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NORM–DECREASING HOMOMORPHISMS BETWEEN IDEALS OF C(G)

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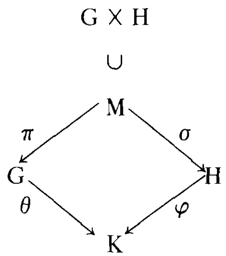
0. Introduction. The aim of this paper is to provide a complete classification of all norm-decreasing algebra homomorphisms between ideals of the group algebras of continuous functions on compact groups.

If G and K are compact groups and θ: G → K is a epimorphism, then θ induces two natural algebra homomorphisms: a monomorphism of C(K) into C(G) and an epimorphism of C(G) onto C(K). Furthermore any character on a compact group induces an automorphism of the corresponding group algebra (see Section 2 for details – by "character" we mean a continuous homomorphism into the circle group). We show that any norm-decreasing homomorphism from C(G) into C(H) (where G and H are compact groups) may be factored into the product of three homomorphisms of these types (see Section 5). Such homomorphisms we call sub-canonical. Amongst sub-canonical homomorphisms we distinguish the canonical homomorphisms (introduced for measure algebras by Kerlin and Pepe [5]); these may be described as those homomorphisms T for which there exists a character χ on G for which $T\chi \neq 0$.

Consider the following diagram

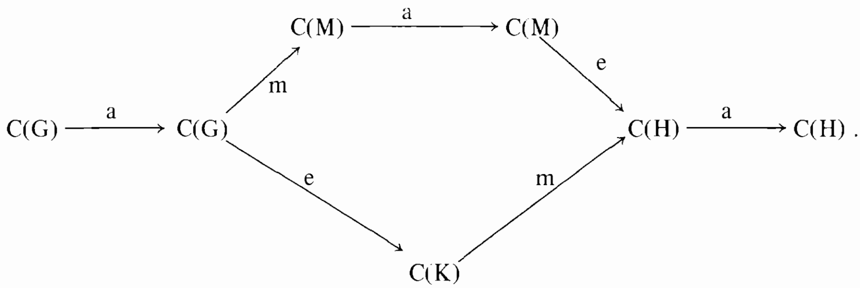


where K is a common quotient, M is a common prequotient of G and H, and they are connected by

$$M = \{(x,y) \in G \times H: \theta x = \varphi y\}$$

$$K = M/(\ker \pi)(\ker \sigma).$$

The corresponding diagram for the algebras is



(Here, m, e and a stand for monomorphism, epimorphism and automorphism respectively.) The canonical homomorphisms are those that factor through C(K), while the sub-canonical ones are those that factor through C(M). A canonical map – along the lower route – can be rerouted along the upper route as a sub-canonical map. However the converse is not true when the automorphism of C(M) onto C(M) is given by a character on M that does not have an extension to G $\times$ H (see Section 2). We give an example of this in Section 3, thus answering a question of Kerlin and Pepe.

If we consider homomorphisms defined only on a two-sided ideal I of C(G), then it is not true that every homomorphism is the restriction of a sub-canonical homomorphism defined on C(G). However, we are still able to give a complete classification of such homomorphisms by adding a further type of homomorphism, namely the restriction operator R_E: C(G) $\rightarrow$ C(E) where E is a closed subgroup of G. It is shown that on certain ideals R_E is a homomorphism.

The corresponding problems for measure algebras and L^1-spaces for locally compact groups appears in [2] and [5] and for ideals in $L^p(G)$ in [1] and [4]. This paper completes the work started in [9] and [10].

1. **Ideals in C(G).** Let G be a compact group. Suppose H is a closed subgroup of G and χ is a character on H. The set of f $\in$ C(G) satisfying

$$f(ux) = \chi(u)f(x) \qquad u \in H \qquad x \in G$$

is a closed right-ideal of C(G). We denote this ideal by J(H,χ).

There is a linear projection P = P(H,χ) with $\|P\| = 1$ of C(G) onto J(H,χ) given by

$$Pf(x) = \int_H \chi(u)f(u^{-1}x)dm_H(u)$$

where m_H denotes normalized Haar measure on H.

LEMMA 1.1. *If H_1 and H_2 are closed subgroups of G and $\chi_1 \in \hat{H}_1$, $\chi_2 \in \hat{H}_2$, then $J(H_1,\chi_1) = J(H_2,\chi_2)$ if and only if $H_1 = H_2$ and $\chi_1 = \chi_2$.*

PROOF. Suppose $J(H_1,\chi_1) = J(H_2,\chi_2)$ and that H_1 is not contained in H_2. Select $w \in H_1 \backslash H_2$. Define a continuous function φ on $H_2 \cup H_2 w$ so that $\varphi(u) = \chi_2(u)$ $(u \in H_2)$ and $\varphi(uw) = \alpha\chi_2(u)$ where $\alpha \neq \chi_1(w)$. Then φ may be extended by Tietze's theorem to a function $\psi \in C(G)$ and then $f = P(H_2,\chi_2)\psi \in J(H_2,\chi_2)$. Clearly $f(x) = \varphi(x)$ $(x \in H_2 \cup H_2 w)$. However $f(w) \neq \chi_1(w)f(1)$. This contradiction shows that $H_1 \subset H_2$. Similarly $H_2 \subset H_1$ and then it is trivial that $\chi_1 = \chi_2$.

