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AUTOMORPHISMS OF $\mathcal{C}(K)$-SPACES AND EXTENSION OF LINEAR OPERATORS

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Abstract. We study the class of separable (real) Banach spaces X which can be embedded into a space $\mathcal{C}(K)$ (K compact metric) in only one way up to automorphism. We show that in addition to the known spaces c_0 (and all it subspaces) and ℓ_1 (and all its weak*-closed subspaces) the space $c_0(\ell_1)$ has this property. We show on the other hand (answering a question of Castillo and Moreno) that ℓ_p for $1<p<\infty$ fails this property. We also show that ℓ_p can be embedded in a super-reflexive space X so that there is an operator $T:\ell_p\to\mathcal{C}(K)$ which has no extension, answering a question of Zippin.

1. Introduction

Throughout this paper, all Banach spaces will be real. It is well known that injective Banach spaces must be nonseparable, and classical results of Goodner [15], Nachbin [36], and Kelley [26] classify isometrically injective Banach spaces as spaces of continuous functions $\mathcal{C}(K)$ where K is extremally disconnected. On the other hand, c_0 is separably injective (Sobczyk [42]) and it is a deep result of Zippin [44] that c_0 is the only separably injective space.

The problem of determining conditions so that operators with range in an arbitrary $\mathcal{C}(K)$-space can be extended has a long history dating back to the memoir of Lindenstrauss [28]. The general problem is to determine conditions on a pair (E,X) where E is a subspace of X so that every bounded linear operator $T:E\to\mathcal{C}(K)$ can be extended to an operator $\tilde{T}:X\to\mathcal{C}(K)$. Under these hypotheses, we say that (E,X) has the $\mathcal{C}$-extension property; if we can guarantee $\|\tilde{T}\|\le\lambda\|T\|$ we will say that (E,X) has the $(\lambda,\mathcal{C})$-extension

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property. If we restrict attention to the case when X is separable, then we may consider the case K is a compact metric space and indeed we may restrict to a special case such as $K=[0,1]$ without loss of generality by Miljutin's theorem [35], [43].

In 1971, Lindenstrauss and Pełczyński [29] show that for every subspace E of c_0, (E,c_0) has the $((1+\varepsilon),\mathcal{C})$-extension property for every $\varepsilon>0$. Later Zippin [45] showed for every subspace E of ℓ_p for $1<p<\infty$ the pair (E,ℓ_p) has the $(1,\mathcal{C})$-extension property; then Johnson and Zippin [17] showed that if E is a weak*-closed subspace of $\ell_1=c_0^*$ then (E,ℓ_1) has the $\mathcal{C}$-extension property. However, there are subspaces E of ℓ_1 so that the pair (E,ℓ_1) fails the $\mathcal{C}$-extension property; see [19] for results on a partial converse to the Johnson–Zippin result. See also [22] for more recent results on, for example, Orlicz sequence spaces. However, for most spaces very little is known; see the handbook article of Zippin [46] for a full discussion of open problems in the area.

In this paper, we will also consider automorphism problems. In [31], Lindenstrauss and Rosenthal showed that if two subspaces X,Y of c_0 of infinite codimension are linearly isomorphic then there is an automorphism U of c_0 so that $U(X)=Y$. Thus, there is essentially only one way (up to automorphism) to embed a Banach space into c_0. They also proved a dual result for quotients of ℓ_1 and some analogous results for ℓ_∞. The Lindenstrauss–Rosenthal theorem for c_0 depends heavily on separable injectivity. Thus, one cannot expect a similar result for $\mathcal{C}[0,1]$. However, it follows from the Lindenstrauss–Pełczyński theorem cited above that if X and Y are isomorphic subspaces of $\mathcal{C}[0,1]$ and X is isomorphic also to a subspace of c_0 then there must be an automorphism of $\mathcal{C}[0,1]$ mapping X to Y. Let us call a separable Banach space X $\mathcal{C}$-automorphic if whenever X_1 and X_2 are subspaces of $\mathcal{C}[0,1]$ (or any $\mathcal{C}(K)$ when K is uncountable and compact metric) with $X_1\approx X_2\approx X$ then there is an automorphism mapping X_1 to X_2 (i.e., there is only one way to embed X into $\mathcal{C}[0,1]$ up to automorphism).

Recently, Castillo and Moreno [10] proved a result that showed this problem is strongly connected with extension problems and asked specifically if ℓ_2 is $\mathcal{C}$-automorphic. We will slightly improve the Castillo–Moreno result by showing that a separable Banach space X is $\mathcal{C}$-automorphic if and only if it has the universal separable $\mathcal{C}$-extension property, i.e., if whenever Y is any separable Banach space containing X then (X,Y) has the $\mathcal{C}$-extension property.

Of course c_0 and all its subspaces are $\mathcal{C}$-automorphic by [29]. Recently in [22] we showed that ℓ_1 and all its weak*-closed subspaces have the universal separable $\mathcal{C}$-extension property (with constant $1+\varepsilon$), and hence these are also $\mathcal{C}$-automorphic.

In this paper, we study the class of $\mathcal{C}$-automorphic spaces and give some more examples. We show, for example, that $c_0(\ell_1)$ is also $\mathcal{C}$-automorphic.

We then study the spaces ℓ_p for $1<p<\infty$ and give necessary and sufficient conditions on a separable Banach space X containing ℓ_p so that (ℓ_p, X) has the $\mathcal{C}$-extension property. Based on these conditions, we are able to show that if $1<p<\infty$ then ℓ_p fails to be $\mathcal{C}$-automorphic (answering the question of Castillo and Moreno). Indeed, we also answer a question of Zippin [46] by showing that ℓ_p can be embedded in a super-reflexive space with an unconditional basis so that (ℓ_p, X) fails to have the $\mathcal{C}$-extension property. This is the first example of a separable super-reflexive space with a subspace where one cannot extend $\mathcal{C}(K)$-valued operators.

On the other hand, we show that if ℓ_p is embedded in a UMD-space X with an unconditional basis (or a UFDD) then (ℓ_p, X) has the $\mathcal{C}$-extension property. The appearance of the UMD-condition here is quite mysterious.

Terminology. We use standard notation for Banach space theory. We will assume all Banach spaces are real (although our results can easily be extended to the complex case). We write (UFDD) for an *unconditional finite-dimensional decomposition.* We write $X=\ell_p(F_n)$ with (F_n) finite-dimensional to mean that $(F_n)_{n=1}^\infty$ is a (UFDD) of X such that

$$\left\|\sum_{k=1}^n f_k\right\|^p=\sum_{k=1}^n \|f_k\|^p, \quad f_1\in F_1,\dots,f_n\in F_n.$$

(Thus, we regard the (F_n) as subspaces of X.) We write $\sum_{j=n+1}^\infty F_j$ for the linear span of $(F_j)_{j=n+1}^\infty$.

We recall that a Banach space X has the *approximation property* if for every compact set $K\subset X$ and $\varepsilon>0$ there is a finite-rank operator $T:X\to X$ with $\|Tx-x\|<\varepsilon$ for $x\in K$. If T can additionally be chosen with $\|T\|\le 1$ then X is said to have the *metric approximation property (MAP).* If X is separable, then it has (MAP) if and only if there is a sequence of finite-rank operators $T_n:X\to X$ with $\lim_{n\to\infty}\|T_n\|=1$ and $\lim_{n\to\infty}T_nx=x$ for $x\in X$. X is said to have the *unconditional metric approximation property (UMAP)* if there is a sequence of finite-rank operators $T_n:X\to X$ with $\lim_{n\to\infty}\|I-2T_n\|=1$ and $\lim_{n\to\infty}T_nx=x$ for $x\in X$. Note that (UMAP) implies (MAP).

A Banach space X is a UMD-space (for *unconditional martingale differences*) if for some (equivalently every) $1<p<\infty$ there is a constant $C=C(p,X)$ so that for any X-valued martingale $(M_k)_{k=0}^n$ we have (if $dM_0=M_0$ and then $dM_k=M_k-M_{k-1}$),

$$\left(\mathbb{E}\left\|\sum_{j=0}^n a_j\,dM_j\right\|^p\right)^{1/p}\le C\max_{0\le j\le n}|a_j|(\mathbb{E}\|M_n\|^p)^{1/p}, \quad a_1,\dots,a_n\in\mathbb{R}.$$

These spaces were introduced by Burkholder [7].

2. Preliminaries

If X is a Banach space, then an *enlargement* of X is a Banach space Y containing X, or more formally, a Banach space Y and an isometric embedding $J : X \to Y$. If Y is separable, we refer to Y as a separable enlargement (of course X must also be separable for this to be possible). We shall need some facts about separable enlargements.

Proposition 2.1. *Let $J_1 : X \to Y_1$ and $J_2 : X \to Y_2$ be two separable enlargements of X. Then there is a separable Banach space Z and isometries $J_3 : Y_1 \to Z$ and $J_4 : Y_2 \to Z$ so that the following diagram commutes:*

$$\begin{tikzcd} & Z & \\ Y_1 \arrow[ur, "J_3"] & & Y_2 . \arrow[ul, "J_4"'] \\ & X \arrow[ul, "J_1"] \arrow[ur, "J_2"'] & \end{tikzcd}$$

Proof. As pointed out by the referee, this is the standard push-out construction. We define Z to be the quotient of $Y_1 \oplus_1 Y_2$ by the subspace $G = \{(J_1x, -J_2x)\ x \in X\}$. Let Q be the quotient map and $R_1 : Y_1 \to Y_1 \oplus_1 Y_2$ and $R_2 : Y_2 : Y_2 \to Y_1 \oplus_1 Y_2$ be the canonical embeddings. Then let $J_3 = QR_1$ and $J_4 = QR_2$. We leave the details to the reader. □

Z is thus a *common enlargement* of Y_1 and Y_2. We may regard Z as a separable Banach space so that $Z \supset Y_j \supset X$ for $j = 1, 2$. We will use this viewpoint in future.

Proposition 2.2. *Let X be a separable Banach space and let Y_n be a sequence of separable enlargements of X. Then there is a separable Banach space Z so that $Z \supset Y_n \supset X$ for every n.*

Proof. This follows by induction: let Z_2 be a common enlargement of Y_1, Y_2 and then inductively let Z_n be a common enlargement of Z_{n-1}, Y_n for $n \geq 3$. Finally, let Z be the completion of $\bigcup_{n=1}^{\infty} Z_n$. □

Proposition 2.3. *Let X be a separable Banach space and suppose E is a closed subspace and $T : E \to X$ is a bounded operator. Then there is a separable enlargement Z of X and a bounded operator $\tilde{T} : Z \to Z$ with $\|\tilde{T}\| = \|T\|$ and $\tilde{T}|_E = T$.*

Proof. Let $X_0 = X$ and then by embedding X in ℓ_∞ find a separable enlargement X_1 of X so that $T : E \to X$ has an extension $T_1 : X_0 \to X_1$ with $\|T_1\| = \|T\|$. Continuing by induction, we find an increasing sequence of separable Banach spaces (X_n) and operators $T_n : X_{n-1} \to X_n$ with $\|T_n\| = \|T\|$

and $T_n|_{X_{n-2}} = T_{n-1}$ for $n \geq 2$. Let Z be the completion of $\bigcup_{n=0}^{\infty} X_n$ and let $\tilde{T}$ be the induced operator. □

Let us also note that if X is only isomorphically embedded in a superspace Y then Y can always be renormed to be an enlargement of X.

Let X be a Banach space and suppose E is a closed subspace. Then we say that (E,X) has the *(linear)* $(\lambda,\mathcal{C})$*-extension property* if every bounded operator $T : E \to \mathcal{C}(K)$, where K is a compact Hausdorff space, has an extension $\tilde{T} : X \to \mathcal{C}(K)$ with $\|\tilde{T}\| \leq \lambda\|T\|$. (E,X) has the *(linear)* $\mathcal{C}$*-extension property* if every bounded operator $T : E \to \mathcal{C}(K)$ has a bounded extension $\tilde{T} : X \to \mathcal{C}(K)$.

$$\begin{tikzcd} X \arrow[r, dotted, "\tilde{T}"] & \mathcal{C}(K). \\ E \arrow[u] \arrow[ur, "T"'] & \end{tikzcd}$$

Let us here recall the Zippin criterion for the $\mathcal{C}$-extension property (see [45] and [46]):

Proposition 2.4. *Suppose X is a Banach space E is a closed subspace of X. Then X has the $(\lambda,\mathcal{C})$-extension property if and only if there is a weak*-continuous map $\Phi : B_{E^*} \to \lambda B_{X^*}$ such that $\Phi(e^*)|_E = e^*$ for $e^* \in B_E$.*

We shall refer to a map $\Phi : B_{E^*} \to X^*$ as a *Zippin selector* if it is weak*-continuous and $\Phi(e^*)|_E = e^*$ for $e^* \in B_{E^*}$.

It is also true that if (E,X) has the $\mathcal{C}$-extension property, then there exists $\lambda \geq 1$ so that (E,X) has the $(\lambda,\mathcal{C})$-extension property.

We will be primarily interested in the case when X is separable. In this case, it is easy to see that we can restrict K to be metrizable. In fact, every $\mathcal{C}(K)$ is isometric to a 1-complemented subspace of $\mathcal{C}(\Delta)$, where Δ is the Cantor set (cf. Proposition 3.1 of [21] combining results of [4] and [35]). Thus, we may always take $K = \Delta$ in the definition.

We say that a separable Banach space X has the *separable universal (linear)* $(\lambda,\mathcal{C})$*-extension property* if whenever Y is a separable enlargement of X then (X,Y) has the $(\lambda,\mathcal{C})$-extension property. X has the *separable universal* $\mathcal{C}$*-extension property* if for every separable enlargement Y of X then (X,Y) has the *separable universal* $\mathcal{C}$*-extension property.*

It may be shown that if X has the separable universal $\mathcal{C}$-extension property then for some $\lambda \geq 1$ it has the separable universal $(\lambda,\mathcal{C})$-extension property. This follows simply from Proposition 2.2.

We now connect this with ideas of Castillo and Moreno [10]. Suppose $(X_j)_{j=1,2}$ are Banach spaces and E_j is a closed subspace of X_j for $j = 1,2$. Suppose $V : E_1 \to E_2$ is a isomorphism. Then we will say that (E_1,X_1) and (E_2,X_2) are *equivalent* if we can find an invertible operator $U : X_1 \to X_2$ so that $U(E_1)=E_2$; we say that they are *V-equivalent* if we can choose U so that $U|_{E_1}=V$.

