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Remarks on lattice structure in ℓ_p and L_p when $0 < p < 1$.

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1. Introduction.

The object of this note is to give an application of interpolation theory to a problem in uniqueness of structure for quasi-Banach lattices. It was shown over twenty years ago by Lindenstrauss and Pelczynski ([15], [16]) that the Banach space ℓ_1 has, up to equivalence, a unique normalized unconditional basis; this may be rephrased as saying that ℓ_1 has a unique structure as a Banach lattice. The analogous result for the function space L_1, due to Abramovic and Wojtaszczyk [1] is that L_1 has unique structure as a nonatomic Banach lattice (i.e. if Y is a nonatomic Banach lattice isomorphic to L_1, then Y is lattice isomorphic to L_1).

We shall consider the case $0 < p < 1$. Here it is well-known that ℓ_p has, like ℓ_1, a unique normalized unconditional basis. For proofs see [8], [12] and [14]; however, an alternative proof in [20] appears to us incorrect. The corresponding result for the function space L_p seems to be much more elusive, although uniqueness of the symmetric structure has been established [21]; the problem is that all the known methods for the sequence space exploit in some form a reduction to finite dimensions which is apparently not available in the function space case.

In this note we exploit some ideas from complex interpolation of quasi-Banach lattices to give a proof of the uniqueness of structure of L_p for $0 < p < 1$. The basic idea to is reduce the question to the L_1−case and then extrapolate. The same idea yields yet another proof for the uniqueness of the unconditional basis of ℓ_p.

In our final section, we give one further proof of the uniqueness of the unconditional basis of ℓ_p, motivated in part by an attempt to extend the ideas of the original Lindenstrauss-Pelcyznski argument [15]. The problem is to find the correct analogue of Grothendieck's theorem that every operator from ℓ_1 to ℓ_2 is absolutely summing. Our replacement is Theorem 4.4 that every operator from ℓ_2 to itself which factors through ℓ_p (for $0 < p \leq 1$) is in the Schatten ideal $\mathcal{C}_r$ where $1/r = 1/p - 1/2$. For $p = 1$ this can be shown equivalent to Grothendieck's theorem.

2. Interpolation of quasi-Kothe function spaces.

Let K be a Polish space and let μ be a probability measure on K. (For brevity we say that (K, μ) is a Polish probability space.) We shall define a quasi-Kothe function space X to be a dense order-ideal in $L_0(K, \mu)$ equipped with a lattice quasi-norm $\| \, \|_X$ such that the unit ball $B_X = \{x \in X : \|x\|_X \leq 1\}$ is closed for the topology of convergence in measure. We emphasize that we incorporate in our definition this last requirement which is sometimes called the Fatou property (or maximality of X). X is called a Kothe function space if $\| \, \|_X$ is a norm [17].

If $0 < p < \infty$ then X is said to be p-convex if for any $x_1, \ldots, x_n \in X$ we have

$$\|(\sum_{k=1}^{n} |x_k|^p)^{1/p}\|_X \leq (\sum_{k=1}^{n} \|x_k\|^p)^{1/p}.$$

X is p-convex if and only its $1/p-$convexification X^p is a Kothe function space. Here X^p consists of all f such that $|f|^{1/p} \in X$ normed by $\|f\|_{X^p} = \||f|^{1/p}\|^p$.

If X_0 and X_1 are two quasi-Kothe function spaces and $0 < \theta < 1$ we define the space $X_\theta = X_0^\theta X_1^{1-\theta}$ to be the space of all $f \in L_0$ such that $|f|$ can be factored in the form $|f| = |g|^\theta |h|^{1-\theta}$ quasi-normed by

$$\|f\|_{X_\theta} = \inf\{\|g\|_{X_0}^\theta \|h\|_{X_1}^{1-\theta} : |f| = |g|^\theta |h|^{1-\theta}\}.$$

X is said to be g-convex if $X^{1-\theta} X^\theta = X$ (with equality of quasinorms); see [11]. If X is p-convex for some $0 < p < 1$ then X is g-convex. In fact g-convexity implies that X can be equivalently quasi-normed to be p-convex for some $0 < p < 1$ [11].

