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ON BANACH SPACES CONTAINING l_p OR c_0

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Abstract. We use the Gowers block Ramsey theorem to characterize Banach spaces containing isomorphs of ℓ_p (for some $1 \leq p < \infty$) or c_0.

1. Introduction

A result of Zippin [Z] gives a characterization of the unit vector basis of c_0 and l_p. He showed that a normalized basis of a Banach space such that all normalized block bases are equivalent, must be equivalent to the unit vector basis of c_0 or l_p for some $1 \leq p < \infty$. Let $1 \leq p \leq \infty$ A Banach space X with a basis $(x_i)_i$ is called *asymptotic-l_p* (*asymptotic-c_0* if $p = \infty$) [M-TJ] if there exists $K > 0$ and an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that, for all n, if $(y_i)_{i=1}^n$ is a normalized block basis of $(x_i)_{i=f(n)}^\infty$, then $(y_i)_{i=1}^n$ is equivalent to the unit vector basis of l_p^n. In [F-F-K-R] Figiel, Frankiewicz, Komorowski and Ryll-Nardzewski gave necessary and sufficient conditions for finding asymptotic-l_p subspaces, for a fixed $1 \leq p \leq \infty$, in an arbitrary Banach space. More precisely, they proved

Theorem (FFKR). *Let $p \geq 1$ and let X be a Banach space with the following property:*

For any infinite dimensional subspace $Y \subseteq X$ there exists a constant M_Y such that for any n there exist infinite dimensional subspaces $U_1, U_2, \dots, U_n$ of Y with

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the property that any normalized sequence $(u_1, u_2, \dots, u_n)$ *with* $u_i \in U_i$ *for any* $i \le n$*, is* M_Y*-equivalent to the unit vector basis of* l_p^n*.*

Then X *contains an asymptotic-*l_p *subspace.*

In [Tc] we consider similar decompositions for which any two n-tuples as above are uniformly equivalent to each other (with the equivalence constant independent of n) and obtain the existence of asymptotic-l_p subspaces. Our current results are in the same direction. In Theorem 2.4 we show that if a Banach space has the property that every closed subspace contains two sequences of infinite-dimensional closed subspaces which are comparable (see the formal definition below) then it contains a copy of ℓ_p for some $1 \le p < \infty$ or c_0. This may be regarded as a characterization of spaces containing l_p or c_0. In Theorem 2.6 we show that if a Banach space X is saturated with sequences of infinite dimensional subspaces E_n, $n \in \mathbb{N}$, such that all normalized sequences (x_n) with $x_n \in E_n$ are tail equivalent, then X must contain a subspace isomorphic to ℓ_p or c_0. For the proofs we make essential use of Gowers' block Ramsey theorem [G].

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2. The main results

We first recall the Gowers block Ramsey theorem. Let X be a Banach space with a basis $(e_i)_i$ and let Σ be the subset of all finite normalized block sequences of $(e_i)_i$. Given a set $\sigma \in \Sigma$ we consider the following two players game (the Gowers game). S (the subspace player) chooses a block subspace X_1 of X and V (the vector player) chooses a normalized vector $x_1 \in X_1$. Then S chooses a block subspace X_2 and V chooses $x_2 \in X_2$. The play alternates S,V,S,V.... Player V wins the game if at some point the sequence $(x_1, x_2 \dots, x_n)$ belongs to σ. We say that V has a winning strategy if no matter what sequence of subspaces S chooses, V can win the game.

If Y is a block subspace of X we say that σ is *large* for Y if every block subspace of Y contains an element of σ. We say that σ is *strategically large* for Y if player V has a winning strategy for the above game when S is restricted to subspaces of Y.

Let $\Delta = (\delta_1, \delta_2 ...)$ be a sequence of positive numbers. The set σ_Δ, the Δ-enlargement of σ, will stand for the set of all sequences $(x_1, \dots, x_n) \in \Sigma$ such that there exists a sequence $(y_1, \dots, y_n) \in \sigma$ with $\|x_i - y_i\| < \delta_i$, for all $i \le n$.