Suppose $0\to X_j\to Y_j\to Z_j\to 0$ for $j=1,2$ are two short exact sequences of Banach spaces. We say that these sequences are *isomorphically equivalent* if there exists invertible linear operators $U: X_1\to X_2$, $V: Y_1\to Y_2$ and $W: Z_1\to Z_2$ so that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r] \arrow[d, "U"'] & Y_1 \arrow[r] \arrow[d, "V"'] & Z_1 \arrow[r] \arrow[d, "W"'] & 0 \\
0 \arrow[r] & X_2 \arrow[r] & Y_2 \arrow[r] & Z_2 \arrow[r] & 0.
\end{tikzcd}$$

Note that (E_1,X_1) and (E_2,X_2) are equivalent in our sense if and only if the short exact sequences $0\to E_j\to X_j\to X_j/E_j\to 0$ $(j=1,2)$ are isomorphically equivalent.

Now suppose X is a separable Banach space, and K_1,K_2 are uncountable compact metric spaces. Let $S_j: X\to\mathcal{C}(K_j)$ be linear embeddings. We shall say that S_1 and S_2 are *equivalent* if $(S_1(X),\mathcal{C}(K_1))$ and $(S_2(X),\mathcal{C}(K_2))$ are equivalent and *strongly equivalent* if $(S_1(X),\mathcal{C}(K_1))$ and $(S_2(X),\mathcal{C}(K_2))$ are $S_2S_1^{-1}$-equivalent. Both equivalence and strong equivalence are equivalence relations of the set of all possible embeddings.

We now give two results due to Castillo and Moreno (Proposition 4.6 of [10]).

Proposition 2.5. *Suppose X has the separable universal $\mathcal{C}$-extension property, and that $S_j: X\to\mathcal{C}(K_j)$ are linear embeddings such that $\mathcal{C}(K_j)/S_j(X)$ has a nonseparable dual for $j=1,2$. Then S_1 and S_2 are strongly equivalent.*

Proof. For completeness, let us prove this in our language. It is enough to prove the result when $K_1=K_2=\Delta$ say. By a result of Rosenthal [40], the quotient map $q_1:\mathcal{C}(\Delta)\to Y_1:=\mathcal{C}(\Delta)/S_1(X)$ is an isomorphism on some subspace Z isomorphic to $\mathcal{C}(\Delta)$. Now by a result of Pełczyński [39], $q(Z)$ contains a subspace G isomorphic to $\mathcal{C}(\Delta)$ and complemented in Y_1. Furthermore, $G=E\oplus F$, where E and F are each isomorphic to $\mathcal{C}(\Delta)$. Let $\tilde{E}=q^{-1}E\cap Z$ and $\tilde{F}=q^{-1}F\cap Z$; then both $\tilde{E}$ and $\tilde{F}$ are isomorphic to $\mathcal{C}(\Delta)$ and $\mathcal{C}(\Delta)=\tilde{E}\oplus\tilde{F}\oplus H$ for some space $H\supset S_1(X)$. Now $\tilde{E}\oplus\tilde{F}\approx\mathcal{C}(\Delta)$ so that $\tilde{F}\oplus H\approx\tilde{E}\oplus\tilde{F}\oplus H\approx\mathcal{C}(\Delta)$. Let $V_1: X\to\mathcal{C}(\Delta)$ be the map induced by the map $x\to=(0,S_1x)$ of X into $\tilde{F}\oplus H$. Then S_1 is strongly equivalent to $0\oplus V_1: X\to\mathcal{C}(\Delta)\oplus\mathcal{C}(\Delta)$.

Similarly, S_2 is strongly equivalent to an embedding $0\oplus V_2: X\to\mathcal{C}(\Delta)\oplus\mathcal{C}(\Delta)$.

Now consider the embedding $V_1\oplus V_2: X\to\mathcal{C}(\Delta)\oplus\mathcal{C}(\Delta)$. Since X has the separable universal $\mathcal{C}$-extension property, there is a bounded operator $T:\mathcal{C}(\Delta)\to\mathcal{C}(\Delta)$ with $T|_{V_1(X)}=V_2V_1^{-1}$. Consider the automorphism of

$\mathcal{C}(\Delta)\oplus\mathcal{C}(\Delta)$ given by the matrix

$$U=\begin{pmatrix} I & 0\\ T & I\end{pmatrix}.$$

Then $U(V_1\oplus 0)=V_1\oplus V_2$ so that S_1 is strongly equivalent to $V_1\oplus V_2$; similarly S_2 is strongly equivalent to $V_1\oplus V_2$. □

PROPOSITION 2.6. *Suppose all embeddings $S:X\to\mathcal{C}(K)$ are equivalent. Then X has the separable universal $\mathcal{C}$-extension property.*

Proof. Suppose $X\subset Y$ where is a separable Banach space. Then $X\subset Y\subset\mathcal{C}(B_{Y^*})$ via the canonical embedding. But then $(X,\mathcal{C}(B_{Y^*}))$ is equivalent to $(X,\mathcal{C}(B_{X^*}))$ and the later pair has the $(1,\mathcal{C})$-extension property. □

We now show the following theorem.

THEOREM 2.7. *Let K be an uncountable compact metric space. If X is a subspace of $\mathcal{C}(K)$ such that $\mathcal{C}(K)/X$ has separable dual, then X fails the separable universal $\mathcal{C}$-EP.*

Proof. We first observe that by Miljutin's theorem ([35], [41]) we can specialize to the case $K=[0,1]$. We use an example of Aharoni and Lindenstrauss [1]. Let V be the subspace of all bounded functions on $[0,1]$ consisting of all functions which are right-continuous at all $t\in[0,1]$, have left-hand limits at all $t\in[0,1]$ and such that $\lim_{s\to t-}f(t)=f(t-)=f(t)$ except possible at the set $\mathbb{Q}_d$ of all dyadic rationals in $(0,1)$. Then V is isometric to $\mathcal{C}(\Delta)$ where Δ is the Cantor set and $\mathcal{C}[0,1]$ is an uncomplemented subspace of V. Furthermore, $V/\mathcal{C}[0,1]$ can be identified with $c_0(\mathbb{Q}_d)$ with the quotient map given by

$$\pi(f)(q)=\frac{1}{2}\big(f(q)-f(q-)\big),\quad q\in\mathbb{Q}_d.$$

Let us denote by e_q the canonical basis vectors in $c_0(\mathbb{Q}_d)$. Then $e_q=\pi(g_q)$ where

$$g_q=\big(\chi_{[q,1]}-\chi_{[0,q)}\big),\quad q\in\mathbb{Q}_d.$$

We identify X as a subspace of $\mathcal{C}[0,1]$ with $\mathcal{C}[0,1]/X$ having separable dual. Thus $Y=V/X$ also has separable dual. Let $\pi_X:V\to Y$ be the corresponding quotient map.

Let us assume that X has the separable universal $\mathcal{C}$-EP. Then there is a bounded linear operator $T:V\to\mathcal{C}[0,1]$ with $Tf=f$ for $f\in X$. We let

$$f_q=g_q-Tg_q.$$

Note that $I-T$ factors in the form $(I-T)=S\pi_X$, where $S:Y\to V$ is bounded with $\|S\|\le\|T\|$. Hence,

$$f_q=S\pi_X(g_q).$$

Let $(\mathcal{O}_n)_{n\in\mathbb{N}}$ be a countable base for the topology of $(0,1)$ and let $(y_n^*)_{n\in\mathbb{N}}$ be a dense countable subset of Y^*. Then for each n we can find $q_n, r_n \in \mathbb{Q}_d \cap \mathcal{O}_n$ with $q_n \neq r_n$ so that

$$\big|y_k^*\big(\pi_X(g_{q_n} - g_{r_n})\big)\big| \leq 2^{-n}, \quad 1 \leq k \leq n.$$

We now claim the existence of an increasing sequence of natural numbers, $(m_n)_{n=1}^\infty$, and a sequence of signs $\varepsilon_n = \pm 1$ so that

(2.1) $$\mathcal{O}_{m_{n+1}} \subset \mathcal{O}_{m_n}, \quad n \geq 1$$

and

(2.2) $$\varepsilon_n\big(f_{q_{m_n}}(t) - f_{r_{m_n}}(t)\big) \geq 1/2, \quad t \in \mathcal{O}_{m_{n+1}},\ n \geq 1.$$

To start the construction, let $m_1 = 1$. Then if m_n has been chosen, we note that

$$\pi(f_{q_{m_n}} - f_{r_{m_n}}) = \pi(g_{q_{m_n}} - g_{r_{m_n}}) = e_{q_{m_n}} - e_{r_{m_n}}.$$

Hence, $f_{q_{m_n}} - f_{r_{m_n}}$ has a jump of size 2 at $q_{m_n} \in \mathcal{O}_{m_n}$. We may therefore select $m_{n+1} > m_n$ so that $\mathcal{O}_{m_{n+1}} \subset \mathcal{O}_{m_n}$ and a sign ε_n so that (2.2) holds.

Now by construction the sequence $(\pi_X(g_{q_n} - g_{r_n}))_{n=1}^\infty$ is weakly null. Hence, $S\pi_X(g_{q_n} - g_{r_n}) = f_{q_n} - f_{r_n}$ is also weakly null. This contradicts (2.2) which implies the existence of some $v^* \in V^*$ so that $|v^*(f_{q_{m_n}} - f_{r_{m_n}})| \geq 1/2$ for all n. □

The referee points out that an alternative proof of the preceding theorem can be given based on Lemma 2.2 in [8] and a result of Lohman [33].

From this, we will conclude the following improvement of Proposition 4.6 of [10].

Theorem 2.8. *Let X be a separable Banach space. Then the following are equivalent:*

(i) *All embeddings of X into a space $\mathcal{C}(K)$ for K an uncountable metric space are equivalent.*

(ii) *All embeddings of X into a space $\mathcal{C}(K)$ for K an uncountable metric space are strongly equivalent.*

(iii) *X has the separable universal $\mathcal{C}$-extension property.*

Proof. (i) ⇒ (iii). Proposition 2.6 above.

(iii) ⇒ (ii). Combine Proposition 2.5 and Theorem 2.7.

(ii) ⇒ (i). Trivial. □

For simplicity, we refer to spaces X satisfying the conditions (i)–(iii) of Theorem 2.8 as $\mathcal{C}$-automorphic spaces. This is a slight variation of the terminology of Castillo and Moreno [10].

Let us conclude this section by listing the previously known examples of $\mathcal{C}$-automorphic spaces:

THEOREM 2.9. *The following spaces are $\mathcal{C}$-automorphic:*

(i) *Any subspace of c_0.*
(ii) *Any dual of a subspace of c_0.*

Proof. (i) This follows from a result of Lindenstrauss and Pełczyński [29] and the fact that c_0 is separably injective. In fact, each such space has the separable universal $(2+\varepsilon,\mathcal{C})$-extension property for any $\varepsilon > 0$.

(ii) This follows from results in [22]; indeed all such space have the separable universal $(1+\varepsilon,\mathcal{C})$-extension property for any $\varepsilon > 0$. $\square$

3. Remarks on the class of $\mathcal{C}$-automorphic spaces

We now make a few remarks concerning the class of $\mathcal{C}$-automorphic spaces (equivalently space with the separable universal $\mathcal{C}$-EP). Our first remark is well known (and indeed used by Johnson and Zippin in [17]).

PROPOSITION 3.1. *Suppose $0 \to X \to Y \to Z \to 0$ is a short exact sequence. Suppose X and Z are $\mathcal{C}$-automorphic; then Y is also $\mathcal{C}$-automorphic.*

Proof. Suppose $T : Y \to \mathcal{C}(K)$ is a bounded operator and than W is a Banach space containing Y. Then $T|_X$ has a bounded extension $S : W \to \mathcal{C}(K)$ and $T - S : Y \to \mathcal{C}(K)$ factors to an operator $R : Z \to \mathcal{C}(K)$. Thus, R extends to an operator $\tilde{R} : W/X \to \mathcal{C}(K)$. Now $S + \tilde{R}Q : W \to \mathcal{C}(K)$ extends T, where $Q : Y \to Z$ is the quotient map. $\square$

Now it follows that any extension of c_0 by ℓ_1 is $\mathcal{C}$-automorphic and it is known that there are nontrivial examples [8].

Let X, Y be Banach spaces. We write $\operatorname{Ext}(X,Y) = \{0\}$ if every extension of X by Y splits, i.e., every short exact sequence $0 \to Y \to Z \to X \to 0$ splits. This is equivalent to the requirement that whenever Z is a Banach space containing Y and $T_0 : X \to Z/Y$ is bounded then T_0 has a lifting $T : X \to Z$ with $QT = T_0$, where $Q : Z \to Z/Y$ is the quotient map.

$$\begin{tikzcd} & Z \arrow[d, "Q"] \\ X \arrow[ur, "T"] \arrow[r, "T_0"] & Z/Y \end{tikzcd}$$

We shall that X has the *$\mathcal{C}$-lifting property* if $\operatorname{Ext}(X, \mathcal{C}(K)) = \{0\}$ for every compact metric K. We say that X has the $(\lambda,\mathcal{C})-$lifting property if in the above diagram we can find T with $\|T\| \le \lambda \|T_0\|$. The following result is then immediate.

PROPOSITION 3.2. *Let X be a subspace of ℓ_1. Then X is $\mathcal{C}$-automorphic if and only if ℓ_1/X has the $\mathcal{C}$-lifting property.*

Proof. The proof is quite formal and essentially due to Johnson and Zippin [17]. We show that X has the separable universal $\mathcal{C}$-EP. If $T_0 : X \to \mathcal{C}(K)$ is a bounded linear operator then we can extend T_0 to a bounded operator $T_1 : \ell_1 \to \ell_\infty(K)$. If $Q : \ell_\infty(K) \to \ell_\infty(K)/\mathcal{C}(K)$ is the quotient map then $QT_1 : \ell_1 \to \ell_\infty(K)/\mathcal{C}(K)$ factors to a map $S : \ell_1/X \to \ell_\infty(K)/\mathcal{C}(K)$. Thus, $QT_1 = SQ_X$ where $Q_X : \ell_1 \to \ell_1/X$ is the quotient map. Now S has a lifting $\tilde{S} : \ell_1/X \to \ell_\infty(K)$ and then $T = T_1 - \tilde{S}Q_X$ is an extension of T_0 mapping ℓ_1 to $\mathcal{C}(K)$. Since ℓ_1 has the separable universal $\mathcal{C}$-EP, this implies that X has the same property.

$$
\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "T_0"'] & \ell_1 \arrow[r, "Q_X"] \arrow[d, "T_1"'] & \ell_1/X \arrow[r] \arrow[d, "S"'] \arrow[dl, "\tilde{S}"'] & 0 \\
0 \arrow[r] & \mathcal{C}(K) \arrow[r] & \ell_\infty(K) \arrow[r, "Q"] & \ell_\infty(K)/\mathcal{C}(K) \arrow[r] & 0.
\end{tikzcd}
$$
□

Our next results concern the notion of Kadets distance between Banach spaces. If X and Y are subspaces of a Banach space Z, we define the gap between X and Y by

$$\Lambda(X,Y) = \max\Big\{ \sup_{y \in B_Y} d(y, B_X), \sup_{x \in B_X} d(x, B_Y) \Big\}.$$

Now if X and Y are arbitrary Banach spaces we define the *Kadets distance* between X and Y by

$$d_K(X,Y) = \inf\{\Lambda(\tilde{X}, \tilde{Y})\},$$

where the infimum is taken over all Banach spaces Z which contain isometric copies $\tilde{X}, \tilde{Y}$ of X, Y. We refer to the survey [38] and also [23] for further details.

