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ON SPACES OF MEASURABLE VECTOR-VALUED FUNCTIONS

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1. Introduction. Let K be a Polish space and let μ be a (finite) regular Borel measure on K. If X is an F-space (complete metrizable topological vector space) then we denote by $L_0(K,\mu;X)$ the space of all Borel functions $f: K \to X$ which are μ-essentially separable. After the standard identification of functions equal almost everywhere, $L_0(K,\mu;X)$ is an F-space under the topology of convergence in μ-measure. If X is F-normed by an F-norm $\|\cdot\|$ then $L_0(K,\mu;X)$ may be F-normed by

$$\|f\| = \int_K \min(1,\|f(s)\|)d\mu(s).$$

If $K = I$, the unit interval, and $\mu = \lambda$, Lebesgue measure on I, then we write $L_0(K,\lambda;X) = L_0(X)$. We write $L_0(K,\mu)$ for $L_0(K,\mu;\mathbf{R})$ and $L_0 = L_0(\mathbf{R})$.

The purpose of this paper is to prove the following theorem:

THEOREM 1.1. *If* X *is an* F-*space such that* $L_0(X)$ *is isomorphic to* L_0, *then* X *is isomorphic to* $L_0(K,\mu)$ *for some Polish space* K *and probability measure* μ.

The isomorphism class of the space $L_0(K,\mu)$ depends only on the nature of the atoms of μ. Hence Theorem 1.1 is equivalent to

THEOREM 1.2. *Let* X *be an* F-*space. Then* $L_0(X)$ *is isomorphic to* L_0 *if and only if* X *is isomorphic to one of the spaces;* $\mathbf{R}^n$ $(n \geq 1)$, ω, L_0, $L_0 \oplus \mathbf{R}^n$ $(n \geq 1)$, $L_0 \oplus \omega$.

Here ω is the space of all real sequences with the usual product topology.

By way of analogy we point out that for a Banach space the isomorphism of $L_1(X)$ and L_1 is equivalent to the fact that X is isomorphic to a complemented subspace of $L_1[0,1]$. It is an unresolved question in Banach space theory whether this implies that $X \cong L_1(K,\lambda;\mathbf{R})$ for some K and λ. However if X has the Radon-Nikodym Property, the Lewis-Stegall theorem implies that $X \cong \ell_1$ (see [5], [6]). Analogous

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results for $L_p(X)$, $0 < p < 1$, were investigated in [2]. The approach here is similar although the details are necessarily rather different and the final result is more complete.

2. **Preliminaries from measure theory.** If μ is a measure then we use "mod μ" to denote statements true up to sets of μ-measure zero.

If K is a Polish space, we denote by $\mathcal{B}(K)$ the collection of Borel subsets of K. Let μ be a measure on K and let $\phi: \mathcal{B}(I) \to \mathcal{B}(K)$ be a map. We shall say ϕ is a *sub-μ-homomorphism* if

(1) $\phi(A_1 \cup A_2) = \phi(A_1) \cup \phi(A_2) \pmod{\mu}$,

for $A_1, A_2 \in \mathcal{B}(K)$.

(2) Given $\epsilon > 0$ there exists $\delta > 0$ so that $\lambda(A) < \delta$ implies $\mu(\phi(A)) < \epsilon$.

We shall say that ϕ is a *μ-homomorphism* if it satisfies (1) and (2) and

(3) $\phi(A_1 \cap A_2) = \emptyset \pmod{\mu}$

for $A_1, A_2 \in \mathcal{B}(I)$ with $A_1 \cap A_2 = \emptyset$.

A standard result from measure theory, essentially given in Royden [7] Theorem 15.10 is

PROPOSITION 2.1. *If $\phi: \mathcal{B}(I) \to \mathcal{B}(K)$ is a μ-homomorphism there is a Borel map $\sigma: K \to I$ so that*

$$\phi(A) \subset \sigma^{-1}(A) \pmod{\mu} \quad A \in \mathcal{B}(I).$$

(If $\phi(I) = K$ we obtain $\phi(A) = \sigma^{-1}(A) \pmod{\mu}$.)