Before stating our first lemma, we remark that since we assume in our definition of quasi-Kothe function spaces X that the unit ball B_X is L_0-closed (the Fatou property) that X is separable if and only if it contains no copy of c_0; equivalently X is separable if and only if whenever x_n is an increasing bounded sequence in X_+ then x_n converges in X.

LEMMA 2.1. *Suppose $0 < p < 1$ and suppose X_0 and X_1 are p-convex quasi-Kothe spaces. Then $X_\theta = X_0^{1-\theta} X_1^\theta$ is a p-convex quasi-Kothe function space and if $x \in X_\theta$ then there exist $x_0 \in X_0$ and $x_1 \in X_1$ with $|x| = |x_0|^{1-\theta}|x_1|^\theta$. and $\|x_0\|_{X_0} = \|x_1\|_{X_1} = \|x\|_{X_\theta}$.*

If, in addition, either X_0 or X_1 is separable then X_θ is separable for $0 < \theta < 1$.

PROOF: Note first that X_0^p and Y_0^p are Kothe function spaces and so there is a strictly positive $h \in L_1$ so that the unit ball of each space is contained in the unit ball of $L_1(h\, d\mu)$.

Assume first that $w_n \in X_\theta$ with $\|w_n\|_{X_\theta} < 1$ and $w_n \to w$ almost everywhere. Then we can write $|w_n| = |u_n|^{1-\theta}|v_n|^\theta$ where $\|u_n\|_{X_0} \leq 1$ and $\|v_n\|_{X_1} \leq 1$. Now $|u_n|^p$ and $|v_n|^p$ are bounded in $L_1(h\, d\mu)$ and by Komlos's theorem [13] we can, by passing to a subsequence, suppose that the sequences s_n and t_n converge a.e. to some s, t where

$$s_n = \left(\frac{1}{n}\sum_{i=1}^{n} |u_i|^p\right)^{1/p} \quad t_n = \left(\frac{1}{n}\sum_{i=1}^{n} |v_i|^p\right)^{1/p}.$$

Then by p-convexity $\|s_n\|_{X_0}, \|t_n\|_{X_1} \leq 1$. Clearly by Holder's inequality we have

$$\left(\frac{1}{n}\sum_{i=1}^{n} |w_i|^p\right)^{1/p} \leq |s_n|^{1-\theta}|t_n|^\theta.$$

and so $|w| \leq s^{1-\theta}t^\theta$; this implies $\|w\|_{X_\theta} \leq 1$, and that w has an optimal factorization. We conclude that B_{X_θ} is closed in L_0 and that every $x \in X_\theta$ has the required optimal factorization. We omit the simple proof that X_θ is also p-convex which follows from Holder's inequality.

If X_1 is separable and $w \in B_{X_\theta}$ we show that $w\chi_{A_n}$ converges to zero in X_θ whenever $\mu(A_n) \to 0$. In fact $w = uv$ where $u \in B_{X_0}$ and $v \in B_{X_1}$ and the statement follows easily from the fact that $\|v\chi_{A_n}\|_{X_1} \to 0$. Thus X_θ is also separable.

We state first a simple extension of Calderon's complex interpolation of Kothe function spaces to the quasi-Kothe case.

THEOREM 2.2. *Let (K_1, μ_1) and (K_2, μ_2) be Polish probability spaces. Let X_0, X_1 be p-convex quasi-Kothe function spaces on (K_1, μ_1) and let Y_0, Y_1 be p-convex quasi-Kothe function spaces on (K_2, μ_2). Suppose that either X_0 or X_1 is separable. Let $T : X_0 + X_1 \to L_0(\mu_2)$ be a continuous linear operator such that $T(X_0) \subset Y_0$ and $T(X_1) \subset Y_1$. Then for $0 < \theta < 1$, T maps $X_\theta = X_0^{1-\theta} X_1^\theta$ into $Y_\theta = Y_0^{1-\theta} Y_1^\theta$ and*

$$\|T\|_{X_\theta \to Y_\theta} \le \|T\|_{X_0 \to Y_0}^{1-\theta} \|T\|_{X_1 \to Y_1}^{\theta}.$$

PROOF: Pick any strictly positive h in $X \cap Y$. Since X_θ is separable it is easy to see that

$$\|T\|_{X_\theta \to Y_\theta} = \sup\{\|T(\xi u^{1-\theta} v^\theta)\|_{Y_\theta}\}$$

where the norm is computed over all ξ with $|\xi| = 1$ a.e. and all $u \in B_{X_0,+}$, $v \in B_{X_1,+}$, such that there exists some M with $0 \le u, v \le Mh$.