We now discuss the question whether the set of $\mathcal{C}$-automorphic spaces is open for the Kadets metric. Notice first that the collection of spaces isomorphic to a subspace of c_0 is open for this metric.

PROPOSITION 3.3 (cf. [23]). *Let X be a Banach space isomorphic to a subspace X_0 of c_0. If Y is a Banach space with $d_K(X,Y) < (1 + 2d(X,X_0))^{-1}$ then Y is isomorphic to a subspace of c_0.*

Proof. Suppose X is isometrically embedded in a Banach space Z and Y is a subspace of Z with $\Lambda(X,Y) < 1$. Then Y is separable and we can assume Z is separable. For $\delta > 0$, we can find a bounded operator $T : X \to c_0$ with $\|x\| \le \|Tx\|$ for $x \in X$ and $\|T\| < d(X, X_0) + \delta$. T can then extended to a bounded operator $\tilde{T} : Z \to c_0$ with $\|\tilde{T}\| \le 2\|T\|$. Now if $y \in Y$ with $\|y\| = 1$ we may find $x \in B_X$ with $\|y - x\| < \Lambda(X,Y) + \delta$. Thus,

$$\|\tilde{T}y\| > \|x\| - 2\|T\|\big(\Lambda(X,Y) + \delta\big) \ge 1 - (2\|T\| + 1)\big(\Lambda(X,Y) + \delta\big).$$

Thus, $\tilde{T}|_Y$ is an isomorphism if

$$\Lambda(X,Y)+\delta<\big(1+2d(X,X_0)+2\delta\big)^{-1}$$

and so the result follows. $\square$

Theorem 3.4. *Let X be a separable Banach space which is both $\mathcal{C}$-automorphic and has the $\mathcal{C}$-lifting property. Then there exists $\varepsilon>0$ so that if Y is a Banach space with $d_K(X,Y)<\varepsilon$, then Y is also $\mathcal{C}$-automorphic.*

Proof. Suppose X has the separable universal $(\lambda,\mathcal{C})$-EP and the $(\mu,\mathcal{C})$-lifting property. Suppose $\varepsilon<\frac{1}{2}(1+\lambda)^{-1}(1+\mu)^{-1}$ and that $d_K(Y,X)<\varepsilon$. Let Z be a separable Banach space containing Y and let $T_0:Y\to\mathcal{C}(K)$ be an operator with $\|T_0\|=1$.

We can construct a separable Banach space W containing Y and (an isometric copy of) X so that $\Lambda(X,Y)<\varepsilon$. Let Z' be a Banach space containing both W and Z. This can be constructed as in Proposition 2.1 as the quotient of $W\oplus_1 Z$ by the subspace $\{(y,-y):y\in Y\}$. We first extend $T:Y\to\mathcal{C}(K)$ to an operator $S:Z'\to\ell_\infty(K)$ with $\|S\|=1$. Then if $x\in B_X$ there exists $y\in B_Y$ with $\|x-y\|<\varepsilon$ and so $\|Sx-Sy\|<\varepsilon$. Hence, $d(Sx,\mathcal{C}(K))<\varepsilon$. If $Q:\ell_\infty(K)\to\ell_\infty/\mathcal{C}(K)$ is the quotient map we have $\|QS|_X\|\le\varepsilon$. By the $\mathcal{C}$-lifting property for X, we can find an operator $R:X\to\ell_\infty(K)$ with $\|R\|\le\mu\varepsilon$ and $QR=QS_X$. Now $S-R$ maps X into $\mathcal{C}(K)$ and $\|S-R\|\le 1+\mu\varepsilon$. Let $V:Z'\to\mathcal{C}(K)$ be an extension with $\|V\|\le\lambda(1+\mu\varepsilon)$. If $y\in B_Y$ there exists $x\in B_X$ with $\|x-y\|<\varepsilon$. Hence,

$$\begin{aligned}\|Vy-Ty\|&=\|Vy-Sy\|\\&\le\|Vx-Sx\|+\big(1+\lambda(1+\mu\varepsilon)\big)\varepsilon\\&\le(\mu+\lambda+1+\lambda\mu\varepsilon)\varepsilon\\&\le(1+\lambda)(1+\mu)\varepsilon.\end{aligned}$$

This implies that the restriction map $\mathcal{R}:\mathcal{L}(Z',\mathcal{C}(K))\to\mathcal{L}(Y,\mathcal{C}(K))$ has the property that if $\|T\|=1$ there exists V with $\|V\|\le\lambda(1+\mu)$ and $\|T-\mathcal{R}(V)\|<(1+\lambda)(1+\mu)\varepsilon\le\frac{1}{2}$. By a version of the open mapping theorem, this implies that $\mathcal{R}$ is surjective and indeed if $\|T\|=1$ there exists an extension $\tilde{T}$ with $\|\tilde{T}\|\le 2\lambda(1+\mu)$. $\square$

Corollary 3.5. *Let X be isomorphic to the dual of a subspace of c_0. Then there exists $\varepsilon>0$ so that if $d_K(X,Y)<\varepsilon$ then Y is $\mathcal{C}$-automorphic.*

Proof. X is $\mathcal{C}$-automorphic by Theorem 2.9. There is quotient map $Q:\ell_1\to X$ whose kernel is weak*-closed. By Proposition 3.2, X has the $\mathcal{C}$-lifting property. $\square$

We know of no example of a subspace X of ℓ_1 which is $\mathcal{C}$-automorphic, but such that X is not isomorphic to the dual of a subspace of c_0. We also do not know whether the property of being (isomorphically) a dual of a subspace of c_0 is open for the Kadets metric: compare Proposition 3.3 and Corollary 3.5.

One can also ask similar questions about the $\mathcal{C}$-lifting property. In [19], it was shown that a separable Banach space with the $\mathcal{C}$-lifting property and a (UFDD) is isomorphic to the dual of a subspace of c_0. In this context, we may mention also that ℓ_1 has a subspace X isomorphic to ℓ_1 but uncomplemented [5]. Then ℓ_1/X has the $\mathcal{C}$-lifting property. So, we may ask whether every subspace of X which is isomorphic to ℓ_1 has the property that ℓ_1/X is isomorphic to the dual of a subspace of c_0. By the automorphism results of Lindenstrauss and Rosenthal [30], this is equivalent to asking whether every such subspace can be mapped by an automorphism of ℓ_1 to a weak*-closed subspace. Unfortunately, for this purpose, Bourgain's construction in [5] is local in nature and leads to a weak*-closed subspace.

4. The homogeneous Zippin criterion

A map $\Phi : X \to Y$ between normed spaces is called homogeneous if $\Phi(\alpha x) = \alpha\Phi(x)$ for α real and $x \in X$.

Suppose X is a Banach space and E is a closed subspace. We will say that (E, X) satisfies the λ-*homogeneous Zippin condition* if there is a homogeneous map $\Phi : E^* \to X^*$ which is weak*-continuous on bounded sets, satisfies $\|\Phi\| = \sup\{\|\Phi(e^*)\| : \|e^*\| \le 1\} \le \lambda$ and such that such that $\Phi(e^*)|_E = e^*$ for every $e^* \in E^*$. We shall refer to a map $\Phi : E^* \to X^*$ as an *homogeneous Zippin selector* if it is homogeneous, weak*-continuous on bounded sets and $\Phi(e^*)|_E = e^*$ for all $e^* \in E^*$.

Obviously if (E, X) satisfies the λ-homogeneous Zippin condition then by the Zippin criterion, Proposition 2.4 the pair (E, X) has the $(\lambda, \mathcal{C})$-extension property. In the case when E is finite-dimensional, it is clear that these two conditions are equivalent, since the map given by Proposition 2.4 can be homogenized and remains weak*-continuous. Thus, in particular, we have the following lemma.

Lemma 4.1. *If E is a finite-dimensional subspace of a Banach space X, then for any $\varepsilon > 0$ there exists an homogeneous Zippin selector $\Phi : E^* \to X^*$ with $\|\Phi\| < 1 + \varepsilon$.*

In general, as we shall see, the $(\lambda, \mathcal{C})$-condition does not imply the λ-homogeneous Zippin condition. It is natural to consider the case of the canonical embedding of a separable Banach space X into $\mathcal{C}(B_{X^*})$ where B_{X^*} has the weak* topology, as the pair $(X, \mathcal{C}(B_{X^*}))$ always has the $(1, \mathcal{C})$-extension property. In the case of c_0, it is contractively complemented in $\mathcal{C}(B_{\ell_1})$ via the projection

$$Pf = (f(e_n^*))_{n=1}^\infty,$$

where e_n^* denote the biorthogonal functionals to the canonical basis. Thus, it is trivial that $(c_0, \mathcal{C}(B_{\ell_1}))$ satisfies the 1-homogeneous Zippin condition by taking $\Phi(x^*) = x^* \circ P$.

There there is another important special case where we can find an isometric homogeneous Zippin selector. We consider the space ℓ_1 as a subspace of $\mathcal{C}(B_{\ell_\infty})$ (where B_{ℓ_∞} has the weak*-topology) in the natural way. Thus, the unit vector basis $(e_n)_{n=1}^\infty$ of ℓ_1 can be considered as functions on B_{ℓ_∞}.

LEMMA 4.2. *There is a sequence of maps $\varphi_n : \ell_\infty \to [0, \infty)$ and a sequence of maps $\psi_n : \ell_\infty \to B_{\ell_\infty}$ such that*

(i) *Each φ_n is weak*-continuous and satisfies $\varphi_n(\alpha\xi) = |\alpha|\varphi_n(\xi)$ for $\alpha \in \mathbb{R}$, $\xi \in \ell_\infty$.*

(ii) *If $\xi \in \ell_\infty$ with $\xi_n = 0$, then $\varphi_n(\xi) = 0$.*

(iii)

$$\sum_{n=1}^{\infty} \varphi_n(\xi) = \|\xi\|_\infty, \quad \xi \in \ell_\infty. \tag{4.1}$$

(iv) *Each ψ_n is weak*-continuous on $\{\varphi_n > 0\}$ satisfies the conditions that $\psi_n(\alpha\xi) = (\operatorname{sgn} \alpha)\psi_n(\xi)$ and $\langle e_j, \psi_n(\xi)\rangle = 0$ if $j < n$.*

(v)

$$\sum_{j=1}^{\infty} \varphi_j(\xi)\langle e_n, \psi_j(\xi)\rangle = \langle e_n, \xi\rangle = \xi_n, \quad 1 \le n < \infty, \ \xi \in \ell_\infty \tag{4.2}$$

Proof. We define the sequences of maps $h_n : \ell_\infty \to [0, \infty)$ for $n \ge 0$ by $h_0(\xi) = 0$ and then

$$h_n(\xi) = \max(|\xi_1|, \dots, |\xi_n|), \quad n \ge 1.$$

We then define a sequence of maps $f_n : \ell_\infty \to [-1, 1]$ by

$$f_n(\xi) = \begin{cases} \xi_n / h_{n-1}(\xi) & \text{if } 0 \le |\xi_n| < h_{n-1}(\xi), \\ \operatorname{sgn} \xi_n & \text{if } |\xi_n| \ge h_{n-1}(\xi). \end{cases}$$

Each h_n is weak*-continuous and each f_n is weak*-continuous at ξ unless $\xi_n = h_{n-1}(\xi) = 0$ i.e., $h_n(\xi) = 0$.

Now define

$$\varphi_n(\xi) = h_n(\xi) - h_{n-1}(\xi), \quad \xi \in \ell_\infty, \ n = 1, 2, \dots$$

and

$$\psi_n(\xi) = (0, 0, \dots, 0, f_n(\xi), f_{n+1}(\xi), \dots)$$

with the first possibly nonzero entry in the nth position. Then (i)–(iv) are immediate. For (4.2),

$$\sum_{j=1}^{\infty} \varphi_j(\xi)\langle e_n, \psi_j(\xi)\rangle = \sum_{j=1}^{n} \varphi_j(\xi) f_n(\xi) = h_n(\xi) f_n(\xi).$$

If $|\xi_n| < h_{n-1}(\xi)$, then $h_n(\xi)f_n(\xi) = h_{n-1}(\xi)f_n(\xi) = \xi_n$. If $|\xi_n| \geq h_{n-1}(\xi)$, then $h_n(\xi)f_n(\xi) = h_n(\xi)\operatorname{sgn}\xi_n = \xi_n$. This establishes (4.2). $\square$

THEOREM 4.3. *The pair* $(\ell_1, \mathcal{C}(B_{\ell_\infty}))$ *has satisfies* 1*-homogeneous Zippin condition.*

Proof. We define $\Phi : \ell_\infty \to \mathcal{M}(B_{\ell_\infty})$ by

$$\Phi(\xi) = \frac{1}{2}\big(\Phi_0(\xi) - \Phi_0(-\xi)\big),$$

where

$$\Phi_0(\xi) = \sum_{j=1}^{\infty} \varphi_j(\xi)\delta_{\psi_j(\xi)}.$$

The only difficulty to establish weak*-continuity of Φ on bounded sets. In fact, it suffices to show that $\xi \to \int F\, d\Phi(\xi)$ is weak*-continuous when F is a polynomial in $e_1, \dots, e_n$ for some fixed n. To do this, note

$$\int F\, d\Phi_0(\xi) = \sum_{j=1}^{n} \varphi_j(\xi)F(\psi_j(\xi)) + \left(\sum_{j=n+1}^{\infty} \varphi_j(\xi)\right)F(0)$$

while, using (i) of Lemma 4.2,

$$\int F\, d\Phi_0(-\xi) = \sum_{j=1}^{n} \varphi_j(-\xi)F(\psi_j(\xi)) + \left(\sum_{j=n+1}^{\infty} \varphi_j(\xi)\right)F(0)$$

so that

$$\int F\, d\Phi(\xi) = \frac{1}{2}\left(\sum_{j=1}^{n} \varphi_j(\xi)F(\psi_j(\xi)) - \sum_{j=1}^{n} \varphi_j(-\xi)F(\psi_j(-\xi))\right).$$

However, the map

$$\xi \to \sum_{j=1}^{n} \varphi_j(\xi)\delta_{\psi_j(\xi)}$$

is easily seen to be weak*-continuous by Lemma 4.2(i) and (iv) and we are finished. $\square$

THEOREM 4.4. *If* (F_n) *is a sequence of finite-dimensional spaces and* $Y = \ell_1(F_n)$ *then* $(Y, \mathcal{C}(B_{Y^*}))$ *satisfies the* 1*-homogeneous Zippin condition.*

Proof. The proof is very similar. We identify the unit ball of Y^* with the infinite product of $B_{F_n^*}$ for $n \geq 1$. We will use the maps constructed in Lemma 4.2. We define a map $G : Y^* \to \ell_\infty$ by $G((f_n^*)_{n=1}^\infty) = (\|f_n^*\|)_{n=1}^\infty$. We define the maps $r : Y^* \to Y^*$ by $r((f_n^*)_{n=1}^\infty) = ((g_n^*)_{n=1}^\infty)$ where

$$g_n^* = \begin{cases} f_n^*/\|f_n^*\| & \text{if } f_n^* \neq 0, \\ 0 & \text{if } f_n^* = 0. \end{cases}$$

We then define $\rho_n : Y^* \to B_{Y^*}$ by

$$\rho_n(y^*) = \psi_n(G(y^*)).r(y^*)$$

where we take the coordinatewise product.

