) \leq 4K$. Hence $E_z^*$ has 2-convexity constant bounded by $4\sqrt{2}K$ ([2], Proposition 1.f.17) and if $a \geq 0$, using the symmetry of $E_z^*$,

$$\|a\|_2 \|e\|_{E_z^*} \leq 4\sqrt{2}K \sqrt{n} \|a\|_{E_z^*}.$$ 

Now $\|e\|_{E_z^*} = n/\|e\|_{E_z} \geq n/K_0$. Thus we conclude that for some $K_1 = K_1(K, r, \theta)$,

$$\|a\|_2 \leq \frac{K_1}{\sqrt{n}} \|a\|_{E_z^*}.$$
For other values of \( z \) we deduce that \( T_r(E^*_z) \leq CK \), for some \( C \) depending only on \( r \). We have immediately by symmetry

\[
\| \sum_{k=1}^{l} e_k \|_{E^*_z} \geq (CK)^{-r} \frac{l}{2n} \| \sum_{k=1}^{n} e_k \|_{E^*_z}
\]

\[
\geq K_2^{-r} ln^{r-1},
\]

where \( K_2 \) depends only on \( K, r \) and \( \theta \). From this we have that if \( a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \), then

\[
\max_{1 \leq k \leq n} k^{1/r} a_k \leq K_2 n^{1/r-1} \| a \|_{E^*_z}.
\]

Now suppose \( a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \). Then there exists a Borel measurable map \( v : T \to R^n_+ \) so that \( \| v(z) \|_{E^*_z} \leq \| a \|_{E^*_z} = A \), say, and

\[
\log a_k = \int_T \log v_k(z) d\lambda.
\]

Now let \( w(z) \) be the vector obtained from \( v(z) \) by rearranging the co-ordinates in decreasing order. Then let \( b_k \geq 0 \) be defined by

\[
\log b_k = \int_T \log w_k(z) d\lambda.
\]

Then \( b_1 \geq b_2 \geq \cdots \geq b_n \geq 0 \), and clearly

\[
\sum_{j=1}^{k} \log a_j \leq \sum_{j=1}^{k} \log b_j
\]

for \( k = 1, 2, \ldots, n \). For convenience let us assume that \( a_n > 0 \); some minor modifications must be made in the other case. In this case we have equality in the above case when \( k = n \). It follows from a well-known lemma of Hardy, Littlewood and Polya (cf [2] Proposition 2.a.5) that \( (\log a_k)_{k=1}^{n} \) is in the convex hull of all permutations of \( (\log b_k)_{k=1}^{n} \).

Thus since the Lorentz space \( l^{p,2/\theta}_r \) is a Banach space under a suitable renorming we have the existence of a constant \( K_3 \) depending only on \( r, \theta \) such that

\[
\sum_{k=1}^{n} k^{2/(1-\theta)\theta} a_k^{2/\theta} \leq K_3 \sum_{k=1}^{n} k^{2/(1-\theta)\theta} b_k^{2/\theta}.
\]

Now

\[
\log(k^{(1-\theta)/r} |b_k|) = \frac{(1-\theta)}{r} \log k + \int_T \log w_k(z) d\lambda(z)
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log w(e^{i\tau}) d\tau + \frac{1}{2\pi} \int_{\pi\theta < |\tau| < \pi} \log(k^{1/r} w_k(e^{i\tau})) d\tau.
\]
Now we may use the estimate if $|\tau| > \pi \theta$ that $k^{1/r} w_k(e^{i\tau}) \leq K_2 n^{1/r-1} A$. We conclude that

$$k^{1-\theta/r} b_k \leq (K_2 n^{1/r-1} A)^{1-\theta} \exp(\int_{-\pi \theta}^{\pi \theta} \log w(e^{i\tau}) \frac{d\tau}{2\pi}).$$

From this and the geometric-arithmetic mean inequality we deduce that

$$k^{2(1-\theta)/\theta} r_k^{2/\theta} \leq (K_2 n^{1/r-1} A)^{2(1-\theta)/\theta} \int_{-\pi \theta}^{\pi \theta} w(e^{i\tau})^2 \frac{d\tau}{2\pi \theta}.$$

Now summing over $k$,

$$\sum_{k=1}^{n} k^{2(1-\theta)/\theta} r_k^{2/\theta} \leq (K_2 n^{1/r-1} A)^{2(1-\theta)/\theta} \int_{-\pi \theta}^{\pi \theta} \|w(e^{i\tau})\|_2^2 \frac{d\tau}{2\pi \theta}.$$

