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TYPE AND ORDER CONVEXITY OF MARCINKIEWICZ AND LORENTZ SPACES AND APPLICATIONS

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Abstract. We consider order and type properties of Marcinkiewicz and Lorentz function spaces. We show that if $0 < p < 1$, a p-normable quasi-Banach space is natural (i.e. embeds into a q-convex quasi-Banach lattice for some $q > 0$) if and only if it is finitely representable in the space $L_{p,\infty}$. We also show in particular that the weak Lorentz space $L_{1,\infty}$ do not have type 1, while a non-normable Lorentz space $L_{1,p}$ has type 1. We present also criteria for upper r-estimate and r-convexity of Marcinkiewicz spaces.

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1. Introduction. In this note we study the order convexity and type of Marcinkiewicz and Lorentz function spaces. The space weak L_p or $L_{p,\infty}$ is well-known to be p-normable if $0 < p < 1$, but is q-convex as a lattice when $0 < q < p$ (see [**4**] and [**5**]). We prove that a p-normable quasi-Banach space X embeds into a p-normable quasi-Banach lattice which is r-convex for some $r > 0$ (i.e. X is natural) if and only if X is finitely representable in $L_{p,\infty}(0, 1)$.

We then consider more general Lorentz and Marcinkiewicz spaces. In [**6**] it was proved that if a quasi-Banach space $(X, \|\cdot\|)$ has type $0 < p < 1$, then $\|\cdot\|$ is a p-norm, and if X has type $p > 1$ then X is normable. It was also shown that there exist quasi-Banach spaces that have type 1, but they are not normable. In this note we show that Marcinkiewicz spaces have type 1 if and only if they are 1-convex (that is normable), while the class of Lorentz spaces with type 1 coincides to the class of those spaces satisfying an upper 1-estimate. In consequence, there exist Lorentz spaces with type 1 that are not normable.

Let us start with basic definitions and notation. Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the sets of all real, nonnegative real and natural numbers, respectively. Let $r_n : [0, 1] \to \mathbb{R}$, $n \in \mathbb{N}$, be Rademacher functions, that is $r_n(t) = \operatorname{sign}(\sin 2^n \pi t)$. A quasi-Banach space X has type $0 < p \le 2$ if there is a constant $K > 0$ such that, for any choice of finitely many

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vectors $x_1, \dots, x_n$ from X,

$$\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt \leq K \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p},$$

and it has cotype $q \geq 2$ if there is a constant $K > 0$ such that for any finite collection of elements $x_1, \dots, x_n$ from X,

$$\left(\sum_{k=1}^n \|x_k\|^q \right)^{1/q} \leq K \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt.$$

Recall also that a quasi-norm $\|\cdot\|$ in X is a p-norm, $0 < p < 1$, if there exists $C > 0$ such that for any $x_i \in X,\ i = 1, \dots, n$

$$\|x_1 + \cdots + x_n\| \leq C(\|x_1\|^p + \cdots + \|x_n\|^p)^{1/p}.$$

By the Aoki-Rolewicz theorem [**9**], for any quasi-norm $\|\cdot\|$ there exists $0 < p < 1$ such that $\|\cdot\|$ is a p-norm. We say that a quasi-Banach space $(X, \|\cdot\|)$ is *normable* whenever there exists a norm $|\!|\!|\cdot|\!|\!|$ in X such that $C^{-1}\|x\| \leq |\!|\!|x|\!|\!| \leq C\|x\|$ for all $x \in X$ and some $C > 0$.

A quasi-Banach lattice $X = (X, \|\cdot\|)$ is said to be *p-convex*, $0 < p < \infty$, respectively *p-concave*, $0 < p < \infty$, if there are positive constants $C^{(p)}$ and $C_{(p)}$ such that

$$\left\| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right\| \leq C^{(p)} \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p},$$

respectively,

$$\left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p} \leq C_{(p)} \left\| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right\|,$$

for every choice of vectors $x_1, \dots, x_n \in X$. We also say that X satisfies an *upper p-estimate*, $0 < p < \infty$, respectively a *lower p-estimate*, $0 < p < \infty$, if the definition of p-convexity, respectively p-concavity, holds true for any choice of disjointly supported elements $x_1, \dots, x_n$ in X ([**6**, **14**]). We notice here that a quasi-Banach lattice is normable if and only if it is 1-convex. However, while a p-normable quasi-Banach lattice necessarily has an upper p-estimate, it may fail to be q-convex for any choice of $q > 0$. Motivated by this, the first author [**8**] defined a quasi-Banach space X to be *natural* if it is isomorphic to a subspace of a quasi-Banach lattice which is q-convex for some $q > 0$.

Let us recall that a quasi-Banach space X is said to be (*crudely*) *finitely representable* in a quasi-Banach space Y if there is a constant C so that for every $\epsilon > 0$ and every finite-dimensional subspace F of X there is a finite-dimensional subspace G of Y and an isomorphism $T : F \to G$ such that $\|T\|\|T^{-1}\| < C + \epsilon$. If $C = 1$ we say that X is *finitely representable* in Y.

A function $U : I \to \mathbb{R}_+$, where $I = [0, 1]$ or $I = [0, \infty)$, is said to be *pseudo-increasing* (resp. *pseudo-decreasing*) whenever there exists $C > 0$ such that $U(s) \leq CU(t)$ (resp. $U(s) \geq CU(t)$) for all $0 \leq s < t$. We say that the expressions A and B are equivalent, whenever A/B is bounded above and below by positive constants. Given a function $U : I \to \mathbb{R}_+$, we define the *lower* and *upper Matuszewska-Orlicz indices* [**13**, **14**] as follows:

$$\alpha(U) = \sup\{p \in \mathbb{R} : U(as) \le C\, a^p\, U(s) \text{ for some } C > 0 \text{ and all } s \in I,\ 0 < a \le 1\},$$
$$\beta(U) = \inf\{p \in \mathbb{R} : U(as) \le C\, a^p\, U(s) \text{ for some } C > 0 \text{ and all } as \in I,\ a \ge 1\}.$$

If U and V are equivalent, then their corresponding indices coincide.