Theorem 2.1. (Gowers's block Ramsey Theorem) *Let $\sigma \in \Sigma$ be large for X and let Δ be a sequence of positive numbers. Then there exists a block subspace Y of X such that σ_Δ is strategically large for Y.*

The following lemma is an application of Gowers's block Ramsey Theorem.

Lemma 2.2. *Let X be a Banach space with a basis. Then there exists $1 \leq p \leq \infty$ and an infinite dimensional block subspace Y of X such that for every sequence $(Y_i)_{i=1}^\infty$ of infinite dimensional block subspaces of Y and for all n there exists a normalized block sequence $(y_i)_{i=1}^n$ in Y with $y_i \in Y_i$ for $1 \leq i \leq n$ and $(y_i)_{i=1}^n$ is $2-$equivalent to the unit vector basis of l_p^n.*

For the proof of this lemma we need the definition of the stabilized Krivine set of a Banach space. If X is a Banach space with a basis and W is an infinite dimensional block subspace of X, let $K(W)$ be the set of p's in $[1, \infty]$ such that l_p is finitely block represented in W, ($p = \infty$ if c_0 is finitely block represented in W). Then $K(W)$ is a closed non-empty subset of $[1, \infty]$ [K]. Note that if W_2 is an infinite dimensional block subspace of W_1 then $K(W_2) \subseteq K(W_1)$. Moreover, if W_2 has finite codimension in W_1 then $K(W_2) = K(W_1)$. Using these properties it is easy to show that there exists an infinite dimensional block subspace W of X such that $K(W) = K(V)$ for all infinite dimensional block subspaces V of W, (else a transfinite induction gives a contradiction). The set $K(W)$ is called a *stabilized Krivine set* for X.

Proof. Let W be an infinite dimensional block subspace of X such that $K(W)$ is a stabilized Krivine set for X. Fix $p \in K(W)$ and for any $n \in \mathbb{N}$ let σ_n be the set of all finite normalized block sequences of W of length n such that they are 2-equivalent to the unit vector basis of l_p^n. The conclusion of the Lemma follows easily if we can find a block subspace Y such that, for any n, σ_n is strategically large for Y.

Note that for any n, σ_n is large in any infinite dimensional block subspace V of W. By applying Theorem 2.1 repeatedly, we obtain a nested sequence $V_1 \supset V_2 \supset V_3 \supset \cdots \supset V_n \supset \dots$ of block subspaces of W such that, for any n, σ_n is strategically large for V_n. Note that we do not enlarge σ_n since we can replace the 2 in "2-equivalent" by $1 + \varepsilon$ for any $\varepsilon > 0$.

Let Y be a diagonal block subspace, that is a subspace generated by a block basis $v_1, v_2, \dots$ with $v_n \in V_n$ for every n. We claim that σ_n is strategically large for Y, for any value of n. Indeed fix $n \in \mathbb{N}$ and denote by $[Y]_n$ the n-tail of Y, that is the subspace generated by $(v_j)_{j \geq n}$. Note that $[Y]_n \subseteq V_n$. Consider a typical Gowers game in Y. For any choice of a block subspace Z of Y the subspace player makes, the vector player chooses a vector $z \in Z \cap [Y]_n$ as if the game was played inside V_n and the subspace player picked $Z \cap [Y]_n$. Since the vector player has a winning strategy for the game played in V_n, it follows that after finitely many steps the finite block sequence he chooses belongs to σ_n. Therefore the vector player has a winning strategy for the game played in Y as well. This proves that σ_n is strategically large for Y, which finishes the proof of the lemma. $\square$

Let $\mathcal{E} = (E_j)_{j=1}^{\infty}$ be a sequence of nonzero subspaces of a Banach space X and let $\mathcal{F} = (F_j)_{j=1}^{\infty}$ be a sequence of nonzero subspaces in a Banach space Y. We will say that $\mathcal{E}$ is C-*dominated* by $\mathcal{F}$, and we write $\mathcal{E} \overset{C}{\prec} \mathcal{F}$ if for any n we have

$$\|\sum_{j=1}^{n} x_j\| \leq C \|\sum_{j=1}^{n} y_j\|$$

whenever $x_j \in E_j,\ y_j \in F_j$ with $\|x_j\| = \|y_j\|$ for $1 \leq j \leq n$.