LEMMA 1.2. *If $w \in G$, then*

$$J(w^{-1}Hw,\chi_w) = \{ {}_w f \colon f \in J(H_1,\chi)\}$$

where $\chi_w \in \widehat{w^{-1}Hw}$ and

$$\chi_w(w^{-1}uw) = \chi(u) \qquad u \in H.$$

PROOF. Easy.

PROPOSITION 1.3. *$J(H,\chi)$ is a two-sided ideal if and only if*

(1) *H is normal*

and

(2) $\chi(x^{-1}ux) = \chi(u) \quad u \in H \quad x \in G.$

In these circumstances $P(H,\chi)$ is an algebra homomorphism.

PROOF. $J(H,\chi)$ is an ideal if and only if it is invariant under left translations. This is equivalent by Lemma 1.2 to

$$J(w^{-1}Hw,\chi_w 0) = J(H,\chi) \qquad w \in G$$

and hence by Lemma 1.1, conditions (1) and (2) follow.

Now P is a linear projection and $P(C(G))$ is an ideal. To show P is an algebra homomorphism it is enough to show that $P^{-1}(0)$ is also an ideal. Clearly

$$Pf_w(x) = Pf(xw)$$

so that $P^{-1}(0)$ is a right-ideal.

$$\begin{aligned} P({}_w f)(x) &= \int_H \chi(u) f(wu^{-1}x)\,dm_H(u) \\ &= \int_H \chi(u) f(wu^{-1}w^{-1}wx)\,dm_H(u) \\ &= \int_H \chi(w^{-1}uw) f(u^{-1}wx)\,dm_H(u) \end{aligned}$$

(since H is normal),

$$= (Pf)(wx) \quad (\text{by } (2)).$$

Hence P is an algebra homomorphism.

We shall call an ideal of the form $J(H,\chi)$ where H,χ satisfy the conditions of Proposition 1.3, a *normal ideal.*

LEMMA 1.4. *The non-trivial intersection of two normal ideals is a normal ideal. In fact*

$$\begin{aligned} J(H,\chi) \cap J(K,\sigma) &= 0 \text{ if } \chi \neq \sigma \text{ on } H \cap K \\ &= J(HK,\tau) \text{ otherwise} \end{aligned}$$

where τ is the unique common extension of χ and σ to HK.

PROOF. Suppose $f \in J(H,\chi) \cap J(K,\sigma)$. Then

$$f(ux) = \chi(u)f(x) \quad u \in H, \quad x \in G$$

and

$$f(vx) = \sigma(v)f(x) \quad v \in K, \quad x \in G.$$

Thus if $\chi(u) \neq \sigma(u)$ for $u \in H \cap K$, $f \equiv 0$.

Otherwise, define $\tau(uv) = \chi(u)\sigma(v)$.

Then $f(wx) = \tau(w)f(x) \quad w \in HK, \quad x \in G$.

Hence $f \in J(HK,\tau)$. Conversely, if $f(wx) = \tau(w)f(x) \quad w \in HK \quad x \in G$ then in particular $f(ux) = \tau(u)f(x) = \chi(u)f(x)$ for $u \in H$, $x \in G$ i.e. $f \in J(H,\chi)$. Similarly $f \in J(K,\sigma)$ and the result is proved.

If $L \neq 0$ is a linear subspace of $C(G)$ we define $L^X = J(H,\chi)$ where H is the group of all $u \in G$ such that for some constant $\chi(u)$ we have

$$f(ux) = \chi(u)f(x) \quad f \in L.$$

It is clear that χ is then a continuous character on H, and that L^X is the intersection of all sets $J(M,\rho)$ which contain L.

PROPOSITION 1.5. *If L is a left-ideal of C(G) then L^X is a normal ideal of C(G).*

PROOF. L^X is clearly a right-ideal. If $w \in G$, then $\{{}_{w^{-1}}f\colon f \in L^X\} \supset L$ and hence by Lemma 1.2 $\{{}_{w^{-1}}f\colon f \in L^X\} \supset L^X$. Hence if $f \in L^X$, ${}_wf \in L^X$ so that L^X is an ideal.

If I is an ideal, then Proposition 1.5 applies and we shall call I^X the *normal hull* of I. Our next result is the basis of our later results.

THEOREM 1.6. *Suppose I is an ideal of C(G) and* $\mu \in M(G)$ *is such that* $\|\mu\| = 1$ *and*

$$\int_G f(x)d\mu(x) = f(1) \qquad f \in I.$$

Then

$$\int_G f(x)d\mu(x) = f(1) \qquad f \in I^X.$$

PROOF. Let Σ be the set of $\lambda \in M(G)$ such that $\|\lambda\| = 1$ and

$$\int_G f(x)d\lambda(x) = f(1) \qquad f \in I.$$

Then Σ is a weak*-closed convex subset of the unit ball U of M(G). We show that Σ is extremal i.e. if $\lambda_1, \lambda_2 \in$ U and $\frac{1}{2}(\lambda_1 + \lambda_2) \in \Sigma$ then $\lambda_1, \lambda_2 \in \Sigma$. If e is a minimal self-adjoint idempotent in I,

$$\|e\| = e(1)$$

and hence

$$\int e\, d\lambda_1 = \int e\, d\lambda_2 = e(1).$$

Since I is the closed linear span of its minimal self-adjoint idempotents we deduce $\lambda_1, \lambda_2 \in \Sigma$.