Next, define $\Phi_0 : Y^* \to \mathcal{M}(B_{Y^*})$ by

$$\Phi_0(y^*) = \sum_{j=1}^{\infty} \varphi_j(G(y^*))\delta_{\rho_j(y^*)}$$

and define

$$\Phi(y^*) = \frac{1}{2}\big(\Phi_0(y^*) - \Phi_0(-y^*)\big).$$

Note that if $f \in F_n$ and $y^* = (f_k^*)_{k=1}^{\infty}$ with $f_n^* \neq 0$,

$$\begin{aligned}\int \langle f, u^*\rangle d\Psi_0(y^*)(u^*) &= \sum_{j=1}^{\infty} \varphi_j(G(y^*))\langle e_n, \psi_j(G(y^*))\rangle f_n^*(f)\|f_n^*\|^{-1} \\ &= f_n^*(f).\end{aligned}$$

Thus, for $y \in Y$, we have

$$\int u^*(y)\, d\Psi(y^*)(u^*) = y^*(y)$$

It is clear that Ψ is homogeneous and weak*-continuity on bounded sets is proved as in the previous theorem. $\square$

We next show that we cannot expect a similar result for ℓ_p when $1 < p < \infty$. The following proposition gives the counterexample promised after Lemma 4.2 since $(\ell_p, \mathcal{C}(B_{\ell_q}))$ has the $(1, \mathcal{C})$-extension property.

Proposition 4.5. *Suppose $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $\lambda_p = (1 + (q-1)q^{-p})^{1/q}$. Then $(\ell_p, \mathcal{C}(B_{\ell_q}))$ fails the λ-homogeneous Zippin condition for any $\lambda < \lambda_p$.*

Proof. Suppose $\Phi : \ell_q \to \mathcal{M}(B_{\ell_q})$ is homogeneous and boundedly weak*-continuous and satisfies

$$\int \eta_n\, d\Phi(\xi)(\eta) = \xi_n, \qquad \xi \in \ell_q,\ n = 1, 2, \ldots.$$

Let $\|\Phi\| = \lambda$.

Let $\tau = q^{1-p} < 1$. We consider the sequence of measures $\mu_n = \Phi(\tau e_1 + (1-\tau^q)^{1/q} e_n)$. Then μ_n converges weak* to $\mu = \Phi(\tau e_1)$.

It follows that

$$\begin{aligned}\tau \le \int |\eta_1|\, d|\mu| &\le \lambda^{1/p}\tau^{1/p}\left(\int |\eta_1|^q\, d|\mu|\right)^{1/q} \\ &\le \lambda^{1/p}\tau^{1/p} \liminf_{n\to\infty}\left(\int |\eta_1|^q\, d|\mu_n|\right)^{1/q}.\end{aligned}$$

Thus, we have

$$\liminf_{n\to\infty} \int |\eta_1|^q \, d|\mu_n| \geq \lambda^{-q/p}\tau.$$

Therefore,

$$\begin{aligned}(1-\tau^q)^{1/q} &\leq \liminf_{n\to\infty} \int |\eta_n| \, d|\mu_n| \\ &\leq \liminf_{n\to\infty} \int (1-|\eta_1|^q)^{1/q} \, d|\mu_n| \\ &\leq \liminf_{n\to\infty} \lambda^{1/p} \left(\int (1-|\eta_1|^q) d|\mu_n| \right)^{1/q}.\end{aligned}$$

Hence,

$$\lambda^{-q/p}(1-\tau^q) \leq \lambda - \lambda^{-q/p}\tau$$

or

$$1+\tau-\tau^q \leq \lambda^q.$$

Substituting the explicit value of τ gives a contradiction. □

However, there is a general positive result if we are prepared to relax the constant. This result is due to Castillo and Suarez [11].

THEOREM 4.6. *Let X be a separable Banach space and suppose Y is an enlargement of X. Then (X,Y) has the $\mathcal{C}$-extension property if and only if there is a homogeneous Zippin selector $\Phi : X^* \to Y^*$.*

Proof ([11]). Let G be the space of weak*-continuous homogeneous maps $f : B_{X^*} \to \mathbb{R}$. Then by a result of Benyamini [3] G is isomorphic to a $\mathcal{C}(K)$-space. Hence, there is a bounded linear operator $T : Y \to G$ with $Tx(x^*) = x^*(x)$ for $x \in X$. Let $\Phi(x^*)(y) = Ty(x^*)$ for $\|x^*\| \leq 1$ and extend by homogeneity. □

REMARK. Observe that if (X,Y) has the $(\lambda,\mathcal{C})$-extension property then (X,Y) satisfies the $\alpha\lambda$-homogeneous Zippin condition where α depends only on X (or more precisely the isomorphism constant of G and a $\mathcal{C}(K)$-space).

The following theorem is now immediate.

THEOREM 4.7. *Let X be a separable Banach space. Then for some $\lambda = \lambda_X$, the pair $(X, \mathcal{C}(B_{X^*}))$ satisfies the λ-homogeneous Zippin condition.*

5. Applications to the separable universal extension property

First note that by Theorem 4.6 and the remark, we have the following proposition.

PROPOSITION 5.1. *Let X be a separable Banach space with the separable universal $\mathcal{C}$-extension property. Then there exists $\lambda \geq 1$ so that whenever Y is a separable enlargement of X then (X,Y) satisfies the λ-homogeneous Zippin condition.*

We now observe that using the results of [22], we can prove the following theorem.

THEOREM 5.2. *Let $X = \ell_1$ or, more generally, $\ell_1(F_n)$ where (F_n) is a sequence of finite-dimensional spaces. Then for every separable enlargement Y of X and every $\varepsilon > 0$, (X,Y) satisfies the $(1+\varepsilon)$-homogeneous Zippin condition.*

Proof. If Y is a separable enlargement of X, then since X has the separable universal $(1+\varepsilon,\mathcal{C})$-extension property [22], there is an extension $T : Y \to \mathcal{C}(B_{X^*})$ of the canonical injection $X \to \mathcal{C}(B_{X^*})$ with $\|T\| < 1+\varepsilon$. If $\Phi : X^* \to \mathcal{M}(B_{X^*})$ is the homogeneous Zippin selector given by Theorem 4.3 or Theorem 4.4 then $T^* \circ \Phi$ is the required homogeneous Zippin selector from X^* to Y^* with $T^* \circ \Phi(x^*)|_X = x^*$ and $\|T^* \circ \Phi\| < 1+\varepsilon$. □

It seems quite likely that this result extends to the case when X is any dual of a subspace of c_0. In the special case when X has the approximation property, such a result comes from the following proposition.

PROPOSITION 5.3. *Let $Z = \ell_p(G_n)$ where $1 < p < \infty$ (respectively $c_0(G_n)$) with $(G_n)_{n=1}^{\infty}$ a sequence of finite-dimensional spaces. Let X be a subspace of Z so that X has the metric approximation property. Then for any $\varepsilon > 0$, there is a sequence of finite-dimensional subspaces (F_n) of X and operators $A : X \to \ell_p(F_n)$ [respectively, $A : X \to c_0(F_n)$] and $B : \ell_p(F_n) \to X$ [respectively, $B : c_0(F_n) \to X$] so that $\|A\|, \|B\| < 1+\varepsilon$ and $BA = I_X$.*

Thus for every $\varepsilon > 0$, X is $(1+\varepsilon)$-isomorphic to a $(1+\varepsilon)$-complemented subspace of a space $\ell_p(F_n)$ (respectively $c_0(F_n)$) where each F_n is a finite-dimensional subspace of X.

REMARK. In [9] page 61, an argument is given which essentially proves an isomorphic version of this result.

Proof of Proposition 5.3. We start with the observation that X has the (UMAP); this follows from, for example, [13] Theorem 9.2. (Note that in this theorem, in the proof of (6) $\implies$ (1), it can be assumed that the 1-unconditional basis is shrinking by the argument of Li [27].) A direct proof is also fairly simple. We note that X^* also has (MAP) even in the non-reflexive case by a result of Godefroy and Saphar [14]. If we denote by $J : X \to Z$ the inclusion then if $(T_n)_{n=1}^{\infty}$ is a shrinking approximating sequence of finite-rank operators on X and S_n are the partial sum operators on Z then $JT_n - S_nJ$ is a weakly null sequence in $\mathcal{K}(X,Z)$. Passing to a sequence of convex combinations, we can then assume $\lim_{n\to\infty} \|I - 2T_n\| = 1$ so that X has (UMAP).

It follows that we can find finite rank operators $V_n : X \to X$ with the property that

$$\Big\| \sum_{j=1}^{n} \delta_j V_j \Big\| \le (1+\varepsilon)^{1/2}, \quad \delta_j = \pm 1,\ n = 1, 2, \dots$$

and if $0 = m_0 < m_1 < m_2 < m_3 < \cdots$,

$$(1+\varepsilon)^{-1/2} \Big(\sum_{j=1}^{n} \|x_j\|^p \Big)^{1/p} \le \Big\| \sum_{j=1}^{n} x_j \Big\| \le (1+\varepsilon)^{1/2} \Big(\sum_{j=1}^{n} \|x_j\|^p \Big)^{1/p}$$

whenever $x_j \in (V_{m_{j-1}+1} + \cdots + V_{m_j - 1})(X)$ for each j (with a corresponding statement in the c_0-case. Fix an integer N so that $N/(N-2) < (1+\varepsilon)^{1/2}$ and then for each $1 \le r \le N$, let $F_{r1} = V_{N+r-1}(X)$ and then for $k \ge 1$ put $F_{rk} = (V_{kN+r-1} - V_{(k-1)N+r})(X)$. We then define an operator $B_r : \ell_p(F_{rk})_{k=1}^{\infty} \to X$ by

$$B_r((x_k)_{k=1}^{\infty}) = \sum_{k=1}^{\infty} x_k$$

(and similarly in the c_0-case). We also define $A_r : X \to \ell_p(F_{rk})$ by

$$A_r(x) = \Bigg(\sum_{j=1}^{N+r-1} V_j x, \sum_{j=N+r+1}^{2N+r-1} V_j x, \dots \Bigg).$$

Then $\|B_r\| \le (1+\varepsilon)^{1/2}$ while $\|A_r\| \le (1+\varepsilon)$. Now let $Y = \ell_p(\ell_p(F_{rk})_{k=1}^{\infty})_{r=1}^{N}$. Define $A : X \to Y$ by

$$Ax = N^{-1/p}(A_1 x, \dots, A_N x)$$

and $\tilde{B} : Y \to X$ by

$$\tilde{B}(u_1, \dots, u_N) = N^{-1/q}(u_1 + \cdots + u_N)$$

so that $\|A\| \le (1+\varepsilon)$ and $\|\tilde{B}\| \le (1+\varepsilon)^{1/2}$. Then for $x \in X$, we have

$$x - \tilde{B}Ax = \frac{1}{N} \sum_{j=N+1}^{\infty} V_j x$$

so that

$$\|I_X - \tilde{B}A\| \le 2/N.$$

It follows that $I_X - \tilde{B}A$ is invertible and setting $B = (I - \tilde{B}A)^{-1}\tilde{B}$ gives the conclusion. □

Corollary 5.4. *If X is the dual of a subspace of c_0 and X has the approximation property then for every separable enlargement Y of X the pair (X, Y) satisfies the $(1+\varepsilon)$-homogeneous Zippin condition for every $\varepsilon > 0$.*

Proof. We need only observe that both X and its predual have (MAP) by a result of Grothendieck [16] (see also [32], page 39). □

THEOREM 5.5. *Suppose X is a separable Banach space with the property that for some $\lambda > 1$ and any separable enlargement Y, the pair (X, Y) satisfies the λ-homogeneous Zippin condition. Then for any $\varepsilon > 0$, $c_0(X)$ has the separable universal $((2+\varepsilon)\lambda, \mathcal{C})$-extension property.*

In particular, $c_0(X)$ is $\mathcal{C}$-automorphic, whenever X is $\mathcal{C}$-automorphic.

Proof. Let $Z = c_0(X)$ and suppose $Z \subset Y$ where Y is separable. Let $Q_n : Z \to Z \subset Y$ be the projection on each coordinate.

Let $(F_k)_{k=1}^{\infty}$ be an increasing sequence of finite-dimensional subspaces of Y with the properties that $F_1 = \{0\}$, $F_k \cap Z = \{0\}$ for every k and $\bigcup_{k\geq 1} F_k$ is dense in Y. Let $W_k = F_k + Z$. We claim that for each k there exists $n_0(k)$ so that if $n \geq n_0$ then the operator $T : W_k \to X$ defined by $T(f+z) = Q_n z$ for $f \in F_k$ and $z \in Z$ has norm $\|T\| \leq 2+\varepsilon$.

Suppose not. Then there is a sequence $(f_n)_{n=1}^{\infty} \in F_k$ and $(z_n)_{n=1}^{\infty}$ in Z so that $\|f_n + z_n\| < 1$ but $\|Q_{m(n)} z_n\| \geq 2+\varepsilon$ for some increasing sequence $(m(n))_{n=1}^{\infty}$. Then since $F_k \cap Z = \{0\}$ the sequence $(f_n)_{n=1}^{\infty}$ is bounded, and hence has a convergent subsequence.

It is clear that for each n there exists $r(n) > n$ so that if $s \geq r(n)$ we have $\|z_s - z_n\| \geq 2+\varepsilon$. Thus $\|f_s - f_n\| \geq \varepsilon > 0$. This contradicts the fact that $(f_n)_{n=1}^{\infty}$ has a convergent subsequence.

It follows that we may select an increasing sequence $k(n) \to \infty$ so that the operator $T_n : W_{k(n)} \to X$ defined by

$$T_n(f+z) = Q_n z, \quad f \in F_{k(n)},\ z \in Z$$

has norm $\|T_n\| \leq 2+\varepsilon$. Now let G_n be a separable enlargement of Y such that there is an operator $\tilde{T}_n : G_n \to G_n$ with $\|\tilde{T}_n\| \leq 2+\varepsilon$ and $\tilde{T}_n|_{W_{k(n)}} = T_n$.