We denote by $D(n,k)$ ($1 \leqslant h < \infty$, $1 \leqslant k \leqslant 2^n$) the standard dyadic partition of I, i.e.,

$$D(n,k) = [(k-1)2^{-n}, k\cdot 2^{-n}) \quad 1 \leqslant k \leqslant 2^n - 1$$

$$D(n,2^n) = [1-2^{-n}, 1].$$

The next lemma is essentially due to Kwapien [4] and is also essentially used in [3].

LEMMA 2.2. *Let $\phi: \mathcal{B}(I) \to \mathcal{B}(K)$ be a sub-μ-homomorphism. There is a countable Borel partitioning $(B_n: n \in \mathbb{N})$ of K so that for each k, $1 \leqslant k \leqslant 2^n$ the maps $\phi_{n,k}: \mathcal{B}(I) \to \mathcal{B}(K)$ are μ-homomorphisms where*

$$\phi_{n,k}(A) = \phi(A \cap D(n,k)) \cap B_n.$$

PROOF. Let $g_n \in L_0(K,\mu)$ be defined by

$$g_n = \Sigma_{k=1}^{2^n} 1_{\phi(D(n,k))}.$$

Then $\{g_n\}$ is monotone increasing. If for $\epsilon > 0$ we choose $\delta > 0$ as in condition (2) above, then a calculation as in [3] (Justification of A3)) shows that if $2^n > \delta^{-1}$,

$$\int_0^1 (1 - \tfrac{1}{2}\delta)^{g_n(t)} \, d\mu(t) \geq 1 - \epsilon.$$

Hence $\lim_{n\to\infty} g_n(t) = g(t) < \infty$ almost everywhere. Let

$$B_1 = \{t: g_1(t) = g(t)\} \cup \{t: g(t) = \infty\}$$

and

$$B_n = \{t: g_{n-1}(t) < g_n(t) = g(t)\} \text{ for } n \geq 2.$$

Then if D(m,i) and D(m,j) are contained in D(n,k) and disjoint

$$1_{\phi(D(m,i))}(s) + 1_{\phi(D(m,j))}(s) \leq 1$$

whenever $g_n(s) = g(s)$. Hence

$$\phi_{n,k}(D(m,i)) \cap \phi_{n,k}(D(m,j)) = \emptyset \mod \mu$$

and the conclusion of the lemma follows by a continuity argument using condition (2).

We shall say that a map σ: K → I is *anti-injective* if whenever $B \in \mathcal{B}(K)$ and $\sigma|B$ is injective then $\mu(B) = 0$. The following lemma is proved in [2].

LEMMA 2.3. *If* σ: K → I *is anti-injective there is a compact metric space* M, *a diffuse measure* π *on* M *and a Borel map* τ: K → M *so that*

(i) *There is a Borel map* ρ: I × M → K *so that* $\rho(\sigma s, \tau s) = s \pmod \mu$.

(ii) *If* $B \subset I$ *and* $C \subset M$ *are Borel sets then*

$$\mu(\sigma^{-1}B)\pi(C) = \mu(\sigma^{-1}B \cap \tau^{-1}C).$$

3. **Operators on $L_0(X)$.** Let T: $L_0(X) \to L_0(K,\mu;Y)$ be a linear operator. We associate to T a map $\phi = \phi_T$: $\mathcal{B}(I) \to \mathcal{B}(K)$. We define $\phi(B)$ to be a Borel set of minimal μ-measure so that supp f ⊂ B implies supp Tf ⊂ $\phi(B)$ (mod μ) [supp f = (s: f(s) ≠ 0)].

The following lemma is due to Kwapien [4].

LEMMA 3.1. *If* $f_1,\ldots,f_n \in L_0(K,\mu)$, *then there exist* $\alpha_1,\ldots,\alpha_n \in \mathbf{R}$ *such that* $\mathrm{supp}(\alpha_1 f_1 + \cdots + \alpha_n f_n) = \cup_{i=1}^n \mathrm{supp}\, f_i \pmod \mu$.