Hence the extreme points of Σ are extreme in U, i.e. are of the form $\alpha\delta_u$, $u \in G$. If $\alpha\delta_u \in \partial_e\Sigma$ (the set of extreme points of Σ) then

$$\alpha f(u) = f(1) \qquad f \in I.$$

Hence if $x \in G$

$$\alpha f_x(u) = f_x(1)$$

i.e.

$$\alpha f(ux) = f(x) \qquad f \in I.$$

Hence

$$\partial_e\Sigma = \{\chi(u^{-1})\delta_u : u \in H\}$$

where $I^X = J(H,\chi)$. Theorem 1.6 now follows from the Krein-Milman theorem.

The following corollary is an analogue of Theorem 2 in [7], since H^1 is not a normal ideal of L^1.

COROLLARY 1.7. *Suppose I is a closed ideal in C(G) and there is a projection P of C(G) onto I with* $\|P\| = 1$*; then I is normal.*

REMARK. The converse is immediate from Proposition 1.3.

PROOF. By a standard device we replace P with a projection Q which commutes with left-translations; define

$$Qf = \int_G {}_{x^{-1}}[P({}_x f)]\, dm_G(x).$$

(cf. [8], page 127); then $\|Q\| = 1$ and Q is a projection onto I.

Then by Theorem 1.6

$$Qf(1) = f(1) \qquad f \in I^X$$

and hence

$$\begin{aligned} Qf(x) &= {}_x(Qf)(1) \\ &= Q({}_x f)(1) \\ &= {}_x f(1) \\ &= f(x) \qquad f \in I^X. \end{aligned}$$

Thus $I^X \subset I$ and I is normal.

2. Canonical and sub-canonical homomorphisms. In this section we construct some basic types of homomorphisms between group algebras. Although we confine ourselves to the algebras C(G), it is clear that our remarks apply also the algebras $L_p(G)$ $(1 \leqslant p < \infty)$ or, with slight rewording to $\mathcal{M}(G)$. General results about ideals in C(G) are contained in [6], Chapter VIII.

If G is a compact group and $\chi \in \hat{G}$, then the map A_χ: C(G) → C(G)

$$A_\chi f(x) = \chi(x) f(x)$$

is an isometric algebra automorphism.

If θ: G → H is an epimorphism between compact groups, then we may define two algebra homomorphisms as follows:

$$\Lambda_\theta : C(H) \to C(G)$$

$$\Lambda_\theta f(x) = f(\theta x)$$

$$\Pi_\theta : C(G) \to C(H)$$

$$\Pi_\theta f(\theta x) = \int_K f(xu)\, dm_K(u)$$

where $K = \ker\theta$. It is easy to see that $\|\Pi_\theta\| = \|\Lambda_\theta\| = 1$.

Thus we can construct norm-decreasing homomorphisms by considering compositions of these three types of homomorphisms. Let G and H be compact groups and suppose K is a common quotient group of G and H, i.e. there exist epimorphisms θ: $G \to K$, φ: $H \to K$. Then the map T: $C(G) \to C(H)$ given by $T = A_\chi \Lambda_\varphi \Pi_\theta A_\rho$ where $\chi \in \hat{H}$, $\rho \in \hat{G}$, is a norm decreasing algebra homomorphism. We shall call such a homomorphism *canonical* (see e.g. [4] for the case of $L_p(G)$).

Let $G \times H$ be the cartesian product of G and H and denote by π_G and π_H the co-ordinate projections. We shall say that a closed subgroup M of $G \times H$ is *full* if $\pi_G(M) = G$ and $\pi_H(M) = H$. Given any full subgroup M of $G \times H$ and any $\chi \in \hat{M}$ we may define a homomorphism

$$\Gamma(M,\chi)\colon C(G) \to C(H)$$

by $\Gamma(M,\chi) = \Pi_{\pi_H} A_\chi \Lambda_{\pi_G}$ where π_G: $M \to G$ and π_H: $M \to H$. Any such homomorphism we shall call *sub-canonical.*

THEOREM 2.1. (i) *Any canonical homomorphism is also sub-canonical.*

(ii) *Let M be a full subgroup of $G \times H$ and $\chi \in \hat{M}$;*

then the following three conditions are equivalent:

(1) *$\Gamma(M,\chi)$ is canonical.*

(2) *χ may be extended to a continuous character on $G \times H$*

(3) *There exists $\rho \in \hat{G}$ such that $\Gamma(M,\chi)\rho \neq 0$.*

PROOF. (i) Let T: $C(G) \to C(H)$ be given by $T = A_\chi \Lambda_\varphi \Pi_\theta A_\rho$ where φ: $H \to K$, θ: $G \to K$ are epimorphisms. Define $M \subset G \times H$ to be the set of (x,y) such that $\theta x = \varphi y$, and define $\sigma \in \hat{M}$ by $\sigma(x,y) = \chi(y)\rho(x)$. Then $T = \Pi_{\pi_H} A_\sigma \Lambda_{\pi_G}$.