If f is a real-valued measurable function on I, then we define the *distribution function* of f by $d_f(\theta) = \lambda\{|f| > \theta\}$ for each $\theta \ge 0$, where λ denotes the Lebesgue measure on I. The *non-increasing rearrangement* of f is defined by

$$f^*(t) = \inf\{s > 0 : d_f(s) \le t\}, \quad t \in I.$$

A positive, Lebesgue measurable function $w : I \to (0, \infty)$ is called a *weight function* whenever

$$W(t) := \int_0^t w(s)\, ds = \int_0^t w < \infty,$$

for all $t \in I$. We shall always assume here that W satisfies condition Δ_2, that is for some $K > 0$ and all $t \in I$,

$$W(t) \le K W(t/2).$$

Given a weight w, the *Marcinkiewicz space* $M_{p,w}$, $0 < p < \infty$, also called the *weak Lorentz space*, is the set of all Lebesgue measurable functions $f : I \to \mathbb{R}$ such that

$$\|f\|_M := \sup_t W^{1/p}(t) f^*(t) = \sup_t W^{1/p}(d_f(t))\, t < \infty.$$

The functional $\|\cdot\|_M$ is a quasi-norm and $(M_{p,w}, \|\cdot\|_M)$ is a quasi-Banach space. In the case when $W(t) = t$, we will denote it by $L_{p,\infty}$. As usual $L_{p,\infty}(0, 1)$ or $L_{p,\infty}(0, \infty)$ will denote the spaces on [0, 1] or [0, ∞), respectively. Recall also that the Marcinkiewicz sequence space $\ell_{p,\infty}$, $0 < p < \infty$, consists of all sequences $x = (\alpha_n) \subset c_0$ such that $\|x\|_{p,\infty} = \sup_n\{n^{1/p}\alpha_n^*\} < \infty$, where $\{\alpha_n^*\}$ is a decreasing permutation of $\{\alpha_n\}$. It is well-known that $L_{p,\infty}$ or $\ell_{p,\infty}$ is q-convex whenever $0 < q < p$ [**4**], but is not p-convex.

Given a weight function w with $\int_0^\infty w = \infty$ if $I = [0, \infty)$, recall that the Lorentz space $\Lambda_{p,w}$, $0 < p < \infty$, consists of all real-valued Lebesgue measurable functions f on I such that

$$\|f\|_\Lambda := \left(\int_I f^{*p} w\right)^{1/p} < \infty.$$

It is well known that $(\Lambda_{p,w}, \|\cdot\|_\Lambda)$ is a quasi-Banach space [**3**, **11**].

Observe that the condition Δ_2 imposed on W is necessary in the context of this paper. In fact for W positive on $(0, \infty)$, the spaces $M_{p,w}$ or $\Lambda_{p,w}$ are linear if and only if W satisfies condition Δ_2 ([**2**]). It is also not difficult to verify that the Δ_2-condition of W is necessary and sufficient for $\|\cdot\|_M$ or $\|\cdot\|_\Lambda$ to be a quasi-norm (cf. [**11**, **18**]).

2. Finite representability in $L_{p,\infty}$.

PROPOSITION 2.1. *Suppose that* $0 < p \leq 1$ *and that* F *is a finite-dimensional subspace of* $L_{p,\infty}(0,\infty)$. *Then, given* $\epsilon > 0$, *there exists a measurable subset* B *of* $(0,\infty)$ *of finite measure so that:*

$$\|f\chi_B\|_{p,\infty} > (1-\epsilon)\|f\|_{p,\infty}, \qquad f \in F$$

and so for a suitable constant $K = K(F,\epsilon)$ *we have*

$$\|f\chi_B\|_\infty \leq K\|f\|_{p,\infty}, \qquad f \in F.$$

Proof. Fix $\delta > 0$ so small that $(1-2\delta)(1-2\delta^p)^{1/p} > 1-\epsilon$. Let $\{f_1,\ldots,f_n\}$ be a δ^2–net for the set $\{f \in F : \|f\|_{p,\infty} = 1\}$. For each $1 \leq k \leq n$ there exists t_k so that

$$f_k^*(t_k) \geq (1-\delta)t_k^{-1/p}.$$

Let $h = \max_{1\leq k\leq n}|f_k|$ so that

$$\|h\|_{p,\infty} \leq \sup_t t^{1/p}\left(\sum_{k=1}^n |f_k|(t)\right)^* \leq \sup_t t^{1/p}\sum_{k=1}^n f_k^*(t/n) \leq n^{1/p}.$$

Choose M so large that $M > n^{1/p}\delta^{-1}t_k^{-1/p}$ for $1 \leq k \leq n$ and $\frac{1}{M} < (1-\delta)t_k^{-1/p}$ for $1 \leq k \leq n$. Now let $B = \{s : \ M^{-1} \leq h(s) \leq M\}$. B is clearly of finite measure. Furthermore if $f \in F$ with $\|f\|_{p,\infty} = 1$ then f can be expressed as a series $\sum_{k=0}^\infty \alpha_k f_{j(k)}$ where $|\alpha_k| \leq \delta^{2k}$. Hence $\|f\chi_B\|_\infty \leq (1-\delta^2)^{-1}M$. Thus the second condition is fulfilled.

Now if $\|f\|_{p,\infty} = 1$ choose f_k so that $\|f - f_k\|_{p,\infty} \leq \delta^2$. Then the set $D = \{s : |f_k(s)| \geq (1-\delta)t_k^{-1/p}\}$ has measure at least t_k. Clearly $h(t) \geq |f_k(t)| \geq (1-\delta)t_k^{-1/p} \geq \frac{1}{M}$ for $t \in D$. Hence if $t \in D\backslash B$ then $h(t) > M$ and so $n^{1/p} \geq \|h\|_{p,\infty} \geq \lambda\{|h| > M\}^{1/p}M \geq \lambda(D\backslash B)^{1/p}M$, which yields that $\lambda(D\backslash B) \leq M^{-p}n \leq \delta^p t_k$. Thus $\lambda(D\cap B) \geq (1-\delta^p)t_k$. In view of the choice of f_k we have $\lambda\{|f - f_k| > \delta t_k^{-1/p}\} < \delta^{2p}\delta^{-p}t_k = \delta^p t_k$. Now, if $|f(t) - f_k(t)| \leq \delta t_k^{-1/p}$ and $|f_k(t)| \geq (1-\delta)t_k^{-1/p}$ then $|f(t)| \geq (1-2\delta)t_k^{-1/p}$ and so

$$\lambda\big\{|f\chi_B| \geq (1-2\delta)t_k^{-1/p}\big\} \geq \lambda\big\{|f - f_k| \leq \delta t_k^{-1/p}\big\} \cap B \cap D \geq (1-2\delta^p)t_k.$$

Thus

$$\|f\chi_B\|_{p,\infty} \geq (1-2\delta)(1-2\delta^p)^{1/p}. \qquad \square$$

PROPOSITION 2.2. *Suppose that* $0 < p \leq 1$. *The space* $\ell_\infty(L_{p,\infty}(0,\infty))$ *is finitely representable in* $L_{p,\infty}(0,1)$.