By assumption, there is a homogeneous Zippin selector $\Psi_n : (Q_n(Z))^* \to G_n^*$ with $\|\Psi_n\| \leq \lambda$. We now define $\Phi_n : Z^* \to Y^*$ by

$$\Phi_n(z^*) = \tilde{T}_n^* \Psi_n\big(z^*|_{Q_n(Z)}\big)\big|_Y.$$

Clearly Φ_n is homogeneous, boundedly weak*-continuous, and we have

$$\|\Phi_n(z^*)\| \leq (2+\varepsilon)\lambda \|Q_n^* z^*\|, \quad z^* \in Z^*.$$

Furthermore, if $z \in Z$ we have

$$\Phi_n(z^*)(z) = \Psi_n\big(z^*|_{Q_n(Z)}\big) Q_n z = z^*(Q_n z).$$

Thus, if $\Phi(z^*) = \sum_{n=1}^{\infty} \Phi_n(z^*)$ we have $\|\Phi\| \leq (2+\varepsilon)\lambda$ and

$$\Phi(z^*)|_Z = z^*, \quad z^* \in Z^*.$$

It remains to observe that Φ is also boundedly weak*-continuous. In fact, if $f \in \sum_{k=1}^{\infty} F_k$ then $T_n f = 0$ for all but finitely many n. Hence, $z^* \to \Phi(z^*)(f)$ is weak*-continuous on bounded sets. This implies that Φ is boundedly weak*-continuous. Thus, (Z, Y) satisfies the $(2+\varepsilon)\lambda$-homogeneous Zippin condition. $\square$

Let us note one important corollary.

COROLLARY 5.6. *The space $c_0(\ell_1)$ satisfies the separable universal $(2+\varepsilon,\mathcal{C})$-extension property for every $\varepsilon > 0$ and is $\mathcal{C}$-automorphic.*

6. The spaces ℓ_p when $1 < p < \infty$

We now turn to the spaces ℓ_p for $1 < p < \infty$.

THEOREM 6.1. *Suppose $1 < p < \infty$ and let Y be an enlargement of ℓ_p. Suppose (ℓ_p, Y) has the $\mathcal{C}$-extension property. Then for some λ, (ℓ_p, Y) satisfies the λ-homogeneous Zippin condition.*

More generally, if Y is an enlargement of $\ell_p(F_n)$ where (F_n) is a sequence of finite-dimensional spaces so that (ℓ_p, Y) has the $\mathcal{C}$-extension property, then for some λ, (ℓ_p, Y) satisfies the λ-homogeneous Zippin condition.

Proof. We give the proof for ℓ_p. Consider the canonical embedding of ℓ_p into $\mathcal{C}(B_{\ell_q})$ and suppose this has an extension $T : Y \to \mathcal{C}(B_{\ell_q})$. Then if $\Phi : \ell_q \to \mathcal{M}(B_{\ell_q})$ is the homogeneous Zippin selector given by Theorem 4.7, we consider $\Psi = T^* \circ \Phi : \ell_q \to Y^*$ and this is the required homogeneous Zippin selector. $\square$

We next give a criterion which implies the existence of a homogeneous Zippin selector.

THEOREM 6.2. *Suppose $1 < p < \infty$ and that $X = \ell_p(F_n)$ where $(F_n)_{n=1}^\infty$ is a sequence of finite-dimensional normed spaces. Suppose Y is a separable enlargement of X equipped with a norm so that*

$$\lim_{n\to\infty} \|y + u_n\| \ge \lim_{n\to\infty} (\|y\|^p + c^p \|u_n\|^p)^{1/p} \tag{6.1}$$

whenever $y \in Y$, $(u_n)_{n=1}^\infty$ is weakly null sequence in ℓ_p and both limits exist. Then (X, Y) satisfies the $(1+\varepsilon)c^{-1}$-homogeneous Zippin condition.

Proof. Suppose $\nu_k > 0$ is a decreasing sequence of real numbers such that

$$\prod_{k=1}^\infty (1+\nu_k)^2 < 1+\varepsilon.$$

We will first prove the theorem under an additional condition.

ASSUMPTION. Assume that there is an increasing sequence $(E_n)_{n=1}^\infty$ of finite-dimensional subspaces of Y whose union is dense and such that we have $(E_n + \sum_{j=1}^n F_j) \cap \sum_{j=n+1}^\infty F_j = \{0\}$ and

$$\|u + x\| \ge (1+\nu_n)^{-1} (\|u\|^p + c^p \|x\|^p)^{1/p}, \quad x \in E_n + \sum_{j=1}^n F_j,\ y \in \sum_{j=n+1}^\infty F_j. \tag{6.2}$$

Under this assumption, we define a sequence of weak*-continuous homogeneous maps $\Phi_n : X^* \to Y^*$. First, for each k, by Lemma 4.1, there exists a homogeneous Zippin selector $\Psi_k : (E_{k-1} + \sum_{j=1}^k F_j)^* \to Y^*$ such that $\|\Psi_k\| < 1 + \nu_k$. In the case $k = 1$, we take $E_0 = \{0\}$. Let us denote by $Q_n : X \to F_n$ the canonical projection.

The construction of (Φ_n) is inductive. To start the induction, let $\Phi_1(x^*) = \Psi_1(x^*|_{F_1})$. Note that $\|\Phi_1(x^*)\| \le (1 + \nu_1)\|Q_1^* x^*\|$.

Next suppose $k \ge 2$ and that Φ_{k-1} has been constructed. Then we define for $x^* \in X^*$ we define $\sigma_k(x^*) \in (E_{k-1} + \sum_{j=1}^k F_j)^*$ by

$$\langle u + x, \sigma_n(x^*)\rangle = \langle u, \Phi_{k-1}(x^*)\rangle + \langle x, x^*\rangle, \quad u \in E_{k-1} + \sum_{j=1}^{k-1} F_j,\ x \in F_k.$$

The map σ_k is weak*-continuous and homogeneous and

$$\|\sigma_k(x^*)\| \le (1 + \nu_k)\big(\|\Phi_{k-1}(x^*)\|^q + c^{-q}\|Q_k^* x^*\|^q\big)^{1/q}.$$

Now define

$$\Phi_k(x^*) = \Psi_k(\sigma_k(x^*)), \quad x^* \in X^*.$$

Clearly,

$$\|\Phi_k(x^*)\|^q \le (1 + \nu_k)^{2q}\big(\|\Phi_{k-1}(x^*)\| + c^{-q}\|Q_n x^*\|^q\big).$$

Iterating this condition gives that

$$\|\Phi_k(x^*)\|^q \le c^{-q}\left(\prod_{j=1}^k (1 + \nu_j)^2\right)^q \left(\sum_{j=1}^k \|Q_j^* x^*\|^q\right)$$

and from this it follows that

$$\|\Phi_k(x^*)\| \le (1 + \varepsilon)c^{-1}\|x^*\|.$$

It is also clear from the construction that

$$\langle x, \Phi_k(x^*)\rangle = \langle x, x^*\rangle, \quad x \in F_j,\ 1 \le j \le k$$

and

$$\langle y, \Phi_k(x^*)\rangle = \langle y, \Phi_j(x^*)\rangle, \quad y \in E_j,\ 1 \le j < k.$$

Hence, $\Phi_k(x^*)$ converges weak* to a homogeneous map $\Phi : X^* \to Y^*$ with $\|\Phi\| \le (1+\varepsilon)c^{-1}$ and $\Phi(x^*)|_X = x^*$. Weak*-continuity on bounded sets follows by noting that $\langle f, \Phi(x^*)\rangle = \langle f, \Phi_n(x^*)\rangle$ whenever $f \in E_n$ and, by construction, each Φ_n is weak*-continuous. Thus, Φ is the required homogeneous Zippin selector. This proves the theorem under the assumption.

We now turn to the general case. The sequence $(\nu_k)_{k=1}^\infty$ is chosen as before. We begin with the standard observation that if E is a finite-dimensional closed subspace of Y and $\nu > 0$ then there exists $N = N(F, \nu)$ so that

$$\|f + x\| \ge (1 + \nu)^{-1}(\|f\|^p + c^p\|x\|^p)^{1/p}, \quad f \in E,\ x \in \sum_{j=N+1}^\infty F_j.$$

See, for example Lemma 3.1 of [25].

Fix any increasing sequence (E_n) of finite-dimensional subspaces of Y whose union is dense. Then by induction, we can find $(m_k)_{k=0}^{\infty}$ with $m_0=0$ and $m_1=1$ so that

$$\|u+x\| \ge (1+\nu_k)^{-1}(\|u\|^p + c^p\|x\|^p)^{1/p},$$
$$u \in E_k + \sum_{j=1}^{m_k} F_j,\ x \in \sum_{j=m_{k+1}+1}^{\infty} F_j,\ k=1,2,\dots.$$

Let $G_k = F_{m_{k-1}+1} + \dots + F_{m_k}$.

Now fix N so that

$$\left(1-\frac{2c^{-1}+1}{N^{1/p}}\right)^{-1} < (1+\varepsilon)^{1/2}$$

and suppose $1 \le r \le N$. Let $\mathbb{A}_r$ the complement in $\mathbb{N}$ of the arithmetic sequence $(N(k-1)+r)_{k=1}^{\infty}$ and let $X_r = [\sum_{j\in\mathbb{A}_r} G_j]$. Then it is clear that using the decomposition $H_1 = \sum_{j=1}^{r-1} G_j$ (if $r>1$) and then $H_k = \sum_{j=(k-2)N+r+1}^{(k-1)N+r-1} G_j$, the space X_r satisfies (6.2) and so we have the existence of a homogeneous Zippin selector $\Phi_r : X_r^* \to Y^*$ with $\|\Phi_r\| < (1+\varepsilon)^{1/2}c^{-1}$.

For $x^* \in X^*$, we define

$$\Psi(x^*) = \frac{1}{N}\sum_{r=1}^{N} \Phi_r(x^*|_{X_r}).$$

Then for $x \in X$ we have, denoting the canonical projection $P_r : X \to X_r$,

$$\langle x, \Psi(x^*)\rangle = \frac{1}{N}\sum_{r=1}^{N}\langle x - P_r x, \Phi_r(x^*|_{X_r})\rangle + \frac{1}{N}\sum_{r=1}^{N}\langle P_r x, x^*\rangle.$$

Note that

$$\sum_{r=1}^{N} P_r x = (N-1)x$$

so that for $x \in X$,

$$\begin{aligned}
|\langle x, \Psi(x^*) - x^*\rangle| &\le \frac{1}{N}\|x\|\|x^*\| + \frac{2c^{-1}}{N}\sum_{r=1}^{N}\|x - P_r x\|\|x^*\| \\
&\le \frac{2c^{-1}+1}{N}\sum_{r=1}^{N}\|x - P_r x\|\|x^*\| \\
&\le \frac{2c^{-1}+1}{N^{1/p}}\|x^*\|\left(\sum_{r=1}^{N}\|x - P_r x\|^p\right)^{1/p} \\
&= \frac{2c^{-1}+1}{N^{1/p}}\|x^*\|\|x\|.
\end{aligned}$$

Hence,

$$\|\rho\Psi(x^*) - x^*\| \le \frac{2c^{-1}+1}{N^{1/p}}\|x^*\|$$

where $\rho : Y^* \to X^*$ is the natural restriction. Let $\phi(x^*) = x^* - \rho\Psi(x^*)$. We then can define

$$\Phi(x^*) = \sum_{n=0}^{\infty} \Psi \circ \phi^n x^*.$$

We may then verify easily that Φ is a homogeneous Zippin selector and

$$\|\Phi\| \le c^{-1}(1+\varepsilon)^{1/2}\left(1 - \frac{2c^{-1}+1}{N^{1/p}}\right)^{-1} < (1+\varepsilon)c^{-1}. \qquad \square$$

THEOREM 6.3. *Suppose $1 < p < \infty$ and $X = \ell_p(F_n)$ is a sequence of finite-dimensional spaces. Suppose Y is a separable enlargement of X. The following conditions on Y are equivalent:*

(i) *(X, Y) has the $\mathcal{C}$-extension property.*

(ii) *There is an equivalent norm on Y so that Y is an enlargement of ℓ_p and*

$$\lim_{n\to\infty} \|y + u_n\| \ge \lim_{n\to\infty} (\|y\|^p + \|u_n\|^p)^{1/p} \tag{6.3}$$

whenever $y \in Y$, $(u_n)_{n=1}^\infty$ is weakly null sequence in X and both limits exist.

(iii) *There exists a linear operator $T : Y \to Z$ (for some Banach space Z) so that for some $c > 0$,*

$$\lim_{n\to\infty} \|T(y + u_n)\| \ge \lim_{n\to\infty} (\|Ty\|^p + c^p\|u_n\|^p)^{1/p} \tag{6.4}$$

whenever $y \in Y$, $(u_n)_{n=1}^\infty$ is weakly null sequence in X and both limits exist.

Proof. (i) $\Rightarrow$ (iii): By Corollary 6.1, there is a homogeneous Zippin selector $\Phi : X^* \to Y^*$. Define a seminorm on Y by

$$|y| = \sup_{\substack{\|u^*\| \le 1 \\ u^* \in \ell_p^*}} |\langle y, \Phi(u^*)\rangle|.$$

Note that $|y| \le \|\Phi\|\|y\|$ for $y \in Y$ and $|u| = \|u\|$ if $u \in X$.

Let us assume $y \in Y$ and $(u_n)_{n=1}^\infty$ is a weakly null sequence in X. Passing to a subsequence, we can suppose that the limits

$$\lim_{n\to\infty} \|y + u_n\|, \qquad \lim_{n\to\infty} \|u_n\|, \qquad \lim_{n\to\infty} |y + u_n|$$

all exist.

Let $u_n^* \in X^*$ be chosen to be the norming functionals for u_n. Then (u_n^*) is a weak*-null sequence. Pick $v^* \in X^*$ so that $\|v^*\| = 1$ and $\langle y, \Phi(v^*)\rangle = |y|$. Suppose $\tau > 0$ is fixed. Then

$$\|v^* + \tau u_n^*\||y + u_n| \ge \langle y + u_n, \Phi(v^* + \tau u_n^*)\rangle, \quad n = 1, 2, \ldots.$$

Now

$$\lim_{n\to\infty}\langle y,\Phi(v^*+\tau u_n^*)\rangle=\langle y,\Phi(v^*)\rangle=|y|$$

by the weak*-continuity of Φ. On the other hand,

$$\langle u_n,\Phi(v^*+\tau u_n^*)\rangle=v^*(u_n)+\tau\|u_n\|$$

so that

$$\lim_{n\to\infty}\langle u_n,\Phi(v^*+\tau u_n^*)\rangle=\tau\lim_{n\to\infty}\|u_n\|.$$

Furthermore,

$$\lim_{n\to\infty}\|v^*+\tau u_n^*\|=(1+\tau^q)^{1/q}.$$

Thus,

$$|y|+\tau\lim_{n\to\infty}\|u_n\|\le(1+\tau^q)^{1/q}\lim_{n\to\infty}|y+u_n|,\quad 0<\tau<\infty.$$

This implies

$$\Big(|y|^p+\lim_{n\to\infty}\|u_n\|^p\Big)^{1/p}\le\lim_{n\to\infty}|y+u_n|.$$

Taking Z to be the completion of (the Hausdorff quotient of) $(Y,|\cdot|)$ and T the identity map, we have (6.4) with $c=1$.