(ii) (1) $\Rightarrow$ (3). It is trivial that if $\Gamma(M,\chi) = A_\sigma \Lambda_\varphi \Pi_\theta A_\tau$ that $\Gamma(M,\chi)\tau^{-1} \neq 0$.

(3) $\Rightarrow$ (2). If $\Gamma(M,\chi)\rho \neq 0$, then define $N \subset G$ be the set of x such that $(x,1) \in M$. Then

$$\Pi_{\pi_H} A_\chi \Lambda_{\pi_G} \rho(1) = \int_N (A_\chi \Lambda_{\pi_G} \rho)(x,1) dm_N(x)$$

$$= \int_N \chi(x,1)\rho(x) dm_N(x)$$

is non-zero if and only if $\chi(x,1) = \rho(x)^{-1}$ $(x \in N)$. However $\Gamma(M,\chi)\rho$ is an idempotent of norm one in C(H) and hence is a character on H. Thus $\chi(x,1) = \rho(x)^{-1}$ $(x \in N)$. Now χ may be extended to $G \times H$, for if we define

$$\sigma(y) = \rho(x)\chi(x,y) \qquad y \in H \qquad (x,y) \in M,$$

$\sigma \in \hat{H}$ and

$$\chi(x,y) = \rho(x)^{-1}\sigma(y) \quad (x,y) \in M.$$

(2) ⇒ (1). If χ can be extended to $G \times H$ then χ may be written

$$\chi(x,y) = \tau(x)\sigma(y) \quad (x,y) \in M.$$

Define $N \subset G$ to be the set of x such that $(x,1) \in M$, and let $K = G/N$. Let θ: $G \to K$ be the natural epimorphism. Define φ: $H \to K$ by $\varphi(y) = \theta(x)$ where $(x,y) \in M$. Then

$$\Gamma(M,\chi) = A_\sigma \Lambda_\varphi \Pi_\theta A_\tau.$$

This completes the proof of Theorem 2.1.

If $J(N,\chi)$ is a normal ideal of $C(G)$ then the natural projection $P(N,\chi)$ is sub-canonical. To see this, let $M \subset G \times G$ be the set of (x,y) such that $x^{-1}y \in N$ and define $\rho \in \hat{M}$ by $\rho(x,y) = \chi(x^{-1}y)$. The condition

$$\chi(x^{-1}ux) = \chi(u) \quad u \in N \quad x \in G$$

implies that ρ is indeed a character. Then

$$P(N,\chi) = \Gamma(M,\rho).$$

In general if $\Gamma(M,\rho)$: $C(G) \to C(H)$ is sub-canonical then if we define $N = \{x: (x,1) \in M\}$ and $\chi \in \hat{N}$ by $\chi(x) = \rho(x^{-1},1)$, and if $x \in G$

$$\begin{aligned}\chi(x^{-1}ux) &= \rho(xu^{-1}x^{-1},1)\\ &= \rho(x,y)\rho(u^{-1},1)\rho(x,y)^{-1}\end{aligned}$$

[where $(x,y) \in M$]

$$= \rho(u^{-1},1) = \chi(u).$$

Hence $J(N,\chi)$ is a normal ideal, and it is easily checked that $\Gamma(M,\rho)$ maps $J(N,\chi)$ *isometrically* into $C(H)$ and maps its complementary ideal to zero.

The range of $\Gamma(M,\rho)$ is also a normal ideal. In fact, if $E = \{v: (1,v) \in M\}$ and $\sigma \in \hat{E}$ is defined by $\sigma(v) = \rho(1,v)$, then we have $\sigma(y^{-1}vy) = \sigma(v)$ $(v \in E, y \in H)$. Thus $J(E,\sigma)$ is a normal ideal and is the range of $\Gamma(M,\rho)$.

Using this notation, it is easy to show that:

PROPOSITION 2.2. *The composition of two sub-canonical homomorphisms is sub-canonical when it is not zero.*

PROOF. Consider $C(G) \xrightarrow{\Gamma(M,\rho)} C(H) \xrightarrow{\Gamma(N,\sigma)} C(K)$. Then by above the range of $\Gamma(M,\rho)$ is the normal ideal $J(H_0,\chi_0)$ where $H_0 = \{v: (1,v) \in M\}$ and $\chi_0(v) = \rho(1,v)$.