Now by an application of Nikishin's theorem [19] there exists $0 < q < 1$ and a function $g \in L_1(\mu_2)$ with $g > 0$ a.e. such that T maps $X_0 + X_1$ boundedly into $L_q(g\, d\mu_2)$.

Let $\mathcal{S}$ denote the strip $0 < \Re z < 1$ in the complex plane and let Δ be the unit disk. Let $\varphi : \Delta \to \mathcal{S}$ be a conformal mapping. φ may be extended continously to map the unit circle $\mathbf{T}$ minus two points onto the boundary $\partial\mathcal{S}$. Suppose $|\xi| = 1$ a.e. and $u \in B_{X_0}$, $v \in B_{X_1}$ are such that $0 \le u, v \le Mh$ for suitable $M > 0$. Define $F : \Delta \to X \cap Y$ by $F(z) = \xi u^{1-\varphi(z)} v^{\varphi(z)}$. By an easy calculation $z :\to hF(z)$ is analytic into L_∞ and hence so is F into $X \cap Y$ (in the sense that it has a power series expansion valid in Δ). Thus $T \circ F$ is bounded and analytic into $L_q(g\, d\mu_2)$, and extends continuously to $\mathbf{T}$ minus two points. Now we claim that for a.e. $\tau \in K_2$ the map $z \to TF(z)(\tau)$ may be supposed to be in the Hardy space H_q. Suppose $TF(z) = \sum_{n \ge 0} f_n z^n$ for suitable $f_n \in L_q(g\, d\mu_2)$. It is clear that $f_n = T(F^{(n)}(0)/n!)$ and so f_n is bounded in L_q. Hence if $z \in \Delta$ the series $\sum_{n \ge 0} \|f_n\|^q |z|^{nq}$ converges and so $\sum_{n \ge 0} f_n(\tau) z^n$ converges absolutely a.e. for $\tau \in K_2$. Thus $TF(z)(\tau) = \sum_{n \ge 0} f_n(\tau) z^n$ a.e.; we can thus redefine $TF(z)$ pointwise by this formula. Then for $0 < r < 1$

$$\int_{K_2} \int_0^{2\pi} |TF(re^{it})(\tau)|^q \frac{dt}{2\pi}\, d\mu_2(\tau)$$

is uniformly bounded. Since the integrand is monotone increasing we deduce immediately that $TF(z)(\tau)$ is in H_q for a.e. τ. Further the boundary values $TF(e^{it})(\tau)$ then exist for a.e. t. It is easy then to check that we must have the identity $TF(e^{it}) = T(F(e^{it}))$ for a.e. t. Now from standard H_q–theory we deduce the inequality

$$\log |TF(re^{it})(\tau)| \le \frac{1}{2\pi} \int_0^{2\pi} P(r, t-s) \log |TF(e^{is})(\tau)| ds$$

for μ_2–a.e. τ. Defining $z_0 = \varphi^{-1}(\theta)$ it is easy to see that this implies, by the g-convexity of the spaces Y_0, Y_1, $|TF(z_0)| = y_0^{1-\theta}y_1^{\theta}$ where $y_0, y_1 \geq 0$ in $L_0(\mu_2)$,$\|y_0\|_{Y_0} \leq \|T\|_{X_0\to Y_0}$ and $\|y_1\|_{Y_1} \leq \|T\|_{X_1\to Y_1}$. The result then follows immediately.∎

The other result we shall need concerning interpolation is an extrapolation principle due to Cwikel and Nilsson [4]; see also [11].

