TYPE AND ORDER CONVEXITY OF MARCINKIEWICZ AND LORENTZ SPACES AND APPLICATIONS

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Abstract. We consider order and type properties of Marcinkiewicz and Lorentz function spaces. We show that if $0 < p < 1$, a $p$-normable quasi-Banach space is natural (i.e. embeds into a $q$-convex quasi-Banach lattice for some $q > 0$) if and only if it is finitely representable in the space $L_{p,\infty}$. We also show in particular that the weak Lorentz space $L_{1,\infty}$ do not have type 1, while a non-normable Lorentz space $L_{1,p}$ has type 1. We present also criteria for upper $r$-estimate and $r$-convexity of Marcinkiewicz spaces.

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1. Introduction. In this note we study the order convexity and type of Marcinkiewicz and Lorentz function spaces. The space weak $L_{p}$ or $L_{p,\infty}$ is well-known to be $p$-normable if $0 < p < 1$, but is $q$-convex as a lattice when $0 < q < p$ (see [4] and [5]). We prove that a $p$-normable quasi-Banach space $X$ embeds into a $p$-normable quasi-Banach lattice which is $r$-convex for some $r > 0$ (i.e. $X$ is natural) if and only if $X$ is finitely representable in $L_{p,\infty}(0, 1)$.

We then consider more general Lorentz and Marcinkiewicz spaces. In [6] it was proved that if a quasi-Banach space $(X, \|\cdot\|)$ has type $0 < p < 1$, then $\|\cdot\|$ is a $p$-norm, and if $X$ has type $p > 1$ then $X$ is normable. It was also shown that there exist quasi-Banach spaces that have type 1, but they are not nornable. In this note we show that Marcinkiewicz spaces have type 1 if and only if they are 1-convex (that is normable), while the class of Lorentz spaces with type 1 coincides to the class of those spaces satisfying an upper 1-estimate. In consequence, there exist Lorentz spaces with type 1 that are not nornable.

Let us start with basic definitions and notation. Let $\mathbb{R}$, $\mathbb{R}_{+}$ and $\mathbb{N}$ denote the sets of all real, nonnegative real and natural numbers, respectively. Let $r_{n} : [0, 1] \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, be Rademacher functions, that is $r_{n}(t) = \text{sign} (\sin 2^{n}\pi t)$. A quasi-Banach space $X$ has type $0 < p \leq 2$ if there is a constant $K > 0$ such that, for any choice of finitely many...
vectors \(x_1, \ldots, x_n\) from \(X\),
\[
\int_0^1 \left\| \sum_{k=1}^n r_k(t)x_k \right\| \, dt \leq K \left( \sum_{k=1}^n \|x_k\|^p \right)^{1/p},
\]
and it has cotype \(q \geq 2\) if there is a constant \(K > 0\) such that for any finite collection of elements \(x_1, \ldots, x_n\) from \(X\),
\[
\left( \sum_{k=1}^n \|x_k\|^q \right)^{1/q} \leq K \int_0^1 \left\| \sum_{k=1}^n r_k(t)x_k \right\| \, dt.
\]
Recall also that a quasi-norm \(\|\cdot\|\) in \(X\) is a \(p\)-norm, \(0 < p < 1\), if there exists \(C > 0\) such that for any \(x_i \in X, i = 1, \ldots, n\)
\[
\|x_1 + \cdots + x_n\| \leq C(\|x_1\|^p + \cdots + \|x_n\|^p)^{1/p}.
\]
By the Aoki-Rolewicz theorem [9], for any quasi-norm \(\|\cdot\|\) there exists \(0 < p < 1\) such that \(\|\cdot\|\) is a \(p\)-norm. We say that a quasi-Banach space \((X, \|\cdot\|)\) is normable whenever there exists a norm \(\|\cdot\|\) in \(X\) such that \(C^{-1}\|x\| \leq \|x\| \leq C\|x\|\) for all \(x \in X\) and some \(C > 0\).