LEMMA 3.2. *If* T: $L_0(X) \to L_0(K,\mu)$ *is a linear operator then* ϕ_T *is a* *sub-μ-homomorphism.*

PROOF. For $\epsilon > 0$ there exists $\delta > 0$ so that $\|f\| \leqslant \delta$ implies $\|Tf\| \leqslant \epsilon$. If $\lambda(B) < \delta$ and supp $f \subset B$ then $\|T(mf)\| \leqslant \epsilon$ for every $m \in \mathbf{N}$ and hence $\mu(\text{supp } f) \leqslant \epsilon$. Now by Lemma 3.1, $\mu(\phi(B)) \leqslant \epsilon$. It follows easily that ϕ_T is a sub-μ-homomorphism.

We shall say that $T: L_0(X) \to L_0(K,\mu;Y)$ is *elementary* if supp $f \cap$ supp $g = 0$ implies supp $Tf \cap$ supp $Tg = \phi$ (mod μ). In this case ϕ is a μ-homomorphism and hence there is a Borel map $\sigma: K \to I$ is that supp $Tf \subset \sigma^{-1}(\text{supp } f)$ for every $f \in L_0(X)$.

LEMMA 3.3. *Let* $T: L_0(X) \to L_0$ *be a linear operator. Then there is a Polish space* K, *a probability measure* μ *on* K *and operators* $T_1: L_0(X) \to L_0(K,\mu)$, $V: L_0(K,\mu) \to L_0$ *so that* $VT_1 = T$ *and* T_1 *is elementary.*

PROOF. Define K to be space of triples (s,n,k) where $s \in I$, $n \in \mathbf{N}$ and $1 \leqslant k \leqslant 2^n$. Let μ be the measure on K defined by

$$\mu(B) = \Sigma_{n=1}^{\infty} \Sigma_{k=1}^{2^n} \frac{1}{4^n} \lambda\{s: (s,n,k) \in B\}$$

for $B \in \mathcal{B}(K)$.

By Lemma 2.2 we can partition I into Borel sets (B_n) so that each $\phi_{n,k}$ is a μ-homomorphism, i.e., $P_{B_n} T P_{D(n,k)}$ is elementary for each $n \in \mathbf{N}$ and $1 \leqslant k \leqslant 2^n$. Here P_A is the natural projection on $L_0(M,\upsilon;X)$ defined by $P_A f = 1_A \cdot f$ for $A \in \mathcal{B}(M)$.

Define T_1 by

$$T_1 f(s,n,k) = P_{B_n} T P_{D(n,k)} f(s)$$

and $V: L_0(K,\mu) \to L_0$ by

$$Vf(s) = \Sigma_{k=1}^{2^n} f(s,n,k) \quad s \in B_n.$$

The lemma then follows.

4. Diagonal operators. An operator $T: L_0(X) \to L_0(Y)$ is called *diagonal* if supp $Tf \subset$ supp f (mod λ).

THEOREM 4.1. *If* X *is separable and* $T: L_0(X) \to L_0(Y)$ *is diagonal then there is a family of continuous linear operators* $T_s: X \to Y$ $(s \in I)$ *so that*

$$Tf(s) = T_s(f(s)) \quad \text{a.e.}$$

PROOF. For each $\delta > 0$ let

$$\rho(\delta) = \sup(\|Tf\|: \|f\| \leqslant \delta).$$

Select an increasing sequence F_n of finite-dimensional subspaces of X with $\cup F_n = X_0$ dense in X. Denote by $1 \otimes x$ the constant function $1 \otimes x(s) = x$ for all $s \in I$. Then it is possible to define linear maps $T_s\colon X_0 \to Y$ so that

$$T(1 \otimes x)(s) = T_s(x) \quad \lambda\text{-a.e.}$$

for $x \in X_0$.