Now $\Gamma(N,\sigma)$ is an isometry on the normal ideal $J(H_1,\chi_1)$ and zero on its complementary ideal, where $H_1 = \{u: (u,1) \in N\}$ and $\chi_1(u) = \sigma(u^{-1},1)$. If $J(H_0,\chi_0) \cap J(H_1,\chi_1) = (0)$, then the composition is zero. Otherwise, by Lemma 1.4, $\chi_0 = \chi_1$ on $H_0 \cap H_1$ and $J(H_0,\chi_0) \cap J(H_1,\chi_1) = J(H_0H_1,\chi)$ where χ extends χ_0 and χ_1. If we now define $E \subset G \times K$ by: $(x,y) \in E$ if there exists $u \in H$ such that $(x,u) \in M$ and $(u,y) \in N$, and $\varphi \in \hat{E}$ by $\varphi(x,y) = \rho(x,u)\sigma(u,y)$, then it is easy to check that

$$\Gamma(N,\sigma)\Gamma(M,\rho) = \Gamma(E,\varphi).$$

We leave the reader to check that the direct sum of two sub-canonical homomorphisms is sub-canonical.

3. **Illustrative example.** Let G_1 and G_2 be two groups of order 8 with the following properties:

(P_1) The centre Z_i of G_i coincides with the commutator subgroup and is of order 2.

(P_2) $G_i/Z_i \cong C_2 \times C_2$.

(P_3) If $u,v \in G_i$ and $uv = vu$ then either $u \in Z_i$ or there exists $m \in N$, $w \in Z_i$ such that $v = u^m w$.

In fact there are two groups with these properties namely the quaternions Q and the dihedral group D_4. The algebra $C(G_i)$ is of dimension 8 and has four one-dimensional minimal ideals and one four-dimensional minimal ideal. The four dimensional ideal is normal, being the ideal $J(Z_i,\chi)$ where χ is the unique non-trivial character on Z_i. The algebra projection $P(Z_i,\chi)$ of $C(G_i)$ onto $J(Z_i,\chi)$ is a sub-canonical homomorphism which is not canonical, since $J(Z_i,\chi)$ contains no non-trivial character (χ cannot be extended to G_i since the commutator subgroup of G_i includes Z_i).

We can however go further than this and establish an isometric isomorphism between the four dimensional ideals of $C(G_1)$ and $C(G_2)$ and a sub-canonical homomorphism between $C(G_1)$ and $C(G_2)$ which is not canonical. Let θ_i: $G_i \to C_2 \times C_2$ be an epimorphism whose kernel is Z_i $(i = 1,2)$ and define $K \subset G_1 \times G_2$ to be the group of all (x,y) such that $\theta_1 x = \theta_2 y$. K is full. We now show that there is a character χ on K which cannot be extended to $G_1 \times G_2$. Indeed the commutator subgroup of $G_1 \times G_2$ is $Z_1 \times Z_2$; we show the commutator subgroup of K is a proper subgroup of $Z_1 \times Z_2$. This will be the case if we establish that for any commutator (x,y), x = 1 or y = 1 implies x = y = 1. Suppose (1,y) is a commutator in K i.e.

$$1 = u_1^{-1}u_2^{-1}u_1u_2$$

$$y = v_1^{-1}v_2^{-1}v_1v_2$$

where $(u_1,v_1) \in K$ and $(u_2,v_2) \in K$. Then by (P_3), either $u_1 \in Z_1$ or $u_2 = u_1^m w$ where $w \in Z_1$. If $u_1 \in Z_1$ then $\theta_1 u_1 = 1$ so that $v_1 \in Z_2$ and y = 1. If $u_2 = u_1^m w$ then $\theta_1 u_2 = (\theta_1 u_1)^m$ so that $\theta_2 v_2 = (\theta_2 v_1)^m$ i.e. $v_2 = v_1^m z$ where $z \in Z_2$; again y = 1. It follows that there is a character χ on K which cannot be extended to $G_1 \times G_2$.

Now $\Gamma(K,\chi)$: $C(G_1) \to C(G_2)$ is a norm-decreasing homomorphism and maps the four-dimensional ideal of $C(G_1)$ isometrically onto the four-dimensional ideal of $C(G_2)$. We remark at this point that if we replace the group algebras $C(G_1)$ and $C(G_2)$ by $M(G_1)$ and $M(G_2)$ the same statement is true and $\Gamma(K,\chi)$ provides an example of a non-canonical norm-decreasing homomorphism in answer to a question of Kerlin and Pepe ([5]).

If 1 denotes the identity character on K, then $\Gamma(K,1)$ maps the span of the characters in $C(G_1)$ onto the span of the character in $C(G_2)$ and the four-dimensional ideal to zero. Thus $\Gamma(K,1) + \Gamma(K,\chi)$ is an algebra isomorphism between $C(G_1)$ and $C(G_2)$ or between $M(G_1)$ and $M(G_2)$). A routine calculation shows that in either case $\|\Gamma(K,1) + \Gamma(K,\chi)\| = \sqrt{2}$ (compare [3]).

4. Homomorphisms between ideals. We now consider homomorphisms defined only on closed ideals of C(G). We shall preserve the names *canonical* and *sub-canonical* for those homomorphisms which are restrictions of canonical and sub-canonical homomorphisms. However there is in this case another possiblity; note here that the situation differs substantially from the cases $L_p(G)$ $(1 \leqslant p < \infty)$. If G is a compact group and H is a proper closed subgroup of G, the restriction map R_H: C(G) → C(H) is never an algebra homomorphism ([10]). However restricted to an ideal, R_H may be a homomorphism.