THEOREM 2.3. *Let X_0, X_1 and Y_1 be p-convex quasi-Kothe function spaces on (K,μ). Suppose for some $0 < \theta < 1$ we have $X_0^{1-\theta}X_1^{\theta} = X_0^{1-\theta}Y_1^{\theta}$ up to equivalence of norm; then $X_1 = Y_1$ up to equivalence of norm.*

PROOF: This is established for Kothe function spaces in [4] and [11]. The general result follows by the obvious process of $1/p$-convexification.∎

3. Applications to uniqueness of unconditional structure.

Our first result is an extension of results proved in [6] and [7].

THEOREM 3.1. *Suppose $0 < p < 1$; then there is a constant $C = C(p)$ so that if (K,μ) is a Polish probability space, X is a separable p-convex quasi-Kothe function space on (K,μ) containing χ_K, and $T : X \to X$ is a bounded operator then there exists $w \in L_1(\mu)$ with $w > 0$ a.e. and $\int w\,d\mu = 1$ so that T extends to a bounded linear operator $T : L_2(w\,d\mu) \to L_2(w\,d\mu)$ with*

$$\|T\|_{L_2(w\,d\mu)\to L_2(w\,d\mu)} \leq C\|T\|_{X\to X}.$$

PROOF: Let $Y = X^p$ be the p-convexification of X. Then Y is a separable Kothe function space. Clearly $L_\infty \subset Y \subset X$ with continuity of the injections. We note that according to [22], Proposition 6, or [9] Theorem 3.3, there is a constant C_0 so that if $x_1,\ldots,x_n \in X$ then

$$\|(\sum_{i=1}^{n}|Tx_i|^2)^{1/2}\|_X \leq C_0\|T\|_X\|(\sum_{i=1}^{n}|x_i|^2)^{1/2}\|_X.$$

Let Γ_0 be the set $\phi \in Y$ such that for some $x_1,\ldots,x_n \in L_\infty$ we have

$$\phi \leq \sum_{i=1}^{n}|Tx_i|^2 - 4C_0^2\|T\|_X^2\sum_{i=1}^{n}|x_i|^2.$$

Let Γ be the closure of Γ_0 in Y. We claim that if B is a Borel set in K of positive measure, $\chi_B \notin \Gamma$. Indeed if not, then for any $\epsilon > 0$ we can find $y \geq 0$ with $\|y\|_Y \leq \epsilon$ and $x_1,\ldots,x_n \in L_\infty$ with

$$\chi_B + 4C_0^2\|T\|^2\sum_{i=1}^{n}|x_i|^2 \leq y + \sum_{i=1}^{n}|Tx_i|^2.$$

Thus

$$2C_0\|T\|(\sum_{i=1}^{n}|x_i|^2)^{1/2} \leq y^{1/2} + (\sum_{i=1}^{n}|Tx_i|^2)^{1/2}$$

and so

$$(2^p-1)C_0^p\|T\|^p\|(\sum_{i=1}^n |x_i|^2)^{1/2}\|_X^p \le \|y^{1/2}\|_X^p.$$

On the other hand

$$\chi_B \le y^{1/2} + (\sum_{i=1}^n |Tx_i|^2)^{1/2}$$

so that we have, setting $M=\|\chi_B\|_X$,

$$\begin{aligned} M^p &\le \|y^{1/2}\|_X^p + C_0^p\|T\|^p\|(\sum_{i=1}^n |x_i|^2)^{1/2}\|_X^p \\ &\le \frac{2^p}{2^p-1}\|y^{1/2}\|_X^p. \end{aligned}$$

Thus

$$M \le C_1\|y^{1/2}\|_X$$

where C_1 depends only on p. However

$$\|y^{1/2}\|_X \le \|y^{1/p}\|_X^{p/2}\|\chi_K\|_X^{1-p/2}$$

so that $\|y^{1/2}\|_X \le M^{1-p/2}\epsilon^{1/2}$. For small enough ϵ this yields a contradiction.