Proof. It is enough to prove that if F is a finite-dimensional subspace of $L_{p,\infty}(0,\infty)$ and $n \in \mathbb{N}$ then for any $\epsilon > 0$, $\ell_\infty^n(F)$ $(1+\epsilon)$-embeds into $L_{p,\infty}(0,1)$. By Proposition 2.1 we can find a constant K and an embedding $T : F \to L_{p,\infty}(0,1)$ such that

- $\|T\| \leq 1$,
- $\|Tf\|_{p,\infty} \geq (1-\epsilon)\|f\|_{p,\infty} \qquad f \in F$ and
- $\|Tf\|_\infty \leq K\|f\|_{p,\infty} \qquad f \in F$.

Pick $\delta > 0$ so that $(1-\delta)^{-1} < (1+\epsilon)^p$. Let $a_1 > a_2 > \cdots > a_n > 0$ be chosen so that $\sum_{j=1}^n a_j < 1$ and $a_{j+1} < K^{-p}\delta a_j$ for $j = 1,2,\ldots,n-1$. Now for $j = 1,2,\ldots,n$ let

B_j be disjoint Borel subsets of $(0, 1)$ of measure a_j. For each j there is an embedding $T_j : F \to L_{p,\infty}(B_j) = \{f\chi_{B_j} : f \in L_{p,\infty}(0, \infty)\}$ with

- $\|T_j\| \leq 1$,
- $\|T_j f\|_{p,\infty} \geq (1 - \epsilon)\|f\|_{p,\infty} \qquad f \in F$ and
- $\|T_j f\|_\infty \leq K a_j^{-1/p}\|f\|_{p,\infty} \qquad f \in F$.

Here $(T_1, \ldots, T_n)$ are obtained by dilating and translating the embedding T. Now if $f_1, \ldots, f_n \in F$ with $\max_j \|f_j\|_{p,\infty} = 1$ we have

$$\begin{aligned} \lambda\left(\left|\sum_{j=1}^{n} T_j f_j\right| > r\right) &= \sum_{j=1}^{n} \lambda(|T_j f_j| > r) \\ &= \sum_{a_j \leq K^p r^{-p}} \lambda(|T_j f_j| > r) \\ &\leq \sum_{a_j \leq K^p r^{-p}} \min(a_j, r^{-p}). \end{aligned}$$

Assuming this sum is nonempty let k be the first index such that $a_k \leq K^p r^{-p}$. Then we may estimate it by

$$r^{-p} + \sum_{k<j\leq n} a_j \leq r^{-p} + K^p r^{-p} \sum_{j=1}^{\infty} (K^{-p}\delta)^j < (1 + \epsilon)^p r^{-p}.$$

It follows that the map $(f_1, \ldots, f_n) \to \sum_{j=1}^{n} T_j f_j$ defines the required $(1 + \epsilon)$-embedding of $\ell_\infty^n(F)$ into $L_{p,\infty}(0, 1)$. □

PROPOSITION 2.3. *The spaces $\ell_{1,\infty}$ and $L_{1,\infty}(0, 1)$ are not of type 1.*

Proof. It suffices to show that $L_{1,\infty}(0, 1)$ is not of type 1. It is well-known that $L_{1,\infty}(0, 1)$ is not normable and indeed that for some constant $c > 0$, there exist (see e.g. [**17**]) non-negative functions $f_1, \ldots, f_n \in L_{1,\infty}(0, 1)$ with $\|f_j\|_{1,\infty} = 1$ and

$$\|f_1 + \cdots + f_n\|_{1,\infty} \geq cn \log n.$$

Let F be a subspace spanned by $\{f_1, \ldots, f_n\}$ and let $N = 2^n$. We consider the space $\ell_\infty^N(F)$ with co-ordinates indexed by all n-tuples $(\eta_1, \ldots, \eta_n)$ where $\eta_j = \pm 1$. Define $\phi_j \in \ell_\infty^N(F)$ by the coordinates $\phi_j(\eta_1, \ldots, \eta_n) = \eta_j f_j$ for $j = 1, \ldots, n$. Then for every choice of sign $\epsilon_j = \pm 1$ we have

$$\|\epsilon_1\phi_1 + \cdots + \epsilon_n\phi_n\| = \|f_1 + \cdots + f_n\|_{1,\infty}.$$

Since $\ell_\infty^N(F)$ embeds almost isometrically into $L_{1,\infty}(0, 1)$ this space fails to have type 1. □

We conclude this section with a characterization of natural spaces. The technique is rather similar to that of [**10**], Theorem 4.2. Recall that the weak Lorentz space $L_{p,\infty}(\Omega, \mu)$ over arbitrary measure space (Ω, μ) consists of all μ-measurable real valued functions f such that $\|f\|_{p,\infty} = \sup_{t\geq 0} \mu\{|f| > t\}^{1/p} t < \infty$.

THEOREM 2.4. *Suppose that $0 < p < 1$ and that X is a p-normable quasi-Banach space. The following conditions on X are equivalent:*

(1) *X is natural.*
(2) *X is (crudely) finitely representable in $L_{p,\infty}(0, 1)$.*

(3) *There exists a constant C with the property that given $x \in X$ there exists a compact Hausdorff space Ω, a probability measure μ on Ω and an operator $T : X \to L_{p,\infty}(\Omega, \mu)$ such that $\|T\| \le 1$ and $\|x\| \le C\|Tx\|$.*

(4) *For some (respectively, every) $0 < \delta < 1$ there is a constant $C = C(\delta)$ so that $x_1, \dots, x_n \in X$ and $y \in X$ is such that $y \in \operatorname{co}\{\pm x_k : k \in A\}$ whenever $A \subset \{1, 2, \dots, n\}$ and $|A| > n\delta$ then $\|y\| \le C \max_{1\le k\le n} \|x_k\|$.*