THEOREM 4.1. *Suppose I is a closed ideal in C(G) and H is a closed subgroup of G. Then the map R_H: $I \to C(H)$ is an algebra homomorphism if and only if it is an injection.*

PROOF. Suppose R_H: I → C(H) is a homomorphism. Then $\ker R_H \cap I$ is an ideal in C(G). If $f \in \ker R_H \cap I$ then for any $x \in G$, $f_x \in \ker R_H$ so that $f_x(1) = 0$ i.e. $f(x) = 0$. Hence $\ker R_H \cap I = \{0\}$ i.e. R_H is a monomorphism.

Conversely suppose R_H is an injection. Suppose J is a minimal ideal in I; we show $R_H|J$ is an algebra homomorphism. Suppose $\dim J = n^2$ and $\{e_{ij}: 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n\}$ is an orthonormal basis of J satisfying $e_{ij}^* = e_{ji}$ and $e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$. (See [6], page 158.) Then $x \leftrightarrow (\frac{1}{n}e_{ij}(x))$ is an irreducible representation of G. We show that this representation remains irreducible when restricted to H. Indeed suppose not; then there exist $\xi_1,\ldots,\xi_n, \eta_1 \cdots \eta_n \in \mathbf{C}$ not all zero such that

$$\Sigma_{i=1}^{n} \Sigma_{j=1}^{n} \eta_i e_{ij}(x)\xi_j = 0 \qquad x \in H$$

i.e.

$$\Sigma_{i=1}^{n} \Sigma_{j=1}^{n} \eta_i \xi_j R_H e_{ij} = 0$$

so $R_H|J$ is not an injection. It now follows that $(R_H e_{ij})$ is an orthonormal basis of a minimal ideal in C(H) and satisfies $(R_H e_{ij})^* = R_H e_{ji}$ and $(R_H e_{ij})(R_H e_{k\ell}) = \delta_{jk} R_H(e_{i\ell})$. Hence $R_H|J$ is an algebra homomorphism and it follows that $R_H|I$ is a homomorphism.

THEOREM 4.2. *Suppose I is a closed ideal in C(G) and that $I^X = J(N,\chi)$. Suppose R_H: $I \to C(H)$ is an injection. Then the following conditions are equivalent:*

(i) *R_H is an isometry on I*

(ii) *R_H is sub-canonical*

(iii) *$HN = G$*

(iv) *R_H is an injection on I^X.*

PROOF. (i) ⇒ (iii). Suppose $x \notin HN$. For any minimal self-adjoint idempotent $e \in I$

$$e(1) = \|e\|.$$

Suppose that for some scalar α, $|\alpha| \leqslant 1$

$$e(x) = \alpha\|e\|$$

for all such idempotents. Then, since I is the span of minimal self-adjoint idempotents,

$$f(x) = \alpha f(1) \quad \text{for all } f \in I,$$

and so, by considering translates

$$f(xy) = \alpha f(y) \qquad f \in I.$$

Hence $x \in N$ and $\alpha = \chi(x)$. This is a contradiction so we deduce that there exist two self-adjoint minimal idempotents e_1, e_2 such that

$$e_1(x) = \alpha\|e_1\|$$

$$e_2(x) = \beta\|e_2\|$$

where $\alpha \neq \beta$. Thus

$$|(e_1 + e_2)(x)| < (e_1 + e_2)(1) = \|e_1 + e_2\|.$$

Hence for each $h \in H$, since $xh \notin NH$, there exist $f^h \in I$, self-adjoint, with $f^h(1) = \|f^h\|$ and

$$|f^h(xh)| < f^h(1).$$

Now by a compactness argument there exists $g \in I$, g self-adjoint, $g(1) = \|g\|$, with

$$|g(xh)| < g(1) \quad \text{for all } h \in H.$$

[To see this, if $U_h = \{y \in G: |f^h(xy)| < f^h(1)\}$, then $\{U_h\}$ is an open cover of H. If $U_{h_1}, U_{h_2} \cdots U_{h_n}$ is a finite subcover, put $g = f^{h_1} + f^{h_2} + \cdots + f^{h_n}$.] Thus

$$\|R_H({}_x g)\| < \|g\| = \|{}_x g\|$$

contradicting (i).

(iii) ⇒ (ii). Let $M = \{(x,y) \in G \times H; x^{-1}y \in N\}$ and $\rho \in \hat{M}$ be defined by $\rho(x,y) = \chi(x^{-1}y)$. Then $R_H = \Pi_{\pi_H} A_\rho \Lambda_{\pi_G}$ where π_H and π_G are projections of M onto H and G respectively. For, if $f \in I$ and $x \in H$, we have, since $\ker \pi_H = N \times \{1\}$

$$
\begin{aligned}
(\Pi_{\pi_H} A_\rho \Lambda_{\pi_G} f)(x) &= (\Pi_{\pi_H} A_\rho \Lambda_{\pi_G} f)(\pi(x,x)) \\
&= \int_N (A_\rho \Lambda_{\pi_G} f)(xu,x)du \\
&= \int_N \chi(u^{-1} f(xu)du \\
&= \int_N \chi(u^{-1})\chi(u)f(x)du = f(x).
\end{aligned}
$$

(ii) ⇒ (i). R_H is a monomorphism on I and hence, by the remarks after Theorem 2.1 that a sub-canonical map is an isometry on an ideal and zero on the complementary ideal, R_H must be an isometry on I.