A quasi-Banach lattice \(X = (X, \|\cdot\|)\) is said to be \(p\)-convex, \(0 < p < \infty\), respectively \(p\)-concave, \(0 < p < \infty\), if there are positive constants \(C^{(p)}\) and \(C_{(p)}\) such that
\[
\left\| \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \right\| \leq C^{(p)} \left( \sum_{i=1}^n \|x_i\|^p \right)^{1/p},
\]
respectively,
\[
\left( \sum_{i=1}^n \|x_i\|^p \right)^{1/p} \leq C_{(p)} \left\| \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \right\|,
\]
for every choice of vectors \(x_1, \ldots, x_n \in X\). We also say that \(X\) satisfies an upper \(p\)-estimate, \(0 < p < \infty\), respectively a lower \(p\)-estimate, \(0 < p < \infty\), if the definition of \(p\)-convexity, respectively \(p\)-concavity, holds true for any choice of disjointly supported elements \(x_1, \ldots, x_n\) in \(X\) ([6, 14]). We notice here that a quasi-Banach lattice is normable if and only if it is 1-convex. However, while a \(p\)-normable quasi-Banach lattice necessarily has an upper \(p\)-estimate, it may fail to be \(q\)-convex for any choice of \(q > 0\). Motivated by this, the first author [8] defined a quasi-Banach space \(X\) to be natural if it is isomorphic to a subspace of a quasi-Banach lattice which is \(q\)-convex for some \(q > 0\).

Let us recall that a quasi-Banach space \(X\) is said to be (crudely) finitely representable in a quasi-Banach space \(Y\) if there is a constant \(C\) so that for every \(\epsilon > 0\) and every finite-dimensional subspace \(F\) of \(Y\) there is a finite-dimensional subspace \(G\) of \(Y\) and an isomorphism \(T : F \to G\) such that \(\|T\|\|T^{-1}\| < C + \epsilon\). If \(C = 1\) we say that \(X\) is finitely representable in \(Y\).

A function \(U : I \to \mathbb{R}_+\), where \(I = [0, 1]\) or \(I = [0, \infty)\), is said to be pseudo-increasing (resp. pseudo-decreasing) whenever there exists \(C > 0\) such that \(U(s) \leq CU(t)\) (resp. \(U(s) \geq CU(t)\)) for all \(0 \leq s < t\). We say that the expressions \(A\) and \(B\) are
equivalent, whenever \( A/B \) is bounded above and below by positive constants. Given a function \( U : I \to \mathbb{R}_+ \), we define the lower and upper Matuszewska-Orlicz indices [13, 14] as follows:

\[
\alpha(U) = \sup\{p \in \mathbb{R} : U(as) \leq C a^p U(s) \text{ for some } C > 0 \text{ and all } s \in I, 0 < a \leq 1\},
\]

\[
\beta(U) = \inf\{p \in \mathbb{R} : U(as) \leq C a^p U(s) \text{ for some } C > 0 \text{ and all } as \in I, a \geq 1\}.
\]

If \( U \) and \( V \) are equivalent, then their corresponding indices coincide.

If \( f \) is a real-valued measurable function on \( I \), then we define the distribution function of \( f \) by \( d_f(\theta) = \lambda(\{|f| > \theta\}) \) for each \( \theta \geq 0 \), where \( \lambda \) denotes the Lebesgue measure on \( I \). The non-increasing rearrangement of \( f \) is defined by

\[
f^*(t) = \inf\{s > 0 : d_f(s) \leq t\}, \quad t \in I.
\]

A positive, Lebesgue measurable function \( w : I \to (0, \infty) \) is called a weight function whenever

\[
W(t) := \int_0^t w(s) \, ds = \int_0^t w < \infty,
\]

for all \( t \in I \). We shall always assume here that \( W \) satisfies condition \( \Delta_2 \), that is for some \( K > 0 \) and all \( t \in I \),

\[
W(t) \leq K W(t/2).
\]

Given a weight \( w \), the Marcinkiewicz space \( M_{p,w} \), \( 0 < p < \infty \), also called the weak Lorentz space, is the set of all Lebesgue measurable functions \( f : I \to \mathbb{R} \) such that

\[
\|f\|_M := \sup_t W^{1/p}(t)f^*(t) = \sup_t W^{1/p}(d_f(t)) t < \infty.
\]