Proof. (1) $\Longrightarrow$ (4): It is enough to show that if X is a quasi-Banach lattice which is r-convex for some $r > 0$ then (4) holds for X for every choice of δ. Let us therefore fix $\delta > 0$. Thus we may assume an estimate

$$\left\| \left(\sum_{j=1}^{m} |v_j|^r \right)^{1/r} \right\| \le M \left(\sum_{j=1}^{m} \|v_j\|^r \right)^{1/r} \qquad v_1, \dots, v_m \in X.$$

Now assume $x_1, \dots, x_n, y$ given as in the statement of (4). Then we may represent the ideal Z generated by the order-interval $[-|y|, |y|]$ as an abstract M-space in the sense of Kakutani if we take $[-|y|, |y|]$ as the unit ball. It thus may be identified with a space $C(\Omega)$ in such a way that $|y(s)| = 1$ for all $s \in \Omega$. Let $u_k = |x_k| \wedge |y|$ so that u_k can be identified with a continuous function on Ω. Fix any $s \in \Omega$ and let $A = \{k : u_k(s) < 1\}$. Then it is clear that $y \notin \operatorname{co}\{\pm x_k : k \in A\}$ and so by hypothesis (4), we have $|A| \le n\delta$. Thus $|\{k : u_k(s) \ge 1\}| \ge n(1 - \delta)$.

Thus

$$\left(\sum_{j=1}^{n} |u_j|^r \right)^{1/r} \ge n^{1/r}(1 - \delta)^{1/r}|y|,$$

and so

$$n^{1/r}(1 - \delta)^{1/r}\|y\| \le M n^{1/r} \max_{1\le k\le n} \|x_k\|,$$

i.e.

$$\|y\| \le M(1 - \delta)^{-1/r} \max_{1\le k\le n} \|x_k\|.$$

This establishes (4) with $C(\delta) = M(1 - \delta)^{-1/r}$.

(4) $\Longrightarrow$ (3): This is an argument based on Nikishin's theorem [**16**]. We assume (4) holds for constants C and $0 < \delta < 1$. Let $(g_n)_{n=1}^{\infty}$ be a sequence of independent normalized Gaussians defined on a probability space $(\Omega', \mathbb{P})$. Let $c_p = \mathbb{E}|g_1|^p$ and choose $\theta > 0$ so that $(2C)^p c_p \theta^p < \frac{1}{4}$. Then pick M so that

$$\mathbb{P}\{|g_1| > \sigma\theta^{-1}M^{-1}\} > \frac{1 + \frac{1}{4}\delta}{1 + \frac{1}{2}\delta},$$

where $\sigma = 1 + \frac{1}{2}\delta$.

Fix $u \in X$ with $\|u\| = 1$ and then let Ω_0 be the subset of the algebraic dual $X^{\#}$ of all $x^{\#}$ such that $x^{\#}(u) = 1$. Let Ω be the Stone-Cech compactification of Ω_0 endowed with the weak* topology induced by X. Let $\hat{C}(\Omega)$ be the continuous functions on Ω with values in the two-point compactification $[-\infty, \infty]$ of $\mathbb{R}$. We then define a map $S : X \to \hat{C}(\Omega)$ by letting Sx be the extension of the continuous map $\hat{x} : \Omega_0 \to \mathbb{R}$ given by $\hat{x}(x^{\#}) = x^{\#}(x)$. Note that S has the following linearity property:

$$S\left(\sum_{k=1}^{n}\alpha_k x_k\right)(\omega) = \sum_{k=1}^{n}\alpha_k Sx_k(\omega) \qquad \text{if } \max_{1\le k\le n}|Sx_k(\omega)| < \infty, \quad \omega \in \Omega.$$

Now consider in $C(\Omega)$ (the space of continuous real-valued functions on Ω) the convex hull K of the set of functions $1 - \min(\sigma, |Sx|)$ for $\|x\| \le \frac{1}{2}C^{-1}$. We claim that K does not meet the open negative cone of all $f \in C(\Omega)$ such that $f < 0$ everywhere. Indeed if it does there exist $x_1, \dots, x_n$ with $\|x_k\| < \frac{1}{2}C^{-1}$ such that

$$\frac{1}{n}\sum_{k=1}^{n}(1 - \min(\sigma, |Sx_k(\omega)|)) < 0 \qquad \omega \in \Omega.$$

However by assumption there exists $A \subset \{1, 2, \dots, n\}$ with $|A| > n\delta$ such that $u \notin \operatorname{co}\{\pm 2x_k : k \in A\}$. In particular there exists $x^{\#} \in \Omega_0$ with $|x^{\#}(2x_k)| < 1$ for $k \in A$. Thus

$$\begin{aligned}\sum_{k=1}^{n}(1 - \min(\sigma, |Sx_k(x^{\#})|)) &\ge \frac{1}{2}|A| + (1-\sigma)(n - |A|)\\ &= \left(\sigma - \frac{1}{2}\right)|A| - n(\sigma - 1)\\ &\ge \frac{1}{2}\delta^2 n.\end{aligned}$$

This gives a contradiction. Thus K does not meet the open negative cone and by the Hahn-Banach theorem, we can find a probability measure μ on Ω such that

$$\int (1 - \min(\sigma, |Sx(\omega)|)d\mu \ge 0, \qquad \|x\| \le \frac{1}{2}C^{-1}.$$

Next we inductively construct a sequence $(E_n)_{n=1}^{\infty}$ of disjoint Borel subsets of Ω and a sequence $x_n \in X$ with $\|x_n\| \le 1$. Let $F_0 = \emptyset$ and $F_n = E_1 \cup \dots \cup E_n$. Then if $(E_k)_{k<n}$ have been selected let b_n be the supremum of all t such that there exists a Borel set A with $\mu(A) = t$ disjoint from F_{n-1} and $x \in X$ with $\|x\| \le 1$ such that $|Sx| \ge M\mu(A)^{-1/p}$ on A. If no such t exists we set $b_n = 0$. Then select E_k with $\mu(E_n) = a_n > \frac{1}{2}b_n$ and x_n with $\|x_n\| \le 1$ such that $|Sx_k| \ge Ma_n^{-1/p}$ on E_n. If $b_n = 0$ we put $E_n = \emptyset$ and $x_n = 0$.