(iii) ⇒ (iv). If $R_H f = 0$ then $f(x) = 0$ for $x \in HN$.

(iv) ⇒ (iii). If $x \notin HN$, define $f \in C(G)$ so that $f(HN) = 0$ and $f(ux) = \chi(u)$ $(u \in N)$. Then $P(N,\chi)\ f \in I^X$ is non-zero but vanishes on H.

REMARKS. It is easy enough to construct examples to show that R_H need not be sub-canonical. For let G be an irreducible subgroup of U_n (n × n-unitary matrices), and let I be the minimal ideal of C(G) corresponding to the self-representation of G. Then $N = \{\lambda 1_n : |\lambda| = 1\} \cap G$. If H is a subgroup of G which is also irreducible and $HN \neq G$, then $R_H|I$ is a monomorphism but not an isometry. For example let $G = SU_n$ and let H be any proper irreducible subgroup of G.

5. Classification of norm-decreasing homomorphisms.

LEMMA 5.1. ([9]). *Let I be a minimal ideal of C(G) and let $T: I \to C(H)$ be a norm-decreasing monomorphism. Then T(I) is a minimal ideal of C(H) and for $f \in I$*

$$Tf(1) = f(1).$$

PROOF. This is proved exactly as in Lemma 3 of [9]; note however that this proof should be modified by deleting the conclusions that $T(N_\alpha)$ and $e'_1 \cdots e'_k$ are self-adjoint.

THEOREM 5.2. *Let I be a closed ideal of C(G) and let $T: I \to C(H)$ be a norm-decreasing monomorphism. Then there is a closed subgroup M of G such that for $f \in I$*

$$Tf = SR_M f$$

where $S: C(M) \to C(H)$ is sub-canonical, and $R_M|I$ is a monomorphism. Further any norm-one linear extension of T defined on a subspace of I^X coincides with SR_M.

PROOF. For each $x \in H$, the map $f \mapsto Tf(x)$ is a linear functional of norm less than or equal to one. Let Σ_x be the set of measures $\mu \in M(G)$ such that

$$f*\mu^*(1) = Tf(x) \qquad f \in I$$

and $\|\mu\| \leqslant 1$. Then $\Sigma_x \neq \emptyset$ (by the Hahn-Banach theorem) and is a weak*-compact convex set.

By Theorem 1.6 and Lemma 5.1, $\mu \in \Sigma_1$ if and only if

$$\int f(x)d\bar{\mu}(x) = f(1) \qquad f \in I^X.$$

Hence if $I^X = J(N,\chi)$, Σ_1 is the closed convex hull of the points $\{\chi(u)\delta_u : u \in N\}$.

Now suppose $\mu \in \Sigma_x$ and $\upsilon \in \Sigma_y$; we shall show that $\upsilon*\mu \in \Sigma_{yx}$. Suppose I_0 is a minimal ideal in I with identity e. Then for $f \in I_0$

$$Tf(x) = f*\mu^*(1) = f*e*\mu^*(1) = Tf*T(e*\mu^*)(1) \text{ by Lemma 5.1}$$

so that

$$Tf*\delta_{x^{-1}}(1) = Tf*T(e*\mu^*)(1).$$

Again by 5.1, $T(I_0)$ is a minimal ideal of C(H) with identity Te, and, since Tf = Tf*Te,

$$Tf*[(Te*\delta_{x^{-1}}) - T(e*\mu^*)](1) = 0$$

for all $f \in I_0$. Hence, since $Te*\delta_{x^{-1}} - T(e*\mu^*) \in TI_0$,

$$Te*\delta_{x^{-1}} = T(e*\mu^*).$$

Now if $\upsilon \in \Sigma_y$, since e is a central idempotent,

$$\begin{aligned} T(e*(\upsilon*\mu)^*) &= T(e*\mu^{**}\upsilon^*) \\ &= T(e*\mu^{**}e*\upsilon^*) \\ &= Te*\delta_{x^{-1}}*\delta_{y^{-1}} \\ &= Te*\delta_{(yx)^{-1}}. \end{aligned}$$

Thus for $f \in I_0$, $(Tf*(\upsilon*\mu)^*)(1) = (Tf)(yx)$. Since this is true for each minimal ideal in I, it is valid for each $f \in I$. Hence, since $\|\upsilon*\mu\| \leqslant 1$, $\upsilon*\mu \in \Sigma_{yx}$.

Now let K = G/N and let φ: G → K be the natural quotient map. Let $\hat{\varphi}$: $\boldsymbol{M}$(G) → $\boldsymbol{M}$(K) be the natural induced map.