(iii) $\Rightarrow$ (ii). By scaling we can assume $c=1$. Define a norm on Y by

$$\|y\|_0=\inf\{\|y-v\|+\|T(y-v)\|+\|v\|:v\in X\}.$$

Clearly, $\|y\|_0\ge\|y\|$ and $\|v\|_0=\|v\|$ for $v\in X$. Now assume that $y\in Y$ and (u_n) is a weakly null sequence in X. Assume $\lim_{n\to\infty}\|y+u_n\|_0$ and $\lim_{n\to\infty}(\|y\|_0^p+\|u_n\|^p)^{1/p}$ both exist. Choose $v_n\in X$ so that

$$\lim_{n\to\infty}\big(\|y+u_n-v_n\|+\|T(y+u_n-v_n)\|+\|v_n\|\big)=\lim_{n\to\infty}\|y+u_n\|_0.$$

The sequence (v_n) is bounded and so by passing to a subsequence we can assume it converges weakly to some $v\in X$. Passing to further subsequences so that required limits exist, we have

$$\lim_{n\to\infty}\|v_n\|=\lim_{n\to\infty}(\|v-v_n\|^p+\|v\|^p)^{1/p}.$$

Now, using the triangle law in ℓ_p^2 and again assuming all limits exist,

$$\begin{aligned}
&\lim_{n\to\infty}\|y+u_n\|_0\\
&\quad\ge\lim_{n\to\infty}\big(\|y-v\|+\big(\|T(y-v)\|^p+\|u_n+v-v_n\|^p\big)^{1/p}\\
&\qquad+(\|v\|^p+\|v-v_n\|^p)^{1/p}\big)\\
&\quad\ge\lim_{n\to\infty}\big((\|y-v\|+\|T(y-v)\|+\|v\|)^p+\|u_n\|^p\big)^{1/p}\\
&\quad\ge\lim_{n\to\infty}(\|y\|_0^p+\|u_n\|^p)^{1/p}.
\end{aligned}$$

(ii) $\Rightarrow$ (i). This follows directly from Theorem 6.2. □

We are now in position to show that the ℓ_p-spaces are not $\mathcal{C}$-automorphic.

THEOREM 6.4. *Suppose $1<p<\infty$. Then there is a separable enlargement Z of ℓ_p such that Z is super-reflexive, has an unconditional basis and (ℓ_p, Z) fails to have $\mathcal{C}$-extension property. In particular, ℓ_p fails to have the separable universal $\mathcal{C}$-extension property and is not $\mathcal{C}$-automorphic.*

Proof. Let $\mathcal{T}$ denote the standard dyadic tree. The nodes are indexed by $\emptyset$ and then all finite sequences $a=(t_1,\dots,t_n)$ of zeros and ones. We write $a\preceq b$ if $a=(t_1,\dots,t_m)$ and $b=(s_1,\dots,s_n)$ where $n\ge m$ and $s_j=t_j$ for $j\le m$; we write $a\prec b$ if $a\preceq b$ and $a\neq b$. The depth $d(a)$ of $a\in\mathcal{T}$ is n where $a=(t_1,\dots,t_n)$. A *segment* β is a finite subset of $\mathcal{T}$ of the form $\{a_1,\dots,a_n\}$ with $a_1\prec a_2\prec\cdots\prec a_n$. Let $\mathcal{B}$ be the collection of all segments.

Let $c_{00}(\mathcal{T})$ denote the space of finitely nonzero functions on $\mathcal{T}$. Let $(e_a)_{a\in\mathcal{T}}$ be the canonical basis. The natural bilinear pairing on $c_{00}(\mathcal{T})$ is given by

$$\langle \xi,\eta\rangle=\sum_{a\in\mathcal{T}}\xi_a\eta_a$$

if $\xi=\sum_{a\in\mathcal{T}}\xi_a e_a$ and $\eta_a=\sum_{a\in\mathcal{T}}\eta_a e_a$.

We first define a norm on $c_{00}(\mathcal{T})$ by

$$\Big\|\sum_{a\in\mathcal{T}}\xi_a e_a\Big\|_{\mathcal{Y}}=\max\bigg(\Big(\sum_{a\in\mathcal{T}}|\xi_a|^q\Big)^{1/q},\ \sup_{\beta\in\mathcal{B}}\sum_{a\in\beta}|\xi_a|\bigg).$$

We define a dual norm

$$\|\xi\|_{\mathcal{X}_0}=\sup(|\langle\xi,\eta\rangle|:\|\eta\|_{\mathcal{Y}}\le 1).$$

Note that

$$\|\xi\|_{\mathcal{X}_0}\le\|\xi\|_{\ell_p(\mathcal{T})}.$$

For each $a\in\mathcal{T}$ let $u_a=e_{a'}+e_{a''}$ where a',a'' are the two successors of a, i.e. $a\prec a',a''$ and $d(a'),d(a'')=d(a)+1$. It is clear that

$$\Big\|\sum_{a\in\mathcal{T}}\xi_a u_a\Big\|_{\mathcal{X}_0}\le 2^{1/p}\Big(\sum_{a\in\mathcal{T}}|\xi_a|^p\Big)^{1/p},\quad \xi\in c_{00}(\mathcal{T}).$$

We will now need the following lemma.

LEMMA 6.5. *Suppose $\eta_a\in c_{00}(\mathcal{T})$. Then there exists $\xi\in c_{00}(\mathcal{T})$ such that $\langle\xi,u_a\rangle=\eta_a$ for $a\in\mathcal{T}$ and*

$$\|\xi\|_{\mathcal{Y}}\le C_p\Big(\sum_{a\in\mathcal{T}}|\eta_a|^q\Big)^{1/q},$$

where $C_p=\max(1,(2^p-2)^{-1/p})$.

Proof. Let $C_n = C_{n,p}$ be the best constant such that whenever η is supported on $\{a : a \in \mathcal{T},\ d(a) \le n\}$ then we can find ξ with $\langle \xi, u_a \rangle = \eta_a$ for $a \in \mathcal{T}$ and

$$\|\xi\|_{\mathcal{Y}} \le C_n \left(\sum_{a\in\mathcal{T}} |\eta_a|^q \right)^{1/q}.$$

Observe that $C_0 < 1$. We now obtain an estimate for C_n in terms of C_{n-1} when $n \ge 1$.

For any η, we consider $\eta' = \sum_{(0)\preceq a} \eta_a e_a$ and $\eta'' = \sum_{(1)\preceq a} \eta_a e_a$ where of course (0) and (1) are the two successors of $\emptyset$. By hypothesis, we can find ξ' and $\xi'' \in c_{00}(\mathcal{T})$ so that ξ' is supported on $\{(0) \prec a\}$ and ξ'' is supported on $\{(1) \prec a\}$ and such that

$$\|\xi'\|_{\mathcal{Y}} \le C_{n-1} \|\eta'\|_{\ell_q(\mathcal{T})}, \quad \|\xi''\|_{\mathcal{Y}} \le C_{n-1} \|\eta''\|_{\ell_q(\mathcal{T})}$$

and

$$\langle \xi' + \xi'', u_a \rangle = \eta_a, \quad d(a) \ge 1.$$

Let β' be the segment starting at (0) so that

$$\sum_{a\in\beta'} |\xi'_a| = \max_{\beta\in\mathcal{B}} \sum_{a\in\beta} |\xi'_a|$$

and similarly let β'' be the segment starting at (1) so that

$$\sum_{a\in\beta''} |\xi''_a| = \max_{\beta\in\mathcal{B}} \sum_{a\in\beta} |\xi''_a|.$$

Let us assume (for convenience; the other case is exactly similar) that

$$\sum_{a\in\beta'} |\xi'_a| \le \sum_{a\in\beta''} |\xi''_a|.$$

Then if $|\eta_\emptyset| \le \sum_{a\in\beta''} |\xi''_a| - \sum_{a\in\beta'} |\xi'_a|$, we will set

$$\xi = \eta_\emptyset e_{(0)} + \xi' + \xi''.$$

If $|\eta_\emptyset| \ge \sum_{a\in\beta''} |\xi''_a| - \sum_{a\in\beta'} |\xi'_a|$, we will set

$$\xi = \frac{1}{2}\eta_\emptyset\big(e_{(0)} + e_{(1)}\big) + \frac{1}{2}\left(\sum_{a\in\beta''} |\xi''_a| - \sum_{a\in\beta'} |\xi'_a| \right) \operatorname{sgn} \eta_\emptyset \big(e_{(0)} - e_{(1)}\big) + \xi' + \xi''.$$

In either case, we have

$$\|\xi\|_{\ell_q(\mathcal{T})} \le \big(|\eta_\emptyset|^q + C_{n-1}\|\eta'\|^q_{\ell_q(\mathcal{T})} + C_{n-1}\|\eta''\|^q_{\ell_q(\mathcal{T})} \big)^{1/q} \le \max(1, C_{n-1}) \|\eta\|_{\ell_q(\mathcal{T})}.$$

In the first case,

$$\max_{\beta\in\mathcal{B}} \sum_{a\in\beta} |\xi_a| \le \sum_{a\in\beta''} |\xi''_a| \le C_{n-1} \|\eta'\|_{\ell_q(\mathcal{T})}.$$

In the second case,

$$\begin{aligned}\max_{\beta\in\mathcal{B}}\sum_{a\in\beta}|\xi_a| &= \frac{1}{2}\biggl(|\eta_\emptyset| + \sum_{a\in\beta'}|\xi'_a| + \sum_{a\in\beta''}|\xi''_a|\biggr)\\ &\le \frac{1}{2}\bigl(|\eta_\emptyset| + C_{n-1}\|\eta'\|_{\ell_q(\mathcal{T})} + C_{n-1}\|\eta''\|_{\ell_q(\mathcal{T})}\bigr)\\ &\le \frac{1}{2}(1+2C_{n-1}^p)^{1/p}\|\eta\|_{\ell_q(\mathcal{T})}.\end{aligned}$$

Hence,

$$C_n \le \max\biggl(1, C_{n-1}, \frac{1}{2}(1+2C_{n-1}^p)^{1/p}\biggr).$$

It follows that if $2^p \ge 3$ we have $C_n \le 1$, while if $2^p < 3$ we have $C_n \le (2^p - 2)^{-1/p}$, i.e.,

$$C_n \le \max\bigl(1, (2^p-2)^{-1/p}\bigr), \quad n = 1, 2, \ldots.$$

This completes the proof of Lemma 6.5. □

Thus, we have that $[u_a]_{a\in\mathcal{T}}$ is isomorphic to ℓ_p.

We let $\mathcal{X}_0$ be the completion of $c_{00}(\mathcal{T})$ under the norm $\|\cdot\|_{\mathcal{X}_0}$. Then $(e_a)_{a\in\mathcal{T}}$ is a 1-unconditional basis for $\mathcal{X}_0$. Furthermore, for any $b\in\mathcal{T}$, we have

$$\Bigl\|\sum_{a\preceq b} e_a\Bigr\|_{\mathcal{X}_0} \le 1.$$

LEMMA 6.6. *The block basis $(u_a)_{a\in\mathcal{T}}$ in $\mathcal{X}_0$ is equivalent to the canonical basis of ℓ_p.*

Proof. We claim that

$$C_p^{-1}\biggl(\sum_{a\in\mathcal{T}}|\xi_a|^p\biggr)^{1/p} \le \Bigl\|\sum_{a\in\mathcal{T}}\xi_a u_a\Bigr\|_{\mathcal{X}_0} \le 2^{1/p}\biggl(\sum_{a\in\mathcal{T}}|\xi_a|^p\biggr)^{1/p}, \quad \xi\in c_{00}(\mathcal{T}).$$

Indeed, using Lemma 6.5, we may pick $\zeta\in c_{00}(\mathcal{T})$ with $\langle\zeta, u_a\rangle = (\operatorname{sgn}\xi_a)|\xi_a|^{p-1}$ and $\|\zeta\|_{\mathcal{Y}} \le C_p\sum_{a\in\mathcal{T}}|\xi_a|^{q(p-1)} = C_p\|\xi\|_{\ell_p(\mathcal{T})}^{p/q}$. Hence,

$$\sum_{a\in\mathcal{T}}|\xi_a|^p \le C_p\|\xi\|_{\ell_p(\mathcal{T})}^{p/q}\Bigl\|\sum_{a\in\mathcal{T}}\xi_a u_a\Bigr\|_{\mathcal{X}_0}.$$

Hence,

$$C_p^{-1}\biggl(\sum_{a\in\mathcal{T}}|\xi_a|^p\biggr)^{1/p} \le \Bigl\|\sum_{a\in\mathcal{T}}\xi_a u_a\Bigr\|_{\mathcal{X}_0} \le 2^{1/p}\biggl(\sum_{a\in\mathcal{T}}|\xi_a|^p\biggr)^{1/p}.$$ □

For $0<\theta<1$, we then define $\mathcal{X}_\theta$ to be the real interpolation space $(\mathcal{X}_0, \ell_p(\mathcal{T}))_{\theta,p}$. It is clear that $(e_a)_{a\in\mathcal{T}}$ is also a 1-unconditional basis for $\mathcal{X}_\theta$ and that $(u_a)_{a\in\mathcal{T}}$ is equivalent to the canonical ℓ_p-basis also in $\mathcal{X}_\theta$. Furthermore $\mathcal{X}_\theta$ is super-reflexive (see [2]).

Now we can renorm $\mathcal{X}_\theta$ so that $[u_a]$ is isometric to ℓ_p so that $\mathcal{X}_\theta$ becomes an enlargement of ℓ_p. Hence, if $(\ell_p, \mathcal{X}_\theta)$ has $\mathcal{C}$-extension property, there is an equivalent norm $\|\cdot\|_{\mathcal{X}_\theta,1}$ on $\mathcal{X}_\theta$ so that

$$\liminf_{d(a)\to\infty} \left\|\xi + \frac{1}{2}u_a\right\|_{\mathcal{X}_\theta,1} \geq (\|\xi\|^p_{\mathcal{X}_\theta,1} + 2c^p)^{1/p}$$

for some suitable constant $c > 0$.