The functional \( \|\cdot\|_M \) is a quasi-norm and \( (M_{p,w}, \|\cdot\|_M) \) is a quasi-Banach space. In the case when \( W(t) = t \), we will denote it by \( L_{p,\infty} \). As usual \( L_{p,\infty}(0, 1) \) or \( L_{p,\infty}(0, \infty) \) will denote the spaces on \([0, 1]\) or \([0, \infty)\), respectively. Recall also that the Marcinkiewicz sequence space \( \ell_{p,\infty} \), \( 0 < p < \infty \), consists of all sequences \( x = (\alpha_n) \subset c_0 \) such that \( \|x\|_{p,\infty} = \sup_n \{n^{1/p} \alpha_n^p\} < \infty \), where \( \{\alpha_n^p\} \) is a decreasing permutation of \( \{\alpha_n\} \). It is well-known that \( L_{p,\infty} \) or \( \ell_{p,\infty} \) is \( q \)-convex whenever \( 0 < q < p \) [4], but is not \( p \)-convex.

Given a weight function \( w \) with \( \int_0^\infty w = \infty \) if \( I = [0, \infty) \), recall that the Lorentz space \( \Lambda_{p,w} \), \( 0 < p < \infty \), consists of all real-valued Lebesgue measurable functions \( f \) on \( I \) such that

\[
\|f\|_{\Lambda} := \left( \int_I f^{q/p} w \right)^{1/p} < \infty.
\]

It is well known that \( (\Lambda_{p,w}, \|\cdot\|_{\Lambda}) \) is a quasi-Banach space [3, 11].

Observe that the condition \( \Delta_2 \) imposed on \( W \) is necessary in the context of this paper. In fact for \( W \) positive on \((0, \infty)\), the spaces \( M_{p,w} \) or \( \Lambda_{p,w} \) are linear if and only if \( W \) satisfies condition \( \Delta_2 \) ([2]). It is also not difficult to verify that the \( \Delta_2 \)-condition of \( W \) is necessary and sufficient for \( \|\cdot\|_M \) or \( \|\cdot\|_{\Lambda} \) to be a quasi-norm (cf. [11, 18]).
2. Finite representability in $L_{p,\infty}$

**Proposition 2.1.** Suppose that $0 < p \leq 1$ and that $F$ is a finite-dimensional subspace of $L_{p,\infty}(0, \infty)$. Then, given $\epsilon > 0$, there exists a measurable subset $B$ of $(0, \infty)$ of finite measure so that:

$$\|f \chi_B\|_{p,\infty} > (1 - \epsilon)\|f\|_{p,\infty}, \quad f \in F$$

and so for a suitable constant $K = K(F, \epsilon)$ we have

$$\|f \chi_B\|_{\infty} \leq K\|f\|_{p,\infty}, \quad f \in F.$$

**Proof.** Fix $\delta > 0$ so small that $(1 - 2\delta)(1 - 2\delta^p)^{1/p} > 1 - \epsilon$. Let $\{f_1, \ldots, f_n\}$ be a $\delta^2$-net for the set $\{f \in F : \|f\|_{p,\infty} = 1\}$. For each $1 \leq k \leq n$ there exists $t_k$ so that

$$f_k^*(t_k) \geq (1 - \delta)t_k^{-1/p}.$$

Let $h = \max_{1 \leq k \leq n}|f_k|$ so that

$$\|h\|_{p,\infty} \leq \sup_{t} t^{1/p} \left(\sum_{k=1}^{n}|f_k|(t)\right)^* \leq \sup_{t} t^{1/p} \sum_{k=1}^{n} f_k^*(t/n) \leq n^{1/p}.$$

Choose $M$ so large that $M > n^{1/p}\delta^{-1}t_k^{-1/p}$ for $1 \leq k \leq n$ and $\frac{1}{M} < (1 - \delta)t_k^{-1/p}$ for $1 \leq k \leq n$. Now let $B = \{s : M^{-1} \leq h(s) \leq M\}$. $B$ is clearly of finite measure. Furthermore if $f \in F$ with $\|f\|_{p,\infty} = 1$ then $f$ can be expressed as a series $\sum_{k=0}^{\infty} \alpha_k f(k)$ where $|\alpha_k| \leq \delta 2^k$. Hence $\|f \chi_B\|_{\infty} \leq (1 - \delta^2)^{-1}M$. Thus the second condition is fulfilled.