Now Y^* can be identified as a Kothe function space contained in L_1 in the natural way (since Y is separable). Let H be the set of $v \in Y^*$ so that $v \ge 0$, $\int_K v\,d\mu = 1$ and $\int_K v\phi \le 0$ for all $\phi \in \Gamma$. Clearly H is nonempty by the Hahn-Banach theorem. By considering countable convex combinations it contains w whose support has maximal measure. Let B be the complement of the support of w. If B has positive measure then there exists $v \in H$ with $\int_B v\,d\mu > 0$ by the Hahn-Banach theorem, and then $\frac{1}{2}(w+v)$ has larger support. Hence $w>0$ a.e. and clearly by construction

$$\int_K |Tx|^2 w\,d\mu \le 4C_0^2\|T\|_X^2 \int_K |x|^2 w\,d\mu$$

for all $w \in L_\infty$. T then extends in the obvious way to a linear operator on $L_2(w\,d\mu)$, with norm at most $2C_0\|T\|_X$.■

COROLLARY 3.2. *Suppose $0<p<1$; then there is a constant $C=C(p)$ so that if (K_1,μ_1) and (K_2,μ_2) are Polish probability spaces, X, Y are separable p-convex Kothe function spaces on (K_1,μ_1) and (K_2,μ_2) respectively, containing χ_{K_1}, χ_{K_2} respectively, and $S: X \to Y$, $T: Y \to X$ are bounded linear operators, then there exist $w_1 \in L_1(\mu_1)$, $w_2 \in L_1(\mu_2)$ with $w_1, w_2 > 0$ a.e. and*

$$\|S\|_{L_2(w_1\,d\mu_1)\to L_2(w_2\,d\mu_2)} \le C\|S\|_{X\to Y}, \qquad \|T\|_{L_2(w_2\,d\mu_2)\to L_2(w_1\,d\mu_1)} \le C\|T\|_{X\to Y}.$$

PROOF: We may suppose $\|S\|, \|T\| = 1$. We form a new Polish probability space $K_3 = K_1 \cup K_2$ with measure $\mu_3 = \frac{1}{2}(\mu_1+\mu_2)$. We then consider the obvious direct sum space $Z = X \oplus Y$ on (K_3,μ_3) with $\|x\|_Z = \|x\chi_{K_1}\|_X + \|x\chi_{K_2}\|_Y$. Define $R: Z \to Z$ by

$$Rx = S(x\chi_{K_1}) + T(x\chi_{K_2}).$$

The result then follows immediately from Theorem 3.1.∎

THEOREM 3.3. *Let (K_1,μ_1) and (K_2,μ_2) be Polish probability spaces, and suppose $0<p<1$. Suppose X and Y are p-convex separable Kothe function spaces on K_1 and K_2 respectively. Suppose X and Y are isomorphic as quasi-Banach spaces. Then for $0<\theta<1$, $X^{1-\theta}L_2^\theta$ is isomorphic to $Y^{1-\theta}L_2^\theta$.*

PROOF: By a simple change of density it suffices to consider the case when X, Y contain χ_{K_1}, χ_{K_2} respectively. Let $S: X \to Y$ and $T: Y \to X$ be bounded linear operators such that $ST = I_Y$, $TS = I_X$. By Corollary 3.2, we can find strictly positive weight functions w_1, w_2 in $L_1(\mu_1)$ and $L_1(\mu_2)$ respectively so that $S: L_2(w_1\,d\mu_1) \to L_2(w_2\,d\mu_2)$ and $T: L_2(w_2\,d\mu_2) \to L_2(w_1\,d\mu_1)$ are bounded. The hypotheses of Theorem 2.2 are easily checked and so S, T extend to maps between $X^{1-\theta}L_2(w_1\,d\mu_1)^\theta$ and $Y^{1-\theta}L_2(w_2\,d\mu_2)^\theta$ and clearly remain inverse to each other. Thus these spaces are isomorphic. However it is clear that $X^{1-\theta}L_2(w_1\,d\mu_1)^\theta$ is isomorphic to $X^{1-\theta}L_2^\theta$ by a simple change of density.∎