The two sequences $\mathcal{E}$ and $\mathcal{F}$ are called C-*comparable* if $\mathcal{E} \overset{C}{\prec} \mathcal{F}$ or $\mathcal{F} \overset{C}{\prec} \mathcal{E}$. $\mathcal{E}$ and $\mathcal{F}$ are C-equivalent if there exist constants C_1 and C_2 with $C_1 C_2 = C$ such that $\mathcal{E} \overset{C_1}{\prec} \mathcal{F}$ and $\mathcal{F} \overset{C_2}{\prec} \mathcal{E}$.

Notice that the sequence $\mathcal{E}$ is comparable to itself if and only if it is equivalent to itself and this is in turn equivalent to the fact that $\mathcal{E} = (E_j)_{j=1}^{\infty}$ is an absolute Schauder decomposition of its closed linear span $[\mathcal{E}]$.

Note that $\mathcal{E}$ is C-dominated by the canonical one-dimensional decomposition of ℓ_p (or c_0 when $p = \infty$) if and only if $\mathcal{E}$ satisfies an upper ℓ_p-estimate with constant C, i.e.:

$$\|\sum_{j=1}^{n} x_j\| \leq C(\sum_{j=1}^{n} \|x_j\|^p)^{1/p}, \qquad x_j \in E_j,\ n = 1, 2, \ldots$$

or

$$\|\sum_{j=1}^{n} x_j\| \leq C \max_{1 \leq j \leq n} \|x_j\|, \qquad x_j \in E_j,\ n = 1, 2, \ldots$$

when $p = \infty$. Similarly $\mathcal{E}$ C-dominates the canonical one-dimensional decomposition of ℓ_p if and only if $\mathcal{E}$ satisfies a lower ℓ_p-estimate with constant C.

Recall that a basic sequence $(x_n)_{n=1}^{\infty}$ has a block upper (respectively, block lower) p-estimate if there is a constant C so that for all block basic sequences $(u_n)_{n=1}^{\infty}$ the sequence $([u_n])_{n=1}^{\infty}$ has an upper (respectively, lower) ℓ_p-estimate with constant C.

Theorem 2.3. *Let X be a separable Banach space with a basis $(e_j)_{j=1}^{\infty}$ and suppose $1 \le p \le \infty$.*
(i) Assume that every closed subspace of X contains a sequence $\mathcal{E}$ of infinite-dimensional subspaces with an upper ℓ_p (or c_0 when $p = \infty$)-estimate. Then $(e_j)_{j=1}^{\infty}$ has a block basic sequence with an block upper p-estimate.
(ii) Assume that every closed subspace of X contains a sequence $\mathcal{E}$ of infinite-dimensional subspaces with a lower ℓ_p (or c_0 when $p = \infty$)-estimate. Then $(e_j)_{j=1}^{\infty}$ has a block basic sequence with a block lower p-estimate.