Now if $\|f\|_{p,\infty} = 1$ choose $f_k$ so that $\|f - f_k\|_{p,\infty} \leq \delta^2$. Then the set $D = \{s : |f_k(s)| \geq (1 - \delta)t_k^{-1/p}\}$ has measure at least $t_k$. Clearly $h(t) \geq |f_k(t)| \geq (1 - \delta)t_k^{-1/p} \geq \frac{1}{M}$ for $t \in D$. Hence if $t \in D \setminus B$ then $h(t) > M$ and so $n^{1/p} \geq \|h\|_{p,\infty} \geq \lambda(|h| > M)^{1/p} M \geq \lambda(D \setminus B)^{1/p} M$, which yields that $\lambda(D \setminus B) \leq M^{-p} n \leq \delta^p t_k$. Thus $\lambda(D \cap B) \geq (1 - \delta^p)t_k$.

In view of the choice of $f_k$ we have $\lambda\{|f - f_k| > \delta t_k^{-1/p}\} < \delta^p \delta^{-p} t_k = \delta^p t_k$. Now, if $|f(t) - f_k(t)| \leq \delta t_k^{-1/p}$ and $|f_k(t)| \geq (1 - \delta)t_k^{-1/p}$ then $|f(t)| \geq (1 - 2\delta)t_k^{-1/p}$ and so

$$\lambda\{|f \chi_B| \geq (1 - 2\delta)t_k^{-1/p}\} \geq \lambda\{|f - f_k| \leq \delta t_k^{-1/p}\} \cap B \cap D \geq (1 - 2\delta^p)t_k.$$

Thus

$$\|f \chi_B\|_{p,\infty} \geq (1 - 2\delta)(1 - 2\delta^p)^{1/p}. \quad \square$$

**Proposition 2.2.** Suppose that $0 < p \leq 1$. The space $\ell_\infty(L_{p,\infty}(0, \infty))$ is finitely representable in $L_{p,\infty}(0, 1)$.

**Proof.** It is enough to prove that if $F$ is a finite-dimensional subspace of $L_{p,\infty}(0, \infty)$ and $n \in \mathbb{N}$ then for any $\epsilon > 0$, $\ell_0^n(F)$ ($1 + \epsilon$)-embeds into $L_{p,\infty}(0, 1)$. By Proposition 2.1 we can find a constant $K$ and an embedding $T : F \to L_{p,\infty}(0, 1)$ such that

- $\|T\| \leq 1$,
- $\|Tf\|_{p,\infty} \geq (1 - \epsilon)\|f\|_{p,\infty}$ for $f \in F$ and
- $\|Tf\|_{\infty} \leq K\|f\|_{p,\infty}$ for $f \in F$.

Pick $\delta > 0$ so that $(1 - \delta)^{-1} < (1 + \epsilon)^p$. Let $a_1 > a_2 > \cdots > a_n > 0$ be chosen so that $\sum_{j=1}^{n} a_j < 1$ and $a_{j+1} < K^{-p} \delta a_j$ for $j = 1, 2, \ldots, n - 1$. Now for $j = 1, 2, \ldots, n$ let
Let $B_j$ be disjoint Borel subsets of $(0, 1)$ of measure $a_j$. For each $j$ there is an embedding
\[ T_j : F \to L_{p, \infty}(B_j) = \{ f \chi_{B_j} : f \in L_{p, \infty}(0, \infty) \} \]
with
\begin{itemize}
  \item $\| T_j \| \leq 1$,
  \item $\| T_j f \|_{p, \infty} \geq (1 - \varepsilon) \| f \|_{p, \infty}$ for $f \in F$ and
  \item $\| T_j f \|_{p, \infty} \leq K a_j^{-1/p} \| f \|_{p, \infty}$ for $f \in F$.
\end{itemize}
Here $(T_1, \ldots, T_n)$ are obtained by dilating and translating the embedding $T$. Now if $f_1, \ldots, f_n \in F$ with $\max_j \| f_j \|_{p, \infty} = 1$ we have
\[ \lambda \left( \left| \sum_{j=1}^n T_j f_j \right| > r \right) = \sum_{j=1}^n \lambda (|T_j f_j| > r) \]
\[ = \sum_{a_j \leq K r^{-p}} \lambda (|T_j f_j| > r) \]
\[ \leq \sum_{a_j \leq K r^{-p}} \min(a_j, r^{-p}). \]
Assuming this sum is nonempty let $k$ be the first index such that $a_k \leq K^p r^{-p}$. Then we may estimate it by
\[ r^{-p} + \sum_{k < j \leq n} a_j \leq r^{-p} + K^p r^{-p} \sum_{j=1}^\infty (K^{-p} \delta)^j \leq (1 + \varepsilon)^p r^{-p}. \]
It follows that the map $(f_1, \ldots, f_n) \to \sum_{j=1}^n T_j f_j$ defines the required $(1 + \varepsilon)$-embedding of $\ell^n_\infty(F)$ into $L_{p, \infty}(0, 1)$. \hfill $\square$