If $\mu \in \Sigma_x$ and $\upsilon \in \Sigma_{x^{-1}}$, $\mu*\upsilon \in \Sigma_1$ and hence $\|\mu*\upsilon\| = 1$. Thus $\|\mu\| = \|\upsilon\| = 1$ and $|\mu|*|\upsilon| = |\mu*\upsilon|$. In particular, $|\mu| = \lambda\mu$ for some complex number λ with $|\lambda| = 1$, and

$$\hat{\varphi}(|\mu|)*\hat{\varphi}(|\upsilon|) = \delta_1 \text{ (in } \boldsymbol{M}(K))$$

since the support of $\mu*\upsilon$ is contained in N. Thus

$$\hat{\varphi}(|\mu|) = \delta_{\theta(x)} \quad \mu \in \Sigma_x$$

where θ: H → K is a group homomorphism. θ is continuous; for if $x_\alpha \to x$ in H, and $\mu_x \in \Sigma_{x_\alpha}$, $\{\mu_\alpha\}$ has a convergent subnet, being w*-compact. If μ is the limit of such a subnet – $\{\mu_\beta\}$ say – we have:

$$(Tf)(x) = \lim_\beta (Tf)(x_\beta) = \lim_\beta (f*\mu_\beta^*)(1) = (f*\mu^*)(1)$$

i.e. $\mu \in \Sigma_x$ and so $\hat{\varphi}(|\mu|) = \delta_x$. Since this is true for all subnets and $\hat{\varphi}$ is w*-continuous, $\theta_{x_\alpha} \to \theta_x$.

Let $\varphi^{-1}\theta(H) = M$ and $E \subset M \times H$ be the set of (x,y) such that $\varphi x = \theta y$. If $(x,y) \in E$ and $\mu \in \Sigma_y$, then since $\chi m_N \in \Sigma_1$, $\chi m_N*\mu \in \Sigma_y$. Since $m_N*\mu = m_N*\upsilon$

whenever $\hat{\varphi}(\mu) = \hat{\varphi}(v)$ we have $\chi m_N * \mu = \alpha \chi m_N * \delta_x$ for some constant α. Now since $f * \chi m_N = f$ for all $f \in I$, $\alpha \chi m_N * \delta_x$ and $\alpha \delta_x$ have the same effect on I. Thus we have $\alpha \delta_x \in \Sigma_y$. By comparing norms $|\alpha| = 1$. It follows that

$$\Sigma_y = \overline{co}\{\alpha(x,y)\delta_x : \varphi x = \theta y\}.$$

If $f \in I$,

$$(Tf)(y) = \alpha(x,y)f(x) \qquad (\varphi x = \theta y).$$

α is a character on E and if S: C(M) → C(H) is the subcanonical homomorphism $\Gamma(E,\alpha)$, we have $T = SR_M$ as required. [Here we need that $\chi(t) = \alpha(t^{-1},1)$ – but, by Lemma 5.1, if $f \in I$ and $t \in N$, $f(1) = (Tf)(1) = \alpha(f,1)f(t) = \alpha(t,1)\chi(t)f(1)$.]

R_M is a monomorphism, since T is.

Finally, since all $\mu \in \Sigma_y$ have the same effect on each $f \in I^X$, any linear extension $\tilde{T}$ of T with $\|\tilde{T}\| = 1$ must have

$$(\tilde{T}f)(y) = \alpha(x,y)f(x) \qquad (\varphi x = \theta y)$$

whenever $f \in I^X$.

If T is isometric, we may ignore M.

THEOREM 5.3. *Let I be a closed ideal of C(G) and let T: I → C(H) be an isometric homomorphism. Then T is subcanonical.*

PROOF. Using the notation of 5.2, $T = SR_M$. Now R_M is an isometry on I, since T is, and so, by Theorem 4.2, R_M is sub-canonical. Since the composition of sub-canonical maps is sub-canonical (Proposition 2.2), T is sub-canonical.

Finally we have the general case.

THEOREM 5.4. *Let I be a closed ideal of C(G) and let I → C(H) be a norm-decreasing homomorphism, then there is a closed subgroup M of G, a normal ideal J(Q,ψ) of C(G) such that for f ∈ I,*

$$Tf = SR_M P(Q,\psi)f$$

where S: C(M) → C(H) is subcanonical.

PROOF. Let I_0 be the kernel of T, and I_1 the complementary ideal to I_0 in I. Then, by 5.3, T has the form SR_M on I_1. Now every norm-one linear extension of SR_M is uniquely defined (and non-zero) on I_1^X. Thus $I_0 \cap I_1^X = (0)$. If $I_1^X = J(Q,\psi)$, then $T = SR_M P(Q,\psi)$ as required.

COROLLARY 5.5. *If T is a norm-decreasing homomorphism of C(G) into C(H),* *then T is subcanonical.*

PROOF. In the representation of Theorem 5.4, R_M is injective on $J(Q,\psi)$ and so by Theorem 4.2, R_M is sub-canonical. The result follows from Proposition 2.2.

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