For fixed n we consider $\xi(\omega') = \theta\sum_{k=1}^{n} g_k(\omega')a_k^{1/p}x_k$. Then by p-normability of X

$$\|\xi(\omega')\|^p \le \theta^p\sum_{k=1}^{n} a_k|g_k(\omega')|^p,$$

and so

$$\mathbb{E}\|\xi\|^p \le c_p\theta^p.$$

It follows that

$$\mathbb{P}\{\|\xi\| \ge (2C)^{-1}\} \le (2C)^p c_p\theta^p < \frac{1}{4},$$

and hence

$$\mathbb{E}\int \min(\sigma, |S\xi|)d\mu < 1 + \frac{1}{4}(\sigma - 1) = 1 + \frac{1}{8}\delta.$$

Now fix $\omega \in \Omega$. If $\max_{1\le k\le n} |Sx_k(\omega)| = \infty$ then $\theta \sum_{k=1}^n g_k Sx_k(\omega)$ is finite only on a set of probability zero (when $(g_1, g_2, \dots, g_n)$ belongs to a certain proper linear subspace of $\mathbb{R}^n$). If $\omega \in F_n$ and $\max |Sx_k(\omega)| < \infty$ then $S\xi(\omega)$ is gaussian with variance $\theta^2 \sum_{k=1}^n a_k^{2/p} |Sx_k(\omega)|^2 \ge M^2\theta^2$. Hence

$$\mathbb{P}\{|S\xi(\omega)| > \sigma\} \ge \mathbb{P}\{|g_1| > \sigma\theta^{-1}M^{-1}\}.$$

Thus if $\omega \in F_n$, in view of the choice of M, θ and σ

$$\mathbb{E}\min(\sigma, |S\xi(\omega)|) \ge 1 + \frac{1}{4}\delta.$$

Hence

$$\left(1 + \frac{1}{4}\delta\right)\mu(F_n) \le 1 + \frac{1}{8}\delta.$$

We conclude that $\mu(F_n) \le 1 - \frac{\delta}{16}$.

Let $B = \Omega \setminus \cup_{k=1}^\infty E_k$. Then $\mu(B) \ge \delta/16$. It is clear that for any $x \in X$, $|Sx(\omega)| < \infty$ μ–a.e. on B and further if $\|x\| = 1$ then $\|Sx\chi_B\|_{p,\infty} \le M$. Hence the linear map $T_0 : X \to L_{p,\infty}(\Omega, \mu)$ defined as $T_0x = Sx\chi_B$ is bounded with norm M and $\|T_0u\|_{p,\infty} \ge (\delta/16)^{1/p}$. Letting $T = M^{-1}T_0$ we obtain the implication (4) implies (3) for an appropriate constant.

(3)$\Longrightarrow$(2): Clearly (3) implies that X is isomorphic to a subspace of an ℓ_∞–product of spaces of the type $L_{p,\infty}(\mu)$ and this means it is crudely finitely representable in $\ell_\infty(L_{p,\infty}(0,\infty))$ and so Proposition 2.2 gives the conclusion.

(2)$\Longrightarrow$(1): From (2) we conclude that X embeds into an ultraproduct of spaces $L_{p,\infty}(0,\infty)$ and this is easily seen to be a q-convex quasi-Banach lattice for any $0 < q < p$. (Ultraproducts of quasi-Banach spaces were apparently first considered in [**19**]; the theory is very similar to that of ultraproducts of Banach spaces). □

3. Marcinkiewicz and Lorentz spaces. In the next theorem we characterize an upper-estimate of $M_{p,w}$.

THEOREM 3.1. *For any* $0 < p, r < \infty$, $M_{p,w}$ *satisfies an upper r-estimate if and only if* $W^{r/p}(t)/t$ *is pseudo-decreasing.*

Proof. Since $M_{pr,w}$, $0 < r < \infty$, is the r-convexification of $M_{p,w}$, it is enough to conduct the proof only for $r = 1$. Suppose that $W^{1/p}(t)/t$ is pseudo-decreasing. Then there exists a concave function equivalent to $W^{1/p}$ (cf. Proposition 5.10 in [**1**]), so without loss of generality we assume that $W^{1/p}$ is concave. Consequently, for any disjoint $f_i \in M_{p,w}$, $i = 1, \dots, n$,

$$\begin{aligned}\left\|\sum_{i=1}^n f_i\right\|_M &= \sup_t W^{1/p}(d_{\sum_{i=1}^n f_i}(t))t = \sup_t W^{1/p}\left(\sum_{i=1}^n d_{f_i}(t)\right)t \\ &\le \sum_{i=1}^n \sup_t W^{1/p}(d_{f_i}(t))t = \sum_{i=1}^n \|f_i\|_M,\end{aligned}$$

which shows that $M_{p,w}$ has an upper 1-estimate. Conversely, assume $M_{p,w}$ has an upper 1-estimate, and take for $0 < s < t$, $n = [t/s]$,

$$f_i = \chi_{(\frac{(i-1)t}{2n}, \frac{it}{2n}]}, \quad i = 1, \dots, 2n.$$

Then $f_i^* = \chi_{(0,t/2n]}$ and $\|f_i\|_M = W^{1/p}(t/2n)$. Consequently

$$\left\| \sum_{i=1}^{2n} f_i \right\|_M = W^{1/p}(t) \le C \sum_{i=1}^{2n} \|f_i\|_M = C \sum_{i=1}^{2n} W^{1/p}(t/2n)$$
$$= C2nW^{1/p}(t/2n) \le C2(t/s)W^{1/p}(s),$$

whence $W^{1/p}(t)/t \le 2CW^{1/p}(s)/s$. □

REMARK 3.2. The space $M_{p,w}$ is not order continuous, and so it contains an order copy of ℓ_∞ [**12**]. Hence it does not have any finite lower estimate neither a type r for $r > 1$.

Recall that the *lower* and *upper Boyd* indices of a rearrangement invariant quasi-Banach space X on I are defined as follows:

$$p(E) = \sup\{p > 0 : \text{there exists } C > 0,\ \|D_s\| \le Cs^{1/p} \text{ for all } s > 1\},$$
$$q(E) = \inf\{q > 0 : \text{there exists } C > 0,\ \|D_s\| \le Cs^{1/q} \text{ for all } 0 < s < 1\},$$

where $D_s : X \to X$, $s > 0$, is the dilation operator defined as $D_s f(t) = f(t/s)$ if $t \in [0, \infty)$ and if $I = [0, 1]$ then $D_s f(t) = f(t/s)$ for $t \le \min(1, s)$ and $D_s f(t) = 0$ for $s < t \le 1$ [**13**, **14**, **15**].