For each $\delta > 0$ choose $\{x_n\colon n \in \mathbf{N}\}$ a sequence in X_0 with $\|x_n\| \leqslant \delta$ so that the set $\{x_n\colon x_n \in F_k\}$ is dense in $F_k \cap \{x\colon \|x\| \leqslant \delta\}$. For each $n \in \mathbf{N}$ there is $f \in L_0(X)$ with $f(s) \in \{x_1,\ldots,x_n\}$ for all s so that

$$\|T_s(f(s))\| = \max_{1 \leqslant j \leqslant n} \|T_s x_j\|.$$

By the fact that T is diagonal we have

$$Tf(s) = T_s(f(s)) \quad \text{a.e.}$$

Hence, since $\|f\| \leqslant \delta$,

$$\int_0^1 \min(1, \max_{1 \leqslant j \leqslant n} \|T_s x_j\|) d\lambda(s) \leqslant \rho(\delta).$$

Letting $n \to \infty$

$$\int_0^1 \min(1, \sup_j \|T_s x_j\|) d\lambda(s) \leqslant \rho(\delta).$$

Since each T_s is continuous on each F_k we have

$$\sup_j \|T_s x_j\| = \sup_{\substack{\|x\| \leqslant \delta \\ x \in X_0}} \|T_s x\| = \rho_s(\delta)$$

say. Thus

$$\int_0^1 \min(1, \rho_s(\delta)) d\lambda(s) \leqslant \rho(\delta).$$

Hence $\lim_{\delta \to 0} \rho_s(\delta) = 0$, a.e. and T_s is continuous on X_0 for almost every $s \in I$. We may redefine T_s on a set of measure zero so that $(T_s\colon s \in I)$ is continuous for all $s \in I$. For simple X_0-valued functions

$$Tf(s) = T_s(f(s)) \quad \lambda\text{-a.e.}$$

If $f \in L_0(X)$ there is a sequence f_n of X_0-valued simple functions so that $f_n \to f$, a.e. Thus (if we extend each T_s to X continuously)

$$Tf(s) = T_s(f(s)) \quad \lambda\text{-a.e.}$$

5. **Proof of Theorem 1.2.** Let $T: L_0(X) \to L_0$ be an isomorphism. By Lemma 3.3 we can find a Polish space K with a probability measure μ and operators T_1: $L_0(X) \to L(K,\mu)$, $V: L_0(K,\mu) \to L_0$ so that T_1 is elementary and $T = VT_1$. Let $\sigma: K \to I$ be a Borel map so that supp $Tf \subset \sigma^{-1}(\text{supp } f)$ for $f \in L_0(X)$.

Since $L_0(X) \cong L_0$, X is isomorphic to a subspace of L_0. Let $W: X \to L_0$ be an isomorphic embedding and let $\hat{W}: L_0(X) \to L_0(I \times I, \lambda \times \lambda)$ be the induced embedding, i.e.

$$(\hat{W}f)(s,t) = (Wf(s))(t), \quad s,t \in I.$$

We consider the map $Q = \hat{W}T^{-1}V$. By Kwapien's representation theorem [1], [4] we can write

$$Qf(s,t) = \Sigma_{n=1}^{\infty} a_n(s,t)f(\tau_n(s,t)) \quad (\lambda \times \lambda)\text{-a.e.}$$

where

(i) Each $a_n: I \times I \to \mathbf{R}$ is a Borel function and the set $\{(s,t)a_n(s,t) \neq 0$ infinitely often$\}$ has measure zero.

(ii) Each $\tau_n: I \times I \to K$ is a Borel map and if $B \subset K$ has $\mu(B) = 0$ then $\mu(\tau_n^{-1} B \cap \text{supp } a_n) = 0$.

Now consider $Q_n: L_0(K,\mu) \to L_0(I \times I, \lambda \times \lambda)$ defined by

$$Q_n = \hat{W} \Sigma_{k=1}^{2^n} P_{D(n,k)} T^{-1} V P_{\sigma^{-1} D(n,k)}.$$

Then we have $P_{\sigma^{-1} D(n,k)} T_1 = T_1 P_{D(n,k)}$ since T_1 is elementary and hence

$$Q_n T_1 = \hat{W}, \quad (n \in \mathbf{N}).$$

Now if $s \in D(n,k)$

$$Q_n f(s,t) = \Sigma_{\sigma\tau_j(s,t) \in D(n,k)} a_j(s,t) f(\tau_j(s,t)) \quad \text{a.e.}$$

and hence as $n \to \infty$

$$\lim_{n\to\infty} Q_n f(s,t) = \Sigma_{\sigma\tau_j(s,t)=s} a_j(s,t) f(\tau_j(s,t)) \quad \text{a.e.}$$