We now construct by induction an order-preserving map $\rho : \mathcal{T} \to \mathcal{T}$ as follows. Let $\rho(\emptyset) = \emptyset$. Then if $\rho(b)$ has been determined for $b \preceq a$ we choose $\hat{a}$ with $\rho(a) \prec \hat{a}$ so that

$$\left\|\sum_{b\preceq a} e_{\rho(b)} + \frac{1}{2}u_{\hat{a}}\right\|^p_{\mathcal{X}_\theta,1} > \left\|\sum_{b\preceq a} e_{\rho(b)}\right\|^p_{\mathcal{X}_\theta,1} + c^p.$$

We then define $\rho(a') = \hat{a}'$ and $\rho(a'') = \hat{a}''$ (denoting as before the successors of a by a', a'', etc.).

Consider the Cantor set $\Delta = \{0,1\}^{\mathbb{N}}$ with its canonical measure $\mathbb{P}$. Let Σ_n denote the σ-algebra generated by the sets $\Delta_{t_1,\dots,t_n} = \{s : s_j = t_j,\ 1 \leq j \leq n\}$.

We define functions $F_n : \Delta \to \mathcal{X}_\theta$ by

$$F_n(t) = \sum_{b\preceq(t_1,\dots,t_n)} e_{\rho(b)}.$$

Note that

$$\|F_n(t)\|_{\mathcal{X}_0} \leq 1, \quad t \in \Delta,\ n = 1, 2, \dots.$$

Then we have an estimate:

$$\|F_n(t)\|_{\mathcal{X}_\theta} \leq C_\theta \|F_n(t)\|^{1-\theta}_{\mathcal{X}_0} \|F_n(t)\|^\theta_{\ell_p(\mathcal{T})} \leq C_\theta n^{\theta/p}.$$

On the other hand,

$$\begin{aligned}\mathbb{E}\|F_n\|^p_{\mathcal{X}_\theta,1} &\geq \mathbb{E}\|\mathbb{E}(F_n|\Sigma_{n-1})\|^p_{\mathcal{X}_\theta,1}\\ &\geq c^p + \mathbb{E}\|F_{n-1}\|^p_{\mathcal{X}_\theta,1}.\end{aligned}$$

Thus,

$$(\mathbb{E}\|F_n\|^p_{\mathcal{X}_\theta,1})^{1/p} \geq cn^{1/p}.$$

This yields a contradiction and shows that $([u_a]_{a\in\mathcal{T}}, \mathcal{X}_\theta)$ fails the $\mathcal{C}$-extension property. □

Remark. We used real interpolation in the above proof simply for consistency with our restriction to real scalars. However, since we are interpolating spaces with unconditional bases we could equally have used complex interpolation of the complexifications of $\mathcal{X}_0$ and $\ell_p(\mathcal{T})$.

Remark. Theorem 6.4 is apparently the first known example of a super-reflexive space failing the $\mathcal{C}$-extension property. It therefore answers Problems 4.1 and 4.2 of [17] or, equivalently, Problems 6.7 and 6.8 of [46].

7. Enlargements of ℓ_p-spaces with the $\mathcal{C}$-extension property

In this final section, we give some positive results for extensions from ℓ_p-spaces. We first develop a criterion for the existence of extensions.

THEOREM 7.1. *Suppose $1<p<\infty$ and let $X=\ell_p(F_n)$where (F_n) is a sequence of finite-dimensional normed spaces. Suppose Y is a separable enlargement of X which is (UMD). Then there is an equivalent norm $\|\cdot\|_0$ on Y so that if $y\in Y$ and (u_n) is a weakly null sequence in X then*

$$\lim_{n\to\infty}\left(\frac{1}{2}\|y+u_n\|_0^p+\frac{1}{2}\|y-u_n\|_0^p\right)^{1/p}\geq\lim_{n\to\infty}(\|y\|_0^p+\|u_n\|^p)^{1/p} \tag{7.1}$$

provided both limits exist.

Proof. We begin with some notation. Let $\Delta=\{0,1\}^{\mathbb{N}}$ be the Cantor set with the usual Haar measure, $\mathbb{P}$. We denote by Σ_j the finite-algebra of the Borel sets generated by the set $\Delta_{s_1,\ldots,s_j}=\{t:\,t_i=s_i,\ 1\leq i\leq j\}$. Thus, Σ_0 is the trivial subalgebra.

Since Y has (UMD) there is a constant C with the following property. Let $(M_j)_{j=0}^m$ be a Y-valued martingale on Δ adapted to $(\Sigma_j)_{j=0}^m$. Let $dM_j=M_j-M_{j-1}$ and $dM_0=M_0$. Then for any $\delta_j=\pm1$ for $0\leq j\leq m$, we have

$$\left(\mathbb{E}\left\|\sum_{j=1}^m\delta_j\,dM_j\right\|^p\right)^{1/p}\leq C(\mathbb{E}\|M_m\|^p)^{1/p}.$$

Now we will utilize the notion of a tree map from [12]. We consider the infinite branching tree $\mathcal{T}_\infty$ which we define to consist of all finite subsets $\{n_1,\ldots,n_k\}$ of $\mathbb{N}$ with $n_1<n_2<\cdots<n_k$ ordered by $\{n_1,\ldots,n_k\}\leq\{m_1,\ldots,m_l\}$ if $k\leq l$ and $n_j=m_j$ for $1\leq j\leq k$. The empty set $\emptyset$ is the root of the tree. A *branch* of the tree is a sequence $a_0=\emptyset$ and then $a_k=\{n_1,\ldots,n_k\}$ where $(n_k)_{k=1}^\infty$ is a fixed increasing sequence.

We consider a class $\mathcal{D}$ of *tree maps* $a\to f_a,\ f:\mathcal{T}\to L_p(\Delta,Y)$. By definition, a tree map satisfies the condition that for every branch β the set $\{a\in\beta:\,f_a\neq0\}$ is finite. We additionally require for $f\in\mathcal{D}$ that $f_\emptyset$ is constant, and each $f_{n_1,\ldots,n_k}$ is Σ_k-measurable, and we insist that

$$\mathbb{E}(f_{n_1,\ldots,n_k}|\Sigma_{k-1})=0,\quad k=1,2,\ldots$$

and if k is even,

$$f_{n_1,\ldots,n_k}(t)\in\sum_{j=n_k+1}^\infty F_k\subset X,\quad t\in\Delta.$$

For each branch β generated by $(n_k)_{k=1}^\infty$, we define

$$\Lambda(\beta,f)=(C^p+1)\mathbb{E}\left\|\sum_{j=0}^\infty f_{n_1,\ldots,n_j}\right\|^p-\mathbb{E}\sum_{j=1}^\infty\|f_{n_1,\ldots,n_{2j}}\|^p.$$

Here, the zero term in the first summation is $f_\emptyset$. Note, of course, that our assumptions on tree maps ensure each such sum is finite. We then define

$$\Lambda(f) = \sup_\beta \Lambda(\beta, f).$$

We then define

$$\Theta(y) = \inf\{\Lambda(f) : f_\emptyset = y,\ f \in \mathcal{D}\}, \quad y \in Y.$$

It follows that we have that if $\theta > \Theta(y)$, then there exists a tree map f with $f_\emptyset = y$ and

$$\Lambda(\beta, f) < \theta$$

for every branch β. Now it is we can choose a branch $\beta = (n_k)_{k=1}^\infty$ by induction so that if $k \geq 1$, then $f_{n_1,\dots,n_{2k}}(t) \in \sum_{j=n_{2k}+1}^{n_{2k+1}} F_j$ for all $t \in \Delta$. Then the sequence

$$M_k = y + \sum_{j=1}^k f_{n_1,\dots,n_j}$$

is a martingale adapted to Σ_k. For some m, we have M_k is constant for $k \geq m$. Thus,

$$\begin{aligned}
\Lambda(\beta, f) &= (C^p + 1)\mathbb{E}\left\| \sum_{j=0}^\infty f_{n_1,\dots,n_j} \right\|^p - \mathbb{E}\sum_{j=1}^\infty \|f_{n_1,\dots,n_{2j}}\|^p \\
&= (C^p + 1)\mathbb{E}\left\| \sum_{j=0}^\infty f_{n_1,\dots,n_j} \right\|^p - \mathbb{E}\left\| \sum_{j=1}^\infty f_{n_1,\dots,n_{2j}} \right\|^p \\
&\geq \mathbb{E}\left\| \sum_{j=0}^\infty f_{n_1,\dots,n_j} \right\|^p \\
&\geq \|y\|^p.
\end{aligned}$$

Thus, $\theta \geq \|y\|^p$ and it follows that $\Theta(y) \geq \|y\|^p$.

It is clear that $\Theta(\alpha y) = |\alpha|^p \Theta(y)$ for $y \in Y$ and $\alpha \in \mathbb{R}$. We next claim that

$$\Theta\left(\frac{1}{2}(y+z)\right) \leq \frac{1}{2}\big(\Theta(y) + \Theta(z)\big).$$

To do this, suppose $\varepsilon > 0$ and find $f^{(y)}, f^{(z)} \in \mathcal{D}$ so that $f_\emptyset^{(y)} = y$, $f_\emptyset^{(z)} = z$ and $\Lambda(f^{(y)}) < \Theta(y) + \varepsilon$ while $\Lambda(f^{(z)}) < \Theta(z) + \varepsilon$. Let us introduce the map $\sigma : \Delta \to \Delta$ defined by

$$\sigma(t_1, t_2, \dots) = (t_3, t_4, \dots).$$

We then define $f \in \mathcal{D}$ by $f_\emptyset = \frac{1}{2}(y+z)$ and then

$$f_{n_1}(t) = \begin{cases} \frac{1}{2}(y-z), & t_1 = 1, \\ \frac{1}{2}(z-y), & t_1 = 0, \end{cases}$$

and

$$f_{n_1,n_2} = 0, \quad n_1 < n_2$$

and then for $k \geq 3$,

$$f_{n_1,n_2,\dots,n_k}(t) = \begin{cases} f^{(y)}_{n_3,\dots,n_k}(\sigma(t)), & t_1 = 1, \\ f^{(z)}_{n_3,\dots,n_k}(\sigma(t)), & t_1 = 0. \end{cases}$$

It is then clear that

$$\Lambda(f) < \frac{1}{2}\big(\Theta(y) + \Theta(z) + \varepsilon\big).$$

We deduce that

$$\Theta\left(\frac{1}{2}(y+z)\right) \leq \frac{1}{2}\big(\Theta(y) + \Theta(z)\big). \tag{7.2}$$

Now suppose $y \in Y$ and $(u_n)_{n=1}^{\infty}$ is a sequence in X such that $u_n \in \sum_{k=n+1}^{\infty} F_k$. We claim that

$$\Theta(y) + \liminf_{n\to\infty} \|u_n\|^p \leq \liminf_{n\to\infty} \frac{1}{2}\big(\Theta(y+u_n) + \Theta(y-u_n)\big). \tag{7.3}$$

In this case, for $\varepsilon > 0$ and for each n, we may tree maps $f^{(n,+)}$ and $f^{(n,-)}$ so that $f^{(n,+)}_{\emptyset} = y + u_n$, $f^{(n,-)}_{\emptyset} = y - u_n$, $\Lambda(f^{(n,+)}) < \Theta(y+u_n) + \varepsilon$ and $\Lambda(f^{(n,-)}) < \Theta(y-u_n) + \varepsilon$. We then define $f \in \mathcal{D}$ by

$$f_{\emptyset} = y, \qquad f_{n_1} = 0, \quad 1 \leq n_1 < \infty$$

and then

$$f_{n_1,n_2}(t) = \begin{cases} u_{n_2}, & t_2 = 1, \\ -u_{n_2}, & t_2 = 0, \end{cases}$$

and if $k \geq 3$

$$f_{n_1,\dots,n_k}(t) = \begin{cases} f^{n_2,+}_{n_3,\dots,n_k}(\sigma(t)), & t_2 = 1, \\ f^{n_2,-}_{n_3,\dots,n_k}(\sigma(t)), & t_2 = 0. \end{cases}$$

Then

$$\Lambda(f) \leq \sup_n \left(\frac{1}{2}\big(\Theta(y+u_n) + \Theta(y-u_n)\big) - \|u_n\|^p\right) + \varepsilon.$$

Therefore,

$$\Theta(f) \leq \sup_n \left(\frac{1}{2}\big(\Theta(y+u_n) + \Theta(y-u_n)\big) - \|u_n\|^p\right).$$

This implies (7.3).

We next note that from (7.2) it follows that

$$\Theta\left(\frac{1}{N}(y_1 + \cdots + y_N)\right) \leq \frac{1}{N}\sum_{j=1}^{N} \Theta(y_j)$$

whenever N is a power of 2. Let $U = \{x : \Theta(y) < 1\}$. Since U contains the open unit ball of X, it follows from the above condition that U is also open and thus is convex. Thus, U generates an equivalent norm $\|\cdot\|_0$ on Y and indeed

$$\Theta(y)=\|y\|_0^p, \quad y\in Y.$$

By (7.3), we have that $\|\cdot\|_0$ satisfies the conclusion of the theorem. $\square$

THEOREM 7.2. *Suppose $1<p<\infty$ and $X=\ell_p(F_n)$ where (F_n) is a sequence of finite-dimensional spaces. Let Y be a separable enlargement of X such that Y has the (UMD) property and a (UFDD). Then (X,Y) has the $\mathcal{C}$-extension property.*

Proof. Let $(E_n)_{n=1}^{\infty}$ be the UFDD for Y and let Q_n be the associated projections $Q_n: Y\to E_n$. $y\in Y$. For any sequence $\delta=(\delta_n)_{n=1}^{\infty}$ where $\delta_n=\pm 1$ define

$$S_\delta y=\sum_{j=1}^{\infty}\delta_j Q_j y.$$

Let us suppose $\|\cdot\|_0$ is the norm on Y satisfying (7.1) given by Theorem 7.1. Suppose $\sup_\delta \|S_\delta\|=K$ and that $C^{-1}\|y\|\le\|y\|_0\le C\|y\|$ for $y\in Y$. We then define

$$\|y\|_1=\sup_\delta \|S_\delta y\|_0.$$

Now if $y\in Y$ and (u_n) is a weakly null sequence in X for given $\varepsilon>0$, we can pick δ so that $\|y\|_1<\|S_\delta y\|_0+\varepsilon$. But then

$$\begin{aligned}&\liminf_{n\to\infty}\frac{1}{2}\big(\|S_\delta(y+u_n)\|_0^p+\|S_\delta(y-u_n)\|_0^p\big)\\&\quad\ge\|S_\delta y\|_0^p+\liminf_{n\to\infty}\|S_\delta u_n\|_0^p\\&\quad\ge\|y\|_1^p-\varepsilon+(CK)^{-1}\liminf_{n\to\infty}\|u_n\|^p.\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\frac{1}{2}(\|y+u_n\|_1^p+\|y-u_n\|_1^p)\ge\|y\|_1^p+(CK)^{-1}\lim_{n\to\infty}\|u_n\|^p$$

whenever all the limits exist. But in $(Y,\|\cdot\|_1)$, (E_n) is a 1-UFDD and so

$$\lim_{n\to\infty}\|y+u_n\|_1=\lim_{n\to\infty}\|y-u_n\|_1$$

whenever one limit exists. We therefore can apply Theorem 6.3(iii) to the identity map $Y\to(Y,\|\cdot\|_1)$ to deduce that (X,Y) has the $\mathcal{C}$-extension property. $\square$

REMARKS. We do not know if the above theorem fails without the UFDD-assumption. Let us remark that the example created in Theorem 6.4 is now seen to necessarily fail (UMD). The first example of a super-reflexive Banach lattice failing (UMD) was given by Bourgain in 1983 [6] and is not totally trivial, so this example may be of some interest.