THEOREM 3.4. *Suppose $0<p<1$. Let X be a quasi-Kothe function space on some Polish probability space (K,μ). If X is isomorphic to $L_p[0,1]$ or then μ is nonatomic and there is a weight function $w \in L_0(\mu)$ with $w>0$ a.e. so that $X = L_p(w\,d\mu)$ up to equivalence of norms.*

PROOF: Since X is isomorphic to L_p, it can be assumed p-convex (see [9], remarks following Theorem 4.2; of course we only need to renorm X to be r-convex for some r as this follows from Theorem 4.2 itself). We have that $X^{1-\theta}L_2^\theta$ is isomorphic to L_q where $1/q = \theta/2 + (1-\theta)/p$. Choose θ so that $q=1$. Thus X is isomorphic to the Banach space L_1. Then by the theorem of Abramovic and Wojtaszczyk [1] $X^{1-\theta}L_2^\theta$ is an AL-space and hence is equal to some $L_1(v\,d\mu)$ up to equivalence of norm, where $v>0$ a.e. Now it follows from Theorem 2.3 that X is equal to $L_p(v^\alpha\,d\mu)$ where $\alpha(1-\theta)=1$. The fact that μ is nonatomic is now obvious since L_p has trivial dual.∎

THEOREM 3.5. *Suppose $0<p<1$. Suppose X is a quasi-Banach lattice which is isomorphic to $L_p[0,1]$. Then X is order-isomorphic to $L_p[0,1]$.*

PROOF: It is only necessary here to show that X is order-isomorphic to a quasi-Kothe function space. We may assume that X is p-convex as a lattice, by [9] as in the preceding theorem. Using the method of [10], Theorem 3.1 it is possible to show that for each $x \in X_+$ there is a lattice homomorphism $T_x : X \to L_p[0,1]$ with $T_x x \neq 0$. It then follows from a simple exhaustion argument that since X is separable there exists an injective lattice homomorphism $T: X \to L_p$. It can further be assumed that $T(X)$ is dense in L_p.

Since X contains no copy of c_0, it follows that X is order-complete and the quasi-norm on X is order-continuous. From these facts it now follows that $T(X)$ is a dense order-ideal in $L_0[0,1]$ and that if we let $Z = T(X)$ with $\|f\|_Z = \|T^{-1}f\|_X$ then Z is a p-convex quasi-Kothe function space on $[0,1]$ which is order-isomorphic to X. The result now follows.∎

It is easy to modify this argument to the discrete case. We then essentially demonstrate the following known theorem ([8],[12]). We will give an alternative proof in the last section.

THEOREM 3.6. *Any normalized unconditional basis of ℓ_p $0 < p < 1$, is equivalent to the canonical basis.*

4. An alternative approach for the uniqueness of the unconditional basis of ℓ_p.

We start by quoting an important factorization theorem of Bennett and Maurey [2], [3], [18].

THEOREM 4.1. *Suppose $0 < p < 1$; then there is a constant $C = C(p)$ such that if $T : c_0 \to \ell_p$ is a bounded linear operator then T factors as $T = DT_0$ where $D : \ell_1 \to \ell_p$ is a diagonal operator with $\|D\| \leq 1$ and $T_0 : c_0 \to \ell_1$ satisfies $\|T_0\| \leq C\|T\|$.*

Let us recall that an operator $T : X \to Y$ between Banach spaces is called r-absolutely summing if there is a constant C so that for any $x_1, \ldots, x_n \in X$ we have

$$(\sum_{k=1}^{n} \|Tx_k\|^r)^{1/r} \leq C \sup_{\|x^*\|\leq 1} (\sum_{k=1}^{n} |x^*(x_k)|^r)^{1/r}.$$

The least such constant C is denoted $\pi_r(T)$.