In particular $\lim_{n\to\infty} Q_n f$ exists in $L_0(I \times I)$ and so there is an operator $U: L_0(K,\mu) \to L_0(X)$ defined by

$$Uf = \lim_{n\to\infty} \hat{W}^{-1} Q_n f \quad f \in L_0(K,\mu).$$

U is continuous by the Banach-Steinhaus Theorem. Furthermore if supp $f \subset \sigma^{-1}(B)$ for

$B \in \mathcal{B}(I)$ then supp $Uf \subset B$; this is easily established for sets $D(n,k)$ and follows for all B by continuity. We also have $UT_1 f = f$ for $f \in L_0(X)$.

Now by an exhaustion argument we can find disjoint compact subsets (E_n; $n \in N$) of K so that $\sigma|E_n$ is continuous and one-one for each n and $\sigma|G$ is anti-injective where $G = K \backslash \cup E_n$.

For each $n \in N$ let υ_n be the induced measure on I from σ: $E_n \to I$, i.e.,

$$\upsilon_n(B) = \mu(\sigma^{-1}B \cap E_n).$$

Let $d\upsilon_n/d\lambda = \phi_n \in L_0$ and define S_n: $L_0(K,\mu) \to L_0$ by

$$\begin{aligned} S_n f(t) &= \phi_n(t) f(\sigma^{-1}t) \quad && t \in \sigma(E_n) \\ &= 0 && t \notin \sigma(E_n 0). \end{aligned}$$

Clearly S_n is continuous for each n.

For G we go back to Lemma 2.3 and find a compact metric space M with a diffuse measure π and a Borel map τ: $G \to M$ so that conditions 2.3(i) and (ii) hold. For convenience of exposition we allow the case $\mu(G) = 0$ and $\pi = 0$.

Let υ_0 be the measure on I defined by

$$\upsilon_0(B) = \mu(\sigma^{-1}B \cap G),$$

and let $\phi_0 = d\upsilon_0/d\lambda$.

Define S_0: $L_0(K,\mu) \to L_0(I \times M, \lambda \times \pi)$ by

$$\begin{aligned} S_0 f(s,t) &= \phi_0(s) f(\rho(s,t)) \quad && s \in \text{supp } \phi_0 \\ &= 0 && s \notin \text{supp } \phi_0. \end{aligned}$$

To show S_0 is continuous we suppose $B \subset K$ is a Borel set with $\mu(B) = 0$ and show $(\lambda \times \pi)(\rho^{-1}B \cap \text{supp } \phi_0) = 0$. In fact

$$\begin{aligned} \int \int_{\rho^{-1}B} \phi_0(s) d\lambda(s) d\pi(t) &\leqslant \int \int_{\rho^{-1}B} d\upsilon_0(s) d\pi(t) \\ &= \mu((\sigma \times \tau)^{-1} \rho^{-1} B) \\ &\leqslant \mu(B) \\ &= 0. \end{aligned}$$

Now identify $L_0(I \times M, \lambda \times \pi)$ with $L_0(I,\lambda,L_0(M,\pi))$; we define S: $L_0(K,\mu) \to L_0(I,\lambda,L_0(M,\pi) \oplus \omega)$ by

$$Sf = (S_0 f, (S_n f)_{n=1}^{\infty}).$$

For each $n \geqslant 0$ we choose a Borel subset A_n of I with $A_n \subset \operatorname{supp} \phi_n$ so that

$$\int_{A_n} \phi_n d\lambda = \int_I \phi_n d\lambda = \upsilon_n(A_n)$$

(thus the singular part of υ_n vanishes on A_n).