THEOREM 3.3. *For any* $0 < p < \infty$, *the Boyd indices of* $M_{p,w}$ *are the following*

$$p(M_{p,w}) = p/\beta(W), \quad q(M_{p,w}) = p/\alpha(W).$$

Proof. Let $I = [0, \infty)$. For $s > 1$,

$$\|D_s f\|_M = \sup_t W(t)^{1/p} f^*(t/s) = \sup_u W(su)^{1/p} f^*(u),$$

and for $r > \beta(W)$, $W(su) \le Cs^r W(u)$. Hence

$$\|D_s f\|_M \le C \sup_u s^{r/p} W(u)^{1/p} f^*(u) = Cs^{r/p} \|f\|_M,$$

which yields that $p(M_{p,w}) = p/\beta(W)$. Analogously we obtain a formula for the upper index as well as for $I = [0, 1]$. □

The next result is well known [**17**], but we provide the proof here for the sake of completeness.

LEMMA 3.4. *Let* $V : \mathbb{R}_+ \to \mathbb{R}_+$, $V(0) = 0$ *and let* V *be concave. If* $\beta(V) = 1$, *then there exists a sequence* (t_n) *of positive numbers such that for every* $0 \le a \le 1$

$$\lim_{n\to\infty} \frac{V(at_n)}{V(t_n)} = a.$$

Proof. Since $\beta(V) = 1$, for all $q < 1$

$$\inf_{\substack{t>0,\\ 0<a<1}} \frac{V(at)}{a^q V(t)} = 0.$$

Then, setting $U(t) = \frac{V(t)}{t}$, U is decreasing and for all $0 < \varepsilon < 1$

$$\inf_{\substack{t>0,\\ 0<a<1}} \frac{a^\varepsilon U(at)}{U(t)} = 0.$$

It follows that for every $\delta > 0$

$$\inf_{t>0} \frac{U(\delta t)}{U(t)} \leq 1.$$

Indeed, if the above condition does not hold then there exists $\delta > 0$ such that

$$\theta := \inf_{t>0} \frac{U(\delta t)}{U(t)} > 1.$$

Hence $\delta < 1$ and setting $\varepsilon = -\log\theta/\log\delta$, $\delta^{n\varepsilon} = \theta^{-n}$ for every $n \in \mathbb{N}$. For any $0 < a < 1$ there exists $n \in \mathbb{N} \cup \{0\}$ such that $\delta^{n+1} \leq a < \delta^n$. Thus for any $t > 0$ it holds

$$\frac{a^\varepsilon U(at)}{U(t)} \geq \frac{(\delta^{n+1})^\varepsilon U(\delta^n t)}{U(t)} = \theta^{-n}\delta \frac{U(\delta^n t)}{U(\delta^{n-1} t)} \cdot \dots \cdot \frac{U(\delta t)}{U(t)} \geq \theta^{-n}\delta\theta^n = \delta > 0,$$

which is a contradiction. Therefore for every $\delta = 1/n$, $n \in \mathbb{N}$, there exists t_n such that

$$1 \leq \frac{U\left(\frac{1}{n} t_n\right)}{U(t_n)} < 1 + \frac{1}{n},$$

which implies that

$$1 \leq \frac{U(at_n)}{U(t_n)} \leq 1 + \frac{1}{n}$$

for $0 < a < 1$ and sufficiently large $n \in \mathbb{N}$. Therefore $\frac{U(at_n)}{U(t_n)} \to 1$, and hence for all $0 < a < 1$, $\frac{V(at_n)}{V(t_n)} \to a$ as $n \to \infty$. □

THEOREM 3.5. *If $M_{p,w}$, $0 < p < \infty$, satisfies an upper 1-estimate and $\beta(W) \geq p$, then $L_{1,\infty}(0,1)$ is finitely representable in $M_{p,w}$ (i.e. $M_{p,w}$ contains uniformly copies of $\ell^n_{1,\infty}$). In particular, $M_{p,w}$ does not have type 1.*

Proof. We give the proof only for $I = [0, \infty)$. By Theorem 3.1, $W^{1/p}(t)/t$ is pseudo-decreasing. Then $W^{1/p}$ is equivalent to a concave function V (cf. Proposition 5.10 in [**1**]), that is

$$C^{-1} V(t) \leq W^{1/p}(t) \leq CV(t)$$

for some $C > 0$ and all $t > 0$. Since $\beta(W) \geq p$, so $\beta(W^{1/p}) = \beta(V) \geq 1$. Then by Lemma 3.4, there exists a sequence $(b_j) \subset (0, \infty)$ such that

$$\lim_{j\to\infty} \frac{V(tb_j)}{V(b_j)} = t, \quad t \in [0, 1].$$

Letting

$$f_{i,j}^{(n)} = \frac{n}{W^{1/p}(b_j)} D_{b_j} \chi_{[\frac{i-1}{n}, \frac{i}{n})}, \quad i = 1 \ldots, n,\ j \in \mathbb{N},$$

for any $x = (\alpha_i)_{i=1}^n$ in n-dimensional vector space define a linear operator T_j as

$$T_j x = \sum_{i=1}^n \alpha_i f_{i,j}^{(n)}.$$

Then setting

$$f(t) = \sum_{i=1}^n n\alpha_i \chi_{[\frac{i-1}{n}, \frac{i}{n})}(t), \quad t > 0,$$

we have for $(\alpha_i^*)_{i=1}^n$, a decreasing permutation of $(\alpha_i)_{i=1}^n$, and for all $j \in \mathbb{N}$,

$$\begin{aligned} \|T_j x\|_M &= \sup_t W^{1/p}(t) \left(\sum_{i=1}^n \alpha_i f_{i,j}^{(n)}(t) \right)^* = \sup_t W^{1/p}(t) (W^{-1/p}(b_j) D_{b_j} f(t))^* \\ &= \sup_t \frac{W^{1/p}(b_j t)}{W^{1/p}(b_j)} f^*(t) = \max_{i=1,\ldots,n} \left\{ n\alpha_i^* \frac{W^{1/p}(b_j i/n)}{W^{1/p}(b_j)} \right\}. \end{aligned}$$