For $|\tau| \leq \pi \theta$, we recall $\|w(e^{i\tau})\|_2 \leq K_1 n^{-1/2} A$. Hence

$$\sum_{k=1}^{n} k^{2(1-\theta)/\theta} r_k^{2/\theta} \leq (K_2 n^{1/r-1} A)^{2(1-\theta)/\theta} (K_1 n^{-1/2} A)^2 \leq (K_4 n^{1/p-1} A)^{2/\theta}$$

for some $K_4$ depending only on $K, \theta$ and $r$. Thus

$$\left( \sum_{k=1}^{n} k^{2(1-\theta)/\theta} r_k^{2/\theta} \right)^{\theta/2} \leq K_4 n^{1/p-1} \|a\|_{E_0}^\theta \leq \frac{2K_4}{M} n^{1/p} \| \sum_{k=1}^{n} a_k x_k \|_{X_\theta}^\theta.$$

and the theorem follows.

**THEOREM 2.2.** Suppose $1 < p < 2$ and $q < s < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then given $\epsilon > 0$, $K < \infty$ there is a constant $c = c(K, \epsilon, p, s) > 0$ so that whenever $X_0$ and $X_1$ are $N$-dimensional Banach spaces with $T_2(X_1) \leq K$, and $F$ is an $n$-dimensional subspace of $[X_0, X_1]_\theta$ where $1/p = 1 - \theta/2$ then $d(F, \ell^n(p, s)) \geq c(\log n)^{1/q-1/s-\epsilon}$.

**PROOF:** Pick any $0 < \alpha < \theta$ and consider $[X_0, X_1]_\alpha = X_\alpha$. Then $Y$ is of type $r$ with $T_r(Y_0) \leq K^\alpha \leq K$. Further $\theta = (1 - \beta)\alpha + \beta$ where $0 < \beta < 1$ and by the re-iteration theorem $X_\theta = [X_\alpha, X_1]_\beta$.

Let $S : \ell^n_{p,s} \to F$ be any isomorphism, and let $Se_k = f_k$ for $1 \leq k \leq n$. We will argue that we may suppose that $(f_k)_{k=1}^{n}$ is $1$-(real) symmetric. We can define $\tilde{S} : \ell^n_{p,s} \to L_2(S_n \times D_n; [X_0, X_1]_\theta)$ by $\tilde{S}e_k(\sigma, t) = \epsilon_k(t)f_{\sigma(k)}$ and $\tilde{S}$ is an isomorphism onto a subspace $\tilde{F}$ of $[L_2(X_0), L_2(X_1)]_\theta$ such that $\|\tilde{S}\|\|\tilde{S}^{-1}\| \leq \|S\|\|S^{-1}\|$. This reasoning allows us to suppose that $f_k$ is already $1$-(real) symmetric.

Now there is a constant $c = c(K, \alpha, p) > 0$ so that for any $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ we have

$$\| \sum_{k=1}^{n} a_k f_k \|_{X_\theta} \geq c n^{-1/r} \left( \sum_{k=1}^{n} k^{2(1-\beta)/\theta} r_k^{2/\theta} \right)^{\beta/2} \| \sum_{k=1}^{n} f_k \|_{X_\theta}.$$
Let us choose $a_k = k^{-1/p}$. Then we have
\[
\left\| \sum_{k=1}^{n} a_k f_k \right\|_{X_\sigma} \geq cn^{-1/p}(\log n)^{\beta/2} \left\| \sum_{k=1}^{n} f_k \right\|_{X_\sigma}.
\]
Clearly
\[
\left\| \sum_{k=1}^{n} f_k \right\|_{X_\sigma} \geq c_1 \|S^{-1}\|^{-1} n^{1/p}
\]
for a suitable $c_1 = c_1(p, s)$. Also
\[
\left\| \sum_{k=1}^{n} a_k f_k \right\| \leq C \|S\|(\log n)^{1/s}
\]
for some suitable $C = C(p, s)$. If we combine these we have
\[
\|S\| \|S^{-1}\| \geq c_2 (\log n)^{\beta/2 - 1/s}
\]
where $c_2 > 0$ depends on $\alpha, p, K$. As $\alpha \to 0$, we have $\beta \to \theta$ and $\beta/2 \to 1/q$ whence the result.\(\blacksquare\)

References.