Let $C = \cup_{n=0}^{\infty} \sigma^{-1}(A_n)$. As the measure $B \mapsto \mu(\sigma^{-1}B \cap (K\backslash C))$ is λ-singular T_1 maps $L_0(X)$ into $L_0(C,\mu)$. The map S is injective on $L_0(C,\mu)$, since if $Sh = 0$ then $\upsilon_n(A_n \cap \operatorname{supp} Sh) = 0$ for all $n \geqslant 0$ and this implies $\mu(\operatorname{supp} h) = 0$. In fact S maps $L_0(C,\mu)$ isomorphically onto the set of $g = (g_0,(g_n)) \in L_0(L_0 \oplus \omega)$ where $\operatorname{supp} g_n \subset A_n \pmod{\lambda}$ $(0 \leqslant n < \infty)$. Indeed $g = Sh$ where

$$h(s) = \phi_n(\sigma s)^{-1} g_n(\sigma s) \quad s \in E_n \cap C$$

$$= \phi_0(\sigma s)^{-1} g_0(\sigma s, \tau s) \quad s \in G \cap C.$$

We define a projection P onto $\boldsymbol{R}(S)$ by

$$Pg = (P_{A_0} g_0, P_{A_n} g_n).$$

Consider the composition $US^{-1}P$: $L_0(L_0 \oplus \omega) \to L_0(X)$. Here S^{-1} is the inverse of S: $L_0(C,\mu) \to \boldsymbol{R}(S)$. Then $US^{-1}PST_1 f = UT_1 f = f$ for $f \in L_0(X)$ since $\boldsymbol{R}(T_1) \subset L_0(C,\mu)$.

Furthermore both $US^{-1}P$ and ST_1 are diagonal. For $US^{-1}P$ note that $\operatorname{supp} Pf \subset \operatorname{supp} f$ and $\operatorname{supp} S^{-1}Pf \subset \sigma^{-1}(\operatorname{supp} f)$. The construction of U then yields the conclusion.

In order to show that ST_1 is diagonal it suffices to show that each $S_n T_1$ is diagonal. We shall demonstrate the case $n = 0$. Let $f \in L_0(X)$ and let $\operatorname{supp} f = A$, $\operatorname{supp} T_1 f = B$ and $\operatorname{supp} ST_1 f = A_1$. Then $A_1 \subset \operatorname{supp} \phi_0 \cap \{s: \pi(t; \rho(s,t) \in B) > 0\}$. Thus to show $\lambda(A_1 \backslash A) = 0$ we calculate:

$$\int_{I\backslash A} \phi_0(s) \pi(t: \rho(s,t) \in B) d\lambda(s) \leqslant (\upsilon_0 \times \pi)(\rho^{-1}B \cap ((I\backslash A) \times M))$$

$$= \mu((B \cap \sigma^{-1}(I\backslash A)) \cap G)$$

$$= 0.$$

Here we use the fact that $(\upsilon_0 \times \pi)(D) = \mu((\sigma \times \tau)^{-1} D)$ which follows from Lemma 2.3, and the definition of υ_0. Let

$$US^{-1}Pf(s) = J_s(f(s)) \quad \text{a.e.}$$

$$ST_1 f(s) = H_s(f(s)) \quad \text{a.e.},$$

where J_s: $L_0(M,\pi) \oplus \omega \to X$ and H_s: $X \to L_0(M,\pi) \oplus \omega$ are continuous linear operators. Then for all $f \in L_0(X)$

$$J_s H_s f(s) = f(s) \quad \text{a.e.}$$

Since X is separable there is an $s \in I$ so that

$$J_s H_s x = x \quad x \in X.$$

Thus X is isomorphic to a complemented subspace Y of $L_0 \oplus \omega$.

Let P be a projection on Y. Then $P(g,h) = (P_{11}g + P_{12}h, P_{22}h)$ where P_{11}: $L_0 \to L_0$, P_{12}: $\omega \to L_0$ and P_{22}: $\omega \to \omega$ are continuous operators; we use the fact that any operator from L_0 to ω is zero.

Here P_{11} and P_{22} are projections and the map $(g,h) \to (P_{11}g, P_{22}h)$ maps Y isomorphically onto $P_{11}(L_0) \oplus P_{22}(\omega)$. The inverse map sends (g,h) to $(g + P_{12}h, h)$, and $P_{11}P_{12}h = 0$ since $(P_{12}h,h) \in Y$. Now $P_{11}(L_0) \cong L_0$ or $\{0\}$ [1] and $P_{22}(\omega) \cong \omega$ or is finite-dimensional.

We have thus proved Theorem 1.2.

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