**Proposition 2.3.** The spaces $\ell_1, \infty$ and $L_1, \infty(0, 1)$ are not of type 1.

**Proof.** It suffices to show that $L_1, \infty(0, 1)$ is not of type 1. It is well-known that $L_1, \infty(0, 1)$ is not normable and indeed that for some constant $c > 0$, there exist (see e.g. [17]) non-negative functions $f_1, \ldots, f_n \in L_1, \infty(0, 1)$ with $\| f_j \|_{1, \infty} = 1$ and
\[ \| f_1 + \cdots + f_n \|_{1, \infty} \geq cn \log n. \]
Let $F$ be a subspace spanned by $\{ f_1, \ldots, f_n \}$ and let $N = 2^n$. We consider the space $\ell^N_\infty(F)$ with co-ordinates indexed by all $n$-tuples $(\eta_1, \ldots, \eta_n)$ where $\eta_j = \pm 1$. Define $\phi_j \in \ell^N_\infty(F)$ by the coordinates $\phi_j(\eta_1, \ldots, \eta_n) = \eta_j f_j$ for $j = 1, \ldots, n$. Then for every choice of sign $\epsilon_j = \pm 1$ we have
\[ \| \epsilon_1 \phi_1 + \cdots + \epsilon_n \phi_n \| = \| f_1 + \cdots + f_n \|_{1, \infty}. \]
Since $\ell^N_\infty(F)$ embeds almost isometrically into $L_1, \infty(0, 1)$ this space fails to have type 1. \hfill $\square$

We conclude this section with a characterization of natural spaces. The technique is rather similar to that of [10], Theorem 4.2. Recall that the weak Lorentz space $L_{p, \infty}(\Omega, \mu)$ over arbitrary measure space $(\Omega, \mu)$ consists of all $\mu$-measurable real valued functions $f$ such that $\| f \|_{p, \infty} = \sup_{t \geq 0} \mu(\{|f| > t\}^{1/p} t < \infty$.

**Theorem 2.4.** Suppose that $0 < p < 1$ and that $X$ is a $p$-normable quasi-Banach space. The following conditions on $X$ are equivalent:
\begin{enumerate}
  \item $X$ is natural.
  \item $X$ is (crudely) finitely representable in $L_{p, \infty}(0, 1)$.
\end{enumerate}
There exists a constant $C$ with the property that given $x \in X$ there exists a compact Hausdorff space $\Omega$, a probability measure $\mu$ on $\Omega$ and an operator $T : X \to L_{p, \infty}(\Omega, \mu)$ such that $\|T\| \leq 1$ and $\|x\| \leq C\|Tx\|$. (4) holds for constants $C$ such that $|C| > n\delta$ then $\|y\| \leq C\max_{1 \leq k \leq n} \|x_k\|$. Proof. (1) $\implies$ (4): It is enough to show that if $X$ is a quasi-Banach lattice which is $r$-convex for some $r > 0$ then (4) holds for $X$ for every choice of $\delta$. Let us therefore fix $\delta > 0$. Thus we may assume an estimate

$$\left\| \left( \sum_{j=1}^{m} |v_j|^r \right)^{1/r} \right\| \leq M \left( \sum_{j=1}^{m} \|v_j\|^r \right)^{1/r},$$