THEOREM 4.2. *Suppose $0 < p \leq 1$; then there is a constant $C = C(p)$ such that whenever $T : c_0 \to \ell_p$ is a bounded linear operator then T factors as $T = D_1T_0D_2$ where $D_1 : \ell_2 \to \ell_p$ and $D_2 : c_0 \to \ell_2$ are diagonal operators with $\|D_1\|, \|D_2\| \leq 1$ and $T_0 : \ell_2 \to \ell_2$ satisfies $\|T_0\| \leq C\|T\|$.*

PROOF: In fact the use of Theorem 4.1 allows us to restrict attention to the case $p = 1$. In this case, we have that T is 2-absolutely summing with $\pi_2(T) \leq K_G\|T\|$ where K_G is the Grothendieck constant [16] p.70. The Grothendieck-Pietsch factorization theorem (cf. [16] p.64 or [5] p.55) now yields a factorization $T = T_1D_2$ where $D_2 : c_0 \to \ell_2$ is diagonal with $\|D_2\| \leq 1$ and $T_1 : \ell_2 \to \ell_1$ satisfies $\|T_1\| \leq K_G\|T\|$. Next consider $T_1^* : \ell_\infty \to \ell_2$; then $\pi_2(T_1^*) \leq K_G^2\|T\|$ and so, utilizing weak* continuity we find a diagonal map $D_3 : \ell_\infty \to \ell_2$ so that $T_1^* = T_2D_3$ where $\|D_3\| \leq 1$ and $T_2 : \ell_2 \to \ell_2$ satisfies $\|T_2\| \leq K_G^2\|T\|$. Finally let $D_1 : \ell_2 \to \ell_1$ be defined as the adjoint of D_3 restricted to c_0 and let $T_0 = T_2^*$■

Now let us note a Corollary of Theorem 4.2, which is essentially known, but the author knows no explicit reference.

COROLLARY 4.3. *Suppose $0 < p \leq 1$. Then there is a constant $C = C(p)$, so that whenever $T : \ell_2 \to \ell_p$ is a bounded operator then there exists a diagonal operator $D : \ell_2 \to \ell_p$ with $\|D\| \leq 1$ and a bounded operator $A : \ell_2 \to \ell_2$ with $\|A\| \leq C\|T\|$ and $T = DA$.*

PROOF: It follows from simple compactness reasoning that it suffices to consider the case when T is defined on a finite-dimensional Hilbert space ℓ_2^n. Then for large enough N there is a map $Q : \ell_\infty^N \to \ell_2^n$ so that for all $x^* \in \ell_2^n = (\ell_2^n)^*$, $\|x^*\| \leq \|Q^*x^*\| \leq 2\|x^*\|$, so that Q is an approximate quotient map. If we apply Theorem 4.2 to TQ and write $B = T_0D_2$ we see that for $x \in \ell_2^n$, we have $\|D_1^{-1}Tx\|_2 \leq C\|T\|\|x\|_2$, in the sense that any co-ordinate of D_1 vanishes then the corresponding co-ordinate of Tx vanishes, and, by definition, the same co-ordinate of $D_1^{-1}Tx$ vanishes. Thus we may define $A = D_1^{-1}T$.■

Our next result is a formal extension of Grothendieck's theorem that every operator from ℓ_1 to ℓ_2 is absolutely summing to the case $0 < p < 1$. We recall that if T is a compact operator on a Hilbert space T is said to be in the Schatten ideal $\mathcal{C}_r$ if its singular values s_n satisfy $\|T\|_{\mathcal{C}_r} = (\sum s_n^r)^{1/r} < \infty$.

THEOREM 4.4. *Suppose $0 < p \leq 1$. Then an operator $T : \ell_2 \to \ell_2$ may be factored in the form $T = AB$ where $B : \ell_2 \to \ell_p$ and $A : \ell_p \to \ell_2$ are bounded operators if and only if T is in the Schatten ideal $\mathcal{C}_r$ where $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. Furthermore there is a constant $C = C(p)$ so that $\|T\|_{\mathcal{C}_r} \leq C\|A\|\|B\|$, for any such factorization.*

PROOF: One direction is trivial. For if T belongs to $\mathcal{C}_r$ then we can by selecting appropriate orthonormal bases reduce the problem to the case when T is diagonal, say $T = \text{diag}\{w_1, w_2, \ldots\}$ with $w_n \geq 0$ and $\sum w_n^r < \infty$. But then $\|Tx\|_p \leq \|w\|_r\|x\|_2$ and we may take $A = T : \ell_2 \to \ell_p$ and $B = I : \ell_p \to \ell_2$ as the identity.