Hence for every $x = (\alpha_i)_{i=1}^n$ we have

$$\begin{aligned} C^{-1} \|x\|_{1,\infty} &\leq \liminf_j \|T_j x\|_M \leq \limsup_j \|T_j x\|_M \\ &\leq C \limsup_j \max_{i=1,\ldots,n} \left\{ n\alpha_i^* \frac{V(b_j i/n)}{V(b_j)} \right\} = C\|x\|_{1,\infty}. \end{aligned}$$

Notice also that for all $x \in \ell_{1,\infty}^n$, $j \in \mathbb{N}$,

$$\|T_j x\|_M \leq nC^2 \max_{i=1,\ldots,n} \{i\alpha_i^*\} = nC^2 \|x\|_{1,\infty}.$$

Recall now that since $\|\cdot\|_M$ is a quasi-norm, by the Aoki-Rolewicz theorem [**9**], there exists $0 < r < 1$ such that $\|\cdot\|_M$ is r-norm. Letting

$$|||g|||_M^r = \inf \left\{ \sum_{i=1}^m \|g_i\|_M^r : g = \sum_{i=1}^m g_i \right\},$$

we get for some $D > 0$ and all $g \in M_{p,w}$

$$|||g|||_M^r \leq \|g\|_M^r \leq D\, |||g|||_M^r\,,$$

and $|||g_1+g_2|||_M^r \le |||g_1|||_M^r + |||g_2|||_M^r$ for all $g_1, g_2 \in M_{p,w}$. Clearly, $|\,|||g_1|||_M^r - |||g_2|||_M^r| \le |||g_1-g_2|||_M^r$. Therefore for every $x \in \ell_{1,\infty}^n$,

$$|||T_jx|||_M^r \le \|T_jx\|_M^r \le C^{2r}n^r\|x\|_{1,\infty}^r,$$

and for every $x, y \in \ell_{1,\infty}^n$,

$$|\,|||T_jx|||_M^r - |||T_jy|||_M^r| \le |||T_jx - T_jy|||_M^r \le C^{2r}n^r\|x-y\|_{1,\infty}^r.$$

Thus the family $(|||T_jx|||_M^r)$ is equi-continuous and uniformly bounded on the unit ball $B_{\ell_{1,\infty}^n}$, and so by the Arzeli-Ascoli theorem it is compact in the space $C(B_{\ell_{1,\infty}^n})$. Thus there exists a subsequence $(j_k) \subset \mathbb{N}$ such that $\lim_k |||T_{j_k}x|||_M^r \in C(B_{\ell_{1,\infty}^n})$. Hence for arbitrary small $\epsilon > 0$ and every $n \in \mathbb{N}$ there exists $j(n) \in \mathbb{N}$ such that for all $x \in \ell_{1,\infty}^n$,

$$\lim_k \left|\left|\left| T_{j_k}x \right|\right|\right|_M^r - \epsilon \le \left|\left|\left| T_{j(n)}x \right|\right|\right|_M^r \le \lim_k \left|\left|\left| T_{j_k}x \right|\right|\right|_M^r + \epsilon.$$

Thus for any $x \in B_{\ell_{1,\infty}^n}$,

$$\begin{aligned}\left\| T_{j(n)}x \right\|_M^r &\le D \left|\left|\left| T_{j(n)}x \right|\right|\right|_M^r \le D \lim_k \left|\left|\left| T_{j_k}x \right|\right|\right|_M^r + D\epsilon \\ &\le D \limsup_j \|T_jx\|_M^r + D\epsilon \le DC^r\|x\|_{1,\infty}^r + D\epsilon,\end{aligned}$$

and

$$\begin{aligned}\left\| T_{j(n)}x \right\|_M^r &\ge \left|\left|\left| T_{j(n)}x \right|\right|\right|_M^r \ge \lim_k \left|\left|\left| T_{j_k}x \right|\right|\right|_M^r - \epsilon \\ &\ge D^{-1} \liminf_j \|T_jx\|_M^r - \epsilon \ge D^{-1}C^{-r}\|x\|_{1,\infty}^r - \epsilon.\end{aligned}$$

It is clear now that there exists $A > 0$ such that for all $n \in \mathbb{N}$ and $x \in \ell_{1,\infty}^n$ it holds

$$A^{-1}\|x\|_{1,\infty} \le \left\| T_{j(n)}x \right\|_M \le A\|x\|_{1,\infty}.$$

This shows that $M_{p,w}$ contains uniformly copies of $\ell_{1,\infty}^n$. □

THEOREM 3.6. *Let $0 < p < \infty$. The following conditions are equivalent.*

(1) *The Hardy operator*

$$H^{(1)}f(t) = \frac{1}{t}\int_0^t f^*(s)ds \qquad 0 < t \in I,$$

is bounded in $M_{p,w}$.

(2) $\beta(W) < p$.

(3) *There exists $C > 0$ such that*

$$\int_0^t W^{-1/p} \le Ct/W^{1/p}(t) \qquad 0 < t \in I.$$

Proof. Theorem 2 in [**15**] states that $H^{(1)}$ is bounded in r.i. quasi-Banach space X if and only if $p(X) > 1$. Hence in view of Theorem 3.3 we immediately obtain the equivalence of (1) and (2). In order to show that (1) is equivalent to (3), notice that $f \in M_{p,w}$ if and only if for every $s \in I$, $f^*(s) \le CW^{-1/p}(s)$. Hence $H^{(1)}f \in M_{p,w}$

is equivalent to inequality $H^{(1)}f(t) \leq CW^{-1/p}(t)$, that is to $\int_0^t W^{-1/p} \leq Ct/W^{1/p}(t)$ for all $0 < t \in I$. □

In the next theorem we characterize the Marcinkiewicz spaces that have type 1.

THEOREM 3.7. *Let* $0 < p < \infty$. *The following conditions are equivalent.*

(1) $M_{p,w}$ *is* 1*-convex, that is the space is normable.*
(2) $M_{p,w}$ *has type* 1.
(3) $\beta(W) < p$.