$v_1, \ldots, v_m \in X$. Now assume $x_1, \ldots, x_n, y$ given as in the statement of (4). Then we may represent the ideal $Z$ generated by the order-interval $[-|y|, |y|]$ as an abstract M-space in the sense of Kakutani if we take $[-|y|, |y|]$ as the unit ball. It thus may be identified with a space $C(\Omega)$ in such a way that $|y(s)| = 1$ for all $s \in \Omega$. Let $u_k = |x_k| \wedge |y|$ so that $u_k$ can be identified with a continuous function on $\Omega$. Fix any $s \in \Omega$ and let $A = \{k : u_k(s) < 1\}$. Then it is clear that $y \notin \text{co}\{\pm x_k : k \in A\}$ and so by hypothesis (4), we have $|A| \leq n\delta$. Thus $|\{k : u_k(s) \geq 1\}| \geq n(1 - \delta)$.

Thus

$$\left( \sum_{j=1}^{n} |u_j|^r \right)^{1/r} \geq n^{1/r}(1 - \delta)^{1/r}|y|,$$

and so

$$n^{1/r}(1 - \delta)^{1/r}\|y\| \leq Mn^{1/r} \max_{1 \leq k \leq n} \|x_k\|,$$

i.e.

$$\|y\| \leq M(1 - \delta)^{-1/r} \max_{1 \leq k \leq n} \|x_k\|.$$ 

This establishes (4) with $C(\delta) = M(1 - \delta)^{-1/r}$. (4) $\implies$ (3): This is an argument based on Nikishin’s theorem [16]. We assume (4) holds for constants $C$ and $0 < \delta < 1$. Let $(g_n)_{n=1}^{\infty}$ be a sequence of independent normalized Gaussians defined on a probability space $(\Omega', \mathbb{P})$. Let $e_p = \mathbb{E}|g_1|^p$ and choose $\theta > 0$ so that $(2C)^pe_p\theta^p < \frac{1}{4}$. Then pick $M$ so that

$$\mathbb{P}\{|g_1| > \sigma \theta^{-1}M^{-1}\} > \frac{1 + \frac{1}{4}\delta}{1 + \frac{1}{2}\delta},$$

where $\sigma = 1 + \frac{1}{2}\delta$.

Fix $u \in X$ with $\|u\| = 1$ and then let $\Omega_0$ be the subset of the algebraic dual $X^\#$ of all $x^\#$ such that $x^\#(u) = 1$. Let $\Omega$ be the Stone-Cech compactification of $\Omega_0$ endowed with the weak* topology induced by $X$. Let $\hat{C}(\Omega)$ be the continuous functions on $\Omega$ with values in the two-point compactification $[-\infty, \infty]$ of $\mathbb{R}$. We then define a map $S : X \to \hat{C}(\Omega)$ by letting $Sz$ be the extension of the continuous map $\hat{x} : \Omega_0 \to \mathbb{R}$ given
by $\hat{x}(x^\#) = x^\#(x)$. Note that $S$ has the following linearity property:

$$S\left(\sum_{k=1}^{n} \alpha_k x_k\right)(\omega) = \sum_{k=1}^{n} \alpha_k Sx_k(\omega) \quad \text{if } \max_{1 \leq k \leq n} |Sx_k(\omega)| < \infty, \quad \omega \in \Omega.$$ 

Now consider in $C(\Omega)$ (the space of continuous real-valued functions on $\Omega$) the convex hull $K$ of the set of functions $1 - \min(\sigma, |Sx|)$ for $\|x\| \leq \frac{1}{2} C^{-1}$. We claim that $K$ does not meet the open negative cone of all $f \in C(\Omega)$ such that $f < 0$ everywhere. Indeed if it does there exist $x_1, \ldots, x_n$ with $\|x_k\| < \frac{1}{2} C^{-1}$ such that

$$\frac{1}{n} \sum_{k=1}^{n} (1 - \min(\sigma, |Sx_k(\omega)|)) < 0 \quad \omega \in \Omega.$$ 