For the other direction, we can use Corollary 4.3 to restrict ourselves to the case when B is diagonal, say $B = \text{diag}\{b_1, b_2, \ldots\}$ with $b_k \geq 0$. Then $B_1 = \text{diag}\{b_1^{r/2}, b_2^{r/2}, \ldots\}$ maps ℓ_2 into ℓ_1 and

$$\|B_1\|_{\ell_2 \to \ell_1} \leq \|B\|_{\ell_2 \to \ell_p}^{r/2}.$$

Thus we factor $B = B_2B_1$ where $B_2 = \text{diag}\{b_1^{1-r/2}, b_2^{1-r/2}, \ldots\}$ and

$$\|B_2\|_{\ell_1 \to \ell_p} \leq \|B\|^{1-r/2}.$$

Also notice that A factors as $A = \hat{A}I$ where $I : \ell_p \to \ell_1$ is the natural inclusion and $\hat{A}$ is the natural extension of A to the Banach envelope ℓ_1 of ℓ_p. Thus $\|\hat{A}\| = \|A\|$.

Thus we can write $AB = \hat{A}IB_2B_1$. Here $\hat{A}$ is absolutely summing and $\pi_1(\hat{A}) \leq K_G\|A\|$. We rewrite the remaining factors, which commute, as B_1B_2 where we now interpret $B_2 : \ell_2 \to \ell_2$ and $B_1 : \ell_2 \to \ell_1$. Thus $\hat{A}B_1$ is Hilbert-Schmidt on ℓ_2 ([16] p.67) and $\|\hat{A}B_1\|_{\mathcal{C}_2} \leq C\|A\|\|B\|^{r/2}$. However B_2 is in the Schatten class $\mathcal{C}_{p/(1-p)}$ with $\|B_2\|_{\mathcal{C}_{p/(1-p)}} \leq \|B\|^{r(1/p-1)}$. Combining we conclude that $AB \in \mathcal{C}_r$ and $\|AB\|_{\mathcal{C}_r} \leq C\|A\|\|B\|$■

We now come our promised application. For alternative proofs see [8], [12] and [14].

THEOREM 4.5. *Suppose $0 < p < 1$. Let (f_n) be a normalized unconditional basis of ℓ_p. Then (f_n) is equivalent to the canonical basis of ℓ_p.*

PROOF: Let us suppose that $\sum a_nf_n$ converges; we will show that $\sum |a_n|^p < \infty$. Define the map $T : c_0 \to \ell_p$ by $T(\xi_n) = \sum \xi_na_nf_n$. Then T factors as $T = D_1T_0D_2$ as in Theorem 4.2 where $D_2 : c_0 \to \ell_2$, $T_0 : \ell_2 \to \ell_1$ and $D_1 : \ell_1 \to \ell_p$ are bounded and D_2, D_1 are diagonal operators. Further since ℓ_p has cotype 2, we can define a bounded operator $S : \ell_p \to \ell_2$ by $S(\sum \alpha_nf_n) = (\alpha_n)$. Then SD_1T_0 is in the Schatten ideal $\mathcal{C}_r$ where $1/r = 1/p - 1/2$.

Now observe that $SD_1T_0D_2 = ST$ is diagonal; in fact $ST = \text{diag}\{a_1, a_2, \ldots\}$. Let $D_2 = \text{diag}\{w_1, w_2, \ldots\}$ and let $P = \text{diag}\{u_1, u_2, \ldots\}$ where $u_k = 1$ when $w_k \neq 0$ and

$u_k = 0$ otherwise. Then SD_1T_0P is also diagonal and so $SD_1T_0P = \operatorname{diag}\{v_1, v_2, \ldots\}$ where $\sum |v_k|^r < \infty$. Of course we have $\sum |w_k|^2 < \infty$. Thus we conclude

$$\sum_{n=1}^{\infty} |a_n|^p = \sum_{n=1}^{\infty} |v_n|^p |w_n|^p < \infty$$

by Holder's inequality.∎

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