Proof. We prove only (i) as (ii) is similar. We may assume that every block subspace contains a sequence $\mathcal{E}$ of block subspaces with an upper ℓ_p-estimate. For each block subspace W let $C(W)$ denote the infimum of all constants C so that W contains a sequence $\mathcal{E}$ of block subspaces with an upper ℓ_p-estimate with constant C. We claim that there exists an infinite dimensional block subspace Y of X and a constant $C < \infty$ such that for each infinite dimensional block subspace Z of Y we have that $C(Z) < C$. Indeed, otherwise there exists a decreasing sequence of block subspaces Z_n of X such that $C(Z_n) > n$. If we choose a sequence $(w_j)_{j=1}^{\infty}$ of successive, linearly independent block vectors with $w_j \in Z_j$ for each j, then the infinite dimensional block subspace W spanned by $(w_j)_{j=1}^{\infty}$ contains a sequence of infinite-dimensional subspaces $\mathcal{E} = (E_j)_{j=1}^{\infty}$ with an ℓ_p-upper estimate with constant C, say. Picking $n > C$ and considering the sequence $(E_j \cap Z_n)_{j=1}^{\infty}$, we have a contradiction. This contradiction proves the above claim.

Thus we may assume that for original basis we have the property that $C(W) < C$ for every block subspace. Now define the set σ to consist of all normalized finite block basic sequences $(u_1, \ldots, u_n)$ so that for some $a_1, \ldots, a_n$ with $|a_1|^p + \cdots + |a_n|^p = 1$

$$\|a_1u_1 + \cdots + a_nu_n\| > C + 2.$$

If the conclusion of the Theorem is false, this set is large. Let $\Delta = (2^{-i})_{i=1}^{\infty}$. then by Theorem 2.1 σ_Δ is strategically large for some block subspace Y. Let $\mathcal{E} = (E_j)_{j=1}^{\infty}$ be a sequence of infinite-dimensional block subspaces of Y with an upper ℓ_p-estimate with constant at most C. If the subspace player S uses the strategy $\mathcal{E}$ then the vector player V may select normalized vectors $v_j \in E_j$ so that for some n there exists $(u_1, \ldots, u_n) \in \sigma$ so that $\|u_j - v_j\| < 2^{-j}$ for $j = 1, 2, \ldots, n$. Now for an appropriate $a_1, \ldots, a_n$ with $|a_1|^p + \cdots + |a_n|^p = 1$ we have

$$\|\sum_{j=1}^{n} a_jv_j\| \ge \|\sum_{j=1}^{n} a_ju_j\| - 1 \ge C + 1$$

which gives a contradiction. This proves (i). □

Theorem 2.4. *Let X be a separable Banach space with the property that every infinite-dimensional closed subspace contains two comparable sequences $\mathcal{E}$ and $\mathcal{F}$ of infinite-dimensional closed subspaces. Then X contains a copy of ℓ_p for some $1 \le p < \infty$ or c_0.*

PROOF. We first assume that X has a basis and then by Lemma 2.2, we may pass to a block subspace Y so that for a suitable p, whenever $\mathcal{E} = (E_j)_{j=1}^\infty$ is a sequence of infinite-dimensional subspaces of Y then for any $n \in \mathbb{N}$ there exist $y_j \in E_j$ for $j = 1, 2, \dots, n$ so that $(y_j)_{j=1}^n$ is 2-equivalent to the canonical ℓ_p^n-basis.

By our assumption, for any closed infinite dimensional subspace Z of Y let $\mathcal{E}$ and $\mathcal{F}$ be two comparable sequences of infinite dimensional subspaces of Z, say $\mathcal{E} = (E_j)$ is dominated by $\mathcal{F} = (F_j)$. By Lemma 2.2 for any $n \in \mathbb{N}$ there exists $y_j \in E_j$ $(j = 1, \dots, n)$, such that $(y_j)_{j=1}^n$ is 2-equivalent to the unit vector basis of ℓ_p^n. Thus $\mathcal{F}$ has a block lower p estimate. Similarly, $\mathcal{E}$ has a block upper p-estimate. Applying Theorem 2.3 (i) and (ii) one after another we can pass to a block basic sequence which has both a block upper and a block lower p-estimate, i.e. is equivalent to the canonical basis of ℓ_p. □

Corollary 2.5. *Let X be a separable Banach space with the property that every infinite-dimensional closed subspace contains a subspace with an absolute Schauder decomposition of infinite-dimensional subspaces. Then X contains a copy of ℓ_p for some $1 \le p < \infty$ or c_0.*

We remark that this Corollary could easily be deduced from the result in [Tc].