Proof. It is obvious that condition (1) implies (2). Now, if we assume that $M_{p,w}$ has type 1 and $\beta(W) \geq p$ then $M_{p,w}$ satisfies an upper 1-estimate and so by Theorem 3.5 it contains copies of $\ell^n_{1,\infty}$ uniformly. Thus $M_{p,w}$ can not have type 1, and this contradiction proves the implication from (2) to (3). If (3) is satisfied, that is $\beta(W) < p$, then by Theorem 3.6, the Hardy operator $H^{(1)}$ is bounded in $M_{p,w}$. Then $\|H^{(1)}f\|_M$ is equivalent to the original quasi-norm in $M_{p,w}$. Moreover, $\|H^{(1)}f\|_M$ is a norm on $M_{p,w}$ since it satisfies the triangle inequality in view of the subadditivity of the operator $H^{(1)}$. Thus we showed that (1) holds, and the proof is completed. □

REMARK 3.8. By Theorem 3.6 we see that the condition $\beta(W) < p$ is equivalent to the integral inequality (3). It is well known (cf. Theorem A in [**11**] and references there) that "$\beta(W) < p$" is also equivalent to another integral inequality, namely the B_p-condition [**18**], that is for all $t \in I\backslash\{0\}$ and some $C > 0$

$$\int_t^\infty s^{-p} w(s)\,ds \leq Ct^{-p} \int_0^t w.$$

Soria (Theorem 3.1 in [**20**]) proved that $M_{p,w}$ is normable if and only if w satisfies the B_p-condition.

For any $0 < r < \infty$, the r-convexification of $M_{p,w}$ is $M_{pr,w}$. Hence we get the following corollary.

COROLLARY 3.9. *For any* $0 < p, r < \infty$, *the space* $M_{p,w}$ *is r-convex if and only if* $\beta(W) < p/r$.

REMARK 3.10. As a simple conclusion we also have that $M_{p,w}$ is L-convex (for definition of L-convexity see [**7**]).

COROLLARY 3.11. *Let* $0 < p < \infty$ *and* $0 < r < 1$. *Then the following conditions are equivalent.*

(1) $M_{p,w}$ *has type r.*
(2) *The quasi-norm* $\|\cdot\|_M$ *in* $M_{p,w}$ *is equivalent to an r-norm.*
(3) $W^{r/p}(t)/t$ *is pseudo-decreasing.*

Proof. The equivalence of (1) and (2) is a result of Theorem 4.2 in [**6**]. By the Kalton's result (Theorem 2.3 (ii) in [**7**]) it follows also that for $0 < r < 1$, if a quasi-normed space $(X, \|\cdot\|)$ is L-convex, then $\|\cdot\|$ is an r-norm if X satisfies an upper r-estimate. This and Theorem 3.1 provide the equivalence of the last two conditions. □

We end the paper with conditions on when type 1 and upper 1-estimate are equivalent in quasi-Banach lattices. We then illustrate the obtained result in Lorentz spaces, providing examples of Lorentz spaces with type 1 that are not normable.

THEOREM 3.12. *Let X be a quasi-Banach lattice that is r-convex and q-concave for some $0 < r < q < \infty$. Then X has type 1 if and only if it satisfies an upper 1-estimate.*

Proof. By the Khintchine's inequality for scalars [**14**], for any $0 < s < \infty$, and $x_i \in X$, $i = 1, \dots, n$, we have for some $A_s, B_s > 0$,

$$A_s\left(\sum_{i=1}^n |x_i|^2\right)^{1/2} \le \left(\int_0^1 \left|\sum_{i=1}^n r_i(t)x_i\right|^s dt\right)^{1/s} \le B_s\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}.$$

Then by the monotonicity of the quasi-norm and its r-convexity and q-concavity, we get the following generalized Khintchine's inequality in X

$$\begin{aligned} A_r\left\|\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}\right\| &\le \left\|\left(\int_0^1 \left|\sum_{i=1}^n r_i(t)x_i\right|^r dt\right)^{1/r}\right\| \le C^{(r)}\left(\int_0^1 \left\|\sum_{i=1}^n r_i(t)x_i\right\|^r dt\right)^{1/r} \\ &\le C^{(r)}\left(\int_0^1 \left\|\sum_{i=1}^n r_i(t)x_i\right\|^q dt\right)^{1/q} \le C^{(r)}C_{(q)}\left\|\left(\int_0^1 \left|\sum_{i=1}^n r_i(t)x_i\right|^q dt\right)^{1/q}\right\| \\ &\le C^{(r)}C_{(q)}B_q\left\|\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}\right\|. \end{aligned}$$

Assuming now that X satisfies an upper 1-estimate, we get by Lemma 2.1 in [**7**] that for some $C > 0$ and any $x_i \in X$, $i = 1, \dots, n$,

$$\left\|\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}\right\| \le C\sum_{i=1}^n \|x_i\|.$$

Hence by the generalized Khintchine's inequality and the Kahane's inequality [**6**],

$$\int_0^1 \left\|\sum_{i=1}^n r_i(t)x_i\right\| dt \le B_1\left\|\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}\right\| \le B_1 C\sum_{i=1}^n \|x_i\|,$$

which finishes the proof. □

Applying now Theorem 3.12 and well known characterizations of convexity and concavity of $\Lambda_{p,w}$ [**11**] we get the following description of $\Lambda_{p,w}$ with type 1.

THEOREM 3.13. *Let w be a weight function such that $0 < \alpha(W) \le \beta(W) < \infty$. Then the following conditions are equivalent.*

(1) *$\Lambda_{p,w}$ has type 1.*
(2) *$\Lambda_{p,w}$ satisfies an upper 1-estimate.*
(3) *$W(t)/t^p$ is pseudo-decreasing and $p \ge 1$.*

Proof. By the assumption $0 < \alpha(W) \le \beta(W) < \infty$ it follows by Theorems 2 and 6 in [**11**] that $\Lambda_{p,w}$ is r-convex and q-concave for some $0 < r < q < \infty$. Applying now Theorem 3.12, the conditions (1) and (2) are equivalent. The equivalence of (2) and (3) is a direct consequence of a characterization of upper 1-estimate of $\Lambda_{p,w}$ (Theorem 3 in [**11**]). □

REMARK 3.14. The characterization of the Lorentz spaces $\Lambda_{p,w}$ with type 1 differs substantially from that of $M_{p,w}$. There are Lorentz spaces with type 1 that are not normable. By Theorem A in **[11]**, $\Lambda_{p,w}$, $1 < p < \infty$, is normable if and only if $\beta(W) < p$. Now, letting $w(t) = t^{p-1}$, $1 < p < \infty$, we have $W(t) = t^p/p$, and so $\beta(W) = p$ and $W(t)/t^p$ is pseudo-decreasing. Thus the space $L_{1,p} := \Lambda_{p,w}$ is not normable (see also **[3]**), but it has type 1 by Theorem 3.13.

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