Notice that this Theorem 7.2 applies to the case when $X = \ell_p$ and $Y = L_r$ where $1 < r \leq p \leq 2$. An isometric result of this nature in proved in [22].

COROLLARY 7.3. *Theorem 7.2 holds if X is assumed to be a subspace of a space $\ell_p(F_n)$ with the approximation property.*

Proof. Of course X also has (MAP) and by Proposition 5.3 X is isomorphic to a complemented subspace of a space $Z = \ell_p(G_n)$ where each G_n is a finite-dimensional subspace of X. It follows by the Pełczyński decomposition trick that $X \oplus \ell_p(Z)$ is isomorphic to $\ell_p(Z)$. Thus, $(X \oplus \ell_p(Z), Y \oplus \ell_p(Z))$ has the $\mathcal{C}$-extension property by Theorem 7.2 and this quickly implies that so does (X, Y). □

We can now extend our results to the case of subspaces of L_1 (which is, of course, not a (UMD)-space).

THEOREM 7.4. *Suppose $1 < p \leq 2$ and X is a subspace of $L_1[0,1]$ which is isomorphic to a subspace of ℓ_p with the approximation property. Then (X, L_1) has the $\mathcal{C}$-extension property.*

Proof. Since E has nontrivial type we can apply the Maurey–Nikishin factorization theory (cf. [34], [37], or [43]) and via a change of density assume that X is isomorphically embedded into some $L_p[0,1]$ where $1 < p \leq 2$. Thus, there is a constant C so that $\|f\|_p \leq C\|f\|_1$ for $f \in X$.

Fix $1 < r < p$. Let $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{r} + \frac{1}{s} = 1$. Let $(h_n)_{n=0}^{\infty}$ denote the Haar basis of L_1 and let S_n be the partial sum operators. Then by Doob's inequality, we have an estimate

$$\|f^*\|_s \leq K\|f\|_s, \quad f \in L_r$$

where $f^* = \sup_n |S_n f|$.

Now apply Theorem 7.2 or Corollary 7.3 there is a (homogeneous) Zippin selector $\Phi : B_{X^*} \to L_s$. Thus, Φ is a weak*-continuous map so that $\Phi(x^*)|_X = x^*$. Let $\lambda = \|\Phi\|$. We now choose $\tau > 0$ so that $C\tau^{1-s/q}(K\lambda)^{s/q} < \frac{1}{4}$.

Let us define a map $\sigma : \lambda B_{L_s} \to L_\infty$ as follows. We let

$$\sigma\left(\sum_{n=0}^{\infty} a_n h_n\right) = \sum_{n=1}^{\infty} \theta_n(a_0, \dots, a_n) a_n h_n,$$

where $0 \leq \theta_n(a_0, \dots, a_n) \leq 1$ are chosen inductively to be maximal subject to the condition

$$\left|\sum_{k=0}^{n} \theta_k(a_0, \dots, a_k) a_k h_k(t)\right| \leq \tau, \quad 0 \leq t \leq 1.$$

It is not difficult to see that each map $f \to \theta_k(f)$ is a continuous function of $a_0, \dots, a_k$, and hence that σ is weak*-continuous.

We then define $\Psi = \sigma \circ \Phi : B_{X^*} \to \tau B_{L_\infty}$. If $x^* \in B_{X^*}$, let $g = \Phi(x^*) \in L_s$ and $g' = \Psi(x^*)$. Then we have

$$\|g - g'\|_q \le \big\|(|g| + \tau)\chi_{(g^* \ge \tau)}\big\|_q \le 2\big\|g^*\chi_{(g^* \ge \tau)}\big\|_q.$$

Now

$$\int_{g^* \ge \tau} (g^*)^q \, dt \le \tau^{q-s} \int_{g^* \ge \tau} (g^*)^s \, dt \le (K\lambda)^s \tau^{q-s}.$$

Thus,

$$\|g - g'\|_q \le 2\tau^{1-s/q}(K\lambda)^{s/q}.$$

Now if $f \in X$ we have,

$$\left| \int f(g - g') \, dt \right| \le 2\tau^{1-s/q}(K\lambda)^{s/q}\|f\|_p \le 2C\tau^{1-s/q}(K\lambda)^{s/q}\|f\|_1.$$

Hence, if $\rho : L_\infty \to X^*$ is the standard restriction map we have

$$\|\rho \circ \Psi(x^*) - x^*\| \le 2C\tau^{1-s/q}(K\lambda)^{s/q} < \frac{1}{2}.$$

Let $\phi(x^*) = x^* - \rho \circ \Psi(x^*)$.

The remainder is standard. We define $\tilde{\Psi} : B_{X^*} \to L_\infty$ by

$$\tilde{\Psi}(x^*) = \sum_{k=0}^{\infty} \Psi \circ \phi^k(x^*)$$

and $\tilde{\Psi}$ is the required Zippin selector. $\square$

Let us note that there are subspaces X of L_1 so that (X, L_1) is known to fail the $\mathcal{C}$-extension property. This follows from the fact there are similar subspaces of ℓ_1 (see [19]). On the other hand, it is unknown whether there is any subspace of L_p where $1 < p < \infty$ so that (X, L_p) fails the $\mathcal{C}$-extension property. This is Problem 6.7 of [46]. Johnson and Zippin [17] showed that there is a subspace X of L_p for which one does not have the isometric (or even the almost isometric) $\mathcal{C}$-extension property.

The argument of Theorem 7.4 implies that if for every $1 < p \le 2$ and every $X \subset L_p$ the pair (X, L_p) has the $\mathcal{C}$-extension property then the same will be true for (X, L_1) for every reflexive subspace X of L_1.

We now give some special results concerning Hilbert spaces. If X is a separable Banach space, we denote by $L_2(X)$ the space $L_2(\Delta, \mathbb{P}; X)$.

Proposition 7.5. *Let X be a separable enlargement of ℓ_2. In order that $(L_2(\ell_2), L_2(X))$ has the $\mathcal{C}$-extension property it is necessary that there exists a constant C so that if $(M_n)_{n=0}^m$ is an X-valued dyadic martingale and $\mathbb{A} \subset \{1, 2, \ldots, m\}$ is a subset with $dM_j \in L_2(\ell_2)$ for $j \in \mathbb{A}$ then*

$$\left(\mathbb{E}\left\| \sum_{j \in \mathbb{A}} dM_j \right\|^2 \right)^{1/2} \le C(\mathbb{E}\|M_m\|^2)^{1/2}.$$

Proof. If $(L_2(\ell_2), L_2(X))$ has the $\mathcal{C}$-extension property there is an equivalent norm $\|\cdot\|_0$ on $L_2(X)$ so that $\|f\|_0 = (\mathbb{E}\|f\|^2)^{1/2}$ for $f \in L_2(\ell_2)$ and

$$\lim_{n\to\infty} \|f + g_n\|_0^2 \geq \|f\|_0^2 + \lim_{n\to\infty} \|g_n\|_0^2$$

whenever $g_n \in L_2(\ell_2)$ is weakly null and the limits exist. We suppose that $C^{-1}\|f\|_0 \leq (\mathbb{E}\|f\|^2)^{1/2} \leq C\|f\|_0$ for $f \in L_2(X)$.

Now let $(M_j)_{j=0}^m$ be a dyadic martingale adapted to Σ_j and suppose $dM_j \in L_2(\ell_2)$ for $j \in \mathbb{A}$. Let $M_j = \varphi_j(t_1, \dots, t_j)$ for $j = 1, 2, \dots$. Let $\mathcal{U}$ be any nonprincipal ultrafilter on $\mathbb{N}$.

Then $\lim_{k_j\to\infty} \varphi_j(t_{k_1}, \dots, t_{k_j}) - \varphi_{j-1}(t_{k_1}, \dots, t_{k_{j-1}}) = 0$ weakly for each fixed $k_1 < k_2 < \cdots < k_{j-1}$. Hence,

$$\lim_{k_j\in\mathcal{U}} \|\varphi_j(t_{k_1}, \dots, t_{k_j})\|_0^2 \geq \|\varphi_{j-1}(t_{k_1}, \dots, t_{k_{j-1}})\|_0^2,$$

while if $j \in \mathbb{A}$,

$$\lim_{k_j\in\mathcal{U}} \|\varphi_j(t_{k_1}, \dots, t_{k_j})\|_0^2 \geq \|\varphi_{j-1}(t_{k_1}, \dots, t_{k_{j-1}})\|_0^2 + \mathbb{E}\|dM_j\|^2.$$

Thus,

$$\lim_{k_m\in\mathcal{U}} \lim_{k_{m-1}\in\mathcal{U}} \cdots \lim_{k_1\in\mathcal{U}} \|\varphi_m(t_{k_1}, \dots, t_{k_m})\|_0^2 \geq \sum_{j\in\mathbb{A}} \mathbb{E}\|dM_j\|^2.$$

This implies that

$$\left(\mathbb{E}\left\|\sum_{j\in\mathbb{A}} dM_j\right\|^2\right)^{1/2} = \left(\sum_{j\in\mathbb{A}} \mathbb{E}\|dM_j\|^2\right)^{1/2} \leq C(\mathbb{E}\|M_m\|^2)^{1/2}. \qquad \square$$

We conclude the paper by considering twisted Hilbert spaces. A Banach space X is a *twisted Hilbert space* if it has a subspace E so that E and X/E are both isomorphic to Hilbert spaces; then X can be renormed so that both spaces are isometrically Hilbertian. Note that a twisted Hilbert space with an unconditional basis is (isomorphically) a Hilbert space [20]. However, there are many nontrivial examples where X has a (UFDD) (even into 2-dimensional spaces, [24]).

THEOREM 7.6. *Let X be a separable twisted Hilbert space and let E be a closed subspace of X so that $E, X/E$ are both Hilbertian. If both pairs $(L_2(E), L_2(X))$ and $(L_2(E^\perp), L_2(X^*))$ have the $\mathcal{C}$-extension property, then X is (UMD).*

Proof. By Proposition 7.5, we may assume that there is a constant C_1 so that if $(M_j)_{j=0}^m$ is an X-valued dyadic martingale adapted to Σ_j and $\mathbb{A}$ is a subset of $\{1, 2, \dots, m\}$ so that $dM_j \in L_2(E)$ for $j \in \mathbb{A}$ then

$$\left(\mathbb{E}\left\|\sum_{j\in\mathbb{A}} dM_j\right\|^2\right)^{1/2} \leq C_1(\mathbb{E}\|M_m\|^2)^{1/2}.$$

Similarly, there is a constant C_2 so that if $(M_j^*)_{j=0}^m$ is an X^*-valued dyadic martingale adapted to Σ_j and $\mathbb{A}$ is a subset of $\{1,2,\dots,m\}$ so that $dM_j \in L_2(E^\perp)$ for $j \in \mathcal{A}$ then

$$\left(\mathbb{E}\left\|\sum_{j\in\mathbb{A}} dM_j^*\right\|^2\right)^{1/2} \leq C_2(\mathbb{E}\|M_m^*\|^2)^{1/2}.$$

Now let $(M_j)_{j=0}^m$ be an X-valued dyadic martingale, adapted to Σ_j, with $M_0 = 0$, and let $\mathbb{A}$ be a subset of $\{1,2,\dots,m\}$. We will show that

$$\left(\mathbb{E}\left\|\sum_{j\in\mathbb{A}} dM_j\right\|^2\right)^{1/2} \leq (C_2 + C_1 + C_1C_2)(\mathbb{E}\|M_m\|^2)^{1/2}.$$

This inequality will imply that X has (UMD).

Let $(M_j^*)_{j=0}^m$ be an X^*-valued dyadic martingale adapted to Σ_j so that $\mathbb{E}\|M_m^*\|^2 \leq 1$ and $dM_j \in L_2(E^\perp)$ for $j \in \mathbb{A}$. Then

$$\begin{aligned}\mathbb{E}\left\langle \sum_{j\in\mathbb{A}} dM_j, M_m^*\right\rangle &= \mathbb{E}\left\langle M_m, \sum_{j\in\mathbb{A}} dM_j^*\right\rangle \\ &\leq C_2(\mathbb{E}\|M_m\|^2)^{1/2}.\end{aligned}$$

From the Hahn–Banach theorem, this implies that

$$\inf\{(\mathbb{E}\|M_m - \tilde{M}_m\|^2)^{1/2}\} \leq C_2(\mathbb{E}\|M_m\|^2)^{1/2},$$

where $(\tilde{M}_j)_{j=0}^m$ runs through all martingales in $L_2(E)$ such that $\tilde{M}_0 = 0$ and $d\tilde{M}_j = 0$ for $j \in \mathbb{A}$.

It thus follows that there exists a martingale $(M'_j)_{j=0}^m$ with $dM'_j = 0$ for $j \notin \mathbb{A}$ and $dM'_j \in L_2(E)$ for $j \in \mathbb{A}$ so that

$$\left(\mathbb{E}\left\|\sum_{j\in\mathbb{A}} (dM_j + dM'_j)\right\|^2\right)^{1/2} \leq C_2(\mathbb{E}\|M_m\|^2)^{1/2}.$$

Thus,

$$\left(\mathbb{E}\left\|\sum_{j\notin\mathbb{A}} dM_j + \sum_{j\in\mathbb{A}} dM'_j\right\|^2\right)^{1/2} \leq (C_2 + 1)(\mathbb{E}\|M_m\|^2)^{1/2}$$

and so

$$\left(\mathbb{E}\left\|\sum_{j\in\mathbb{A}} dM'_j\right\|^2\right)^{1/2} \leq C_1(C_2 + 1)(\mathbb{E}\|M_m\|^2)^{1/2}.$$

Finally, we deduce as promised

$$\left(\mathbb{E}\left\|\sum_{j\in\mathbb{A}} dM_j\right\|^2\right)^{1/2} \leq (C_2 + C_1 + C_1C_2)(\mathbb{E}\|M_m\|^2)^{1/2}. \qquad \square$$

REMARK. Note that $L_2(X)$ is also a twisted Hilbert space since $L_2(X)/L_2(E)$ is isometrically isomorphic to $L_2(X/E)$. There are examples of non-UMD twisted Hilbert spaces (see [18]). Indeed the method of [18] can be adapted to show that there are such spaces with (UFDD). This means that there is also a twisted Hilbert space X with a Hilbertian subspace E so that X/E is also Hilbertian but (E, X) fails the $\mathcal{C}$-extension property.

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