A sequence $(x_n)_n$ will be called *C-tail equivalent* if for any $N \in \mathbb{N}$, there exists $k > N$ such that $(x_n)_{n=1}^\infty$ is C-equivalent to $(x_n)_{n=k}^\infty$. In particular, a subsymmetric basic sequence is tail equivalent.

Theorem 2.6. *Let X be a Banach space having the property that for any infinite dimensional closed subspace Y of X there exist a constant C_Y and infinite dimensional subspaces $\mathcal{E} = (E_i)_{i=1}^\infty$ of Y such that all normalized sequences $(x_n)_n$ with $x_n \in E_n$ for all n are C_Y-tail equivalent. Then X contains a basic sequence equivalent to the unit vector basis of l_p or c_0.*

PROOF. By passing to a subspace and relabeling assume that X has a basis and M is its basis constant. According to Lemma 2.2 let Y be a block subspace of X such that for every sequence $(Y_i)_{i=1}^\infty$ of infinite dimensional block subspaces of Y and for all $n \in \mathbb{N}$ there exists a normalized sequence $(y_i)_{i=1}^n$ with $y_i \in Y_i$ for $1 \le i \le n$ and $(y_i)_{i=1}^n$ is 2-equivalent to the unit vector basis of ℓ_p^n.

By our hypothesis there exists a sequence $\mathcal{E} = (E_j)$ of infinite dimensional subspaces of Y such that all normalized sequences (x_n) with $x_n \in E_n$ are C-tail equivalent. Using a standard perturbation arguments we can assume that $\mathcal{E}$ consists of block subspaces. Applying Lemma 2.2 for $(E_i)_{i=1}^{\infty}$ and for $n = 2$ we obtain normalized block vectors $z_1 \in E_1$ and $z_2 \in E_2$ such that $\{z_1, z_2\}$ is 2-equivalent to the unit vector basis of l_p^2. Next we apply again Lemma 2.2 for $(E_i)_{i=3}^{\infty}$ and for $n = 3$ and obtain normalized block vectors $z_3 \in E_3$, $z_4 \in E_4$ and $z_5 \in E_5$ such that $\{z_3, z_4, z_5\}$ is 2-equivalent to the unit vector basis of l_p^3. We continue in this manner; thus we built inductively a normalized block sequence $(z_i)_i$ with $z_i \in E_i$ for all i such that $\{z_1, z_2\}$ is 2-equivalent to the unit vector basis of l_p^2, $\{z_3, z_4, z_5\}$ is 2-equivalent to the unit vector basis of l_p^3, $\{z_6, z_7, z_8, z_9\}$ is 2-equivalent to the unit vector basis of l_p^4 and so on. Clearly $(z_i)_i$ is a block basic sequence with basis constant at most M.

Fix $n \in \mathbb{N}$. From the definition of tail equivalence we can find N large enough such that $(z_i)_{i=1}^{\infty}$ is C-equivalent to $(z_i)_{i=N+1}^{\infty}$ and $\{z_{N+1}, z_{N+2}, \ldots, z_{N+n}\}$ overlaps with at most two finite sequences of z's as above. In other words, there exists $k \leq n$ such that $(z_i)_{y=N+1}^{N+k}$ is 2-equivalent to the unit vector basis of l_p^k and $(z_i)_{y=N+k+1}^{N+n}$ is 2-equivalent to the unit vector basis of l_p^{n-k}. Then it easily follows that $\{z_{N+1}, z_{N+2}, \ldots, z_{N+n}\}$ is $16(M+1)$-equivalent to the unit vector basis of l_p^n. Since $(z_i)_{i=1}^{\infty}$ is C-equivalent to $(z_i)_{i=N+1}^{\infty}$, it follows that $(z_i)_{i=1}^{n}$ is $16C(M+1)$-equivalent to the unit vector basis of l_p^n, and this finishes the proof. □

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