However by assumption there exists $A \subset \{1, 2, \ldots, n\}$ with $|A| > n\delta$ such that $u \notin co \{\pm x_k : \; k \in A\}$. In particular there exists $x^* \in \Omega_0$ with $|x^*(2x_k)| < 1$ for $k \in A$. Thus

$$\sum_{k=1}^{n} (1 - \min(\sigma, |Sx_k(x^*)|)) \geq \frac{1}{2} |A| + (1 - \sigma)(n - |A|) = \left(\sigma - \frac{1}{2}\right) |A| - n(\sigma - 1) \geq \frac{1}{2} \delta^2 n.$$ 

This gives a contradiction. Thus $K$ does not meet the open negative cone and by the Hahn-Banach theorem, we can find a probability measure $\mu$ on $\Omega$ such that

$$\int (1 - \min(\sigma, |Sx(\omega)|))d\mu \geq 0, \quad \|x\| \leq \frac{1}{2} C^{-1}.$$ 

Next we inductively construct a sequence $(E_n)_{n=1}^{\infty}$ of disjoint Borel subsets of $\Omega$ and a sequence $x_n \in X$ with $\|x_n\| \leq 1$. Let $F_0 = \emptyset$ and $F_n = E_1 \cup \cdots \cup E_n$. Then if $(E_k)_{k<n}$ have been selected let $b_n$ be the supremum of all $t$ such that there exists a Borel set $A$ with $\mu(A) = t$ disjoint from $F_{n-1}$ and $x \in X$ with $\|x\| \leq 1$ such that $|Sx| \geq M\mu(A)^{1/p}$ on $A$. If no such $t$ exists we set $b_n = 0$. Then select $E_k$ with $\mu(E_n) = d_n > \frac{1}{2} b_n$ and $x_n$ with $\|x_n\| \leq 1$ such that $|Sx_k| \geq M\mu(A)^{1/p}$ on $E_n$. If $b_n = 0$ we put $E_n = \emptyset$ and $x_n = 0$.

For fixed $n$ we consider $\xi(\omega) = \theta \sum_{k=1}^{n} g_k(\omega) a_k^{1/p} x_k$. Then by $p$-normability of $X$

$$\|\xi(\omega)\| \leq \theta p \sum_{k=1}^{n} a_k |g_k(\omega)|^{1/p},$$

and so

$$E\|\xi\|^{p} \leq c_p \theta^p.$$ 

It follows that

$$\mathbb{P}\{\|\xi\| \geq (2C)^{-1}\} \leq (2C)^p c_p \theta^p < \frac{1}{4},$$

and hence

$$E \int \min(\sigma, |S\xi|)d\mu < 1 + \frac{1}{4}(\sigma - 1) = 1 + \frac{1}{8}\delta.$$
Now fix \( \omega \in \Omega \). If \( \max_{1 \leq k \leq n} |S_{x_k}(\omega)| = \infty \) then \( \theta \sum_{k=1}^{n} g_k S_{x_k}(\omega) \) is finite only on a set of probability zero (when \( (g_1, g_2, \ldots, g_n) \) belongs to a certain proper linear subspace of \( \mathbb{R}^n \)). If \( \omega \in F_n \) and \( \max |S_{x_k}(\omega)| < \infty \) then \( S_{\xi}(\omega) \) is gaussian with variance \( \theta^2 \sum_{k=1}^{n} d_k^2 |S_{x_k}(\omega)|^2 \geq M^2 \theta^2 \). Hence

\[
\mathbb{P}(\{|S_{\xi}(\omega)| > \sigma\} \geq \mathbb{P}(\{|g_1| > \sigma \theta^{-1} M^{-1}\}).
\]

Thus if \( \omega \in F_n \), in view of the choice of \( M, \theta \) and \( \sigma \)

\[
\mathbb{E} \min(\sigma, |S_{\xi}(\omega)|) \geq 1 + \frac{1}{4} \delta.
\]

Hence

\[
\left(1 + \frac{1}{4} \delta\right) \mu(F_n) \leq 1 + \frac{1}{8} \delta.
\]

We conclude that \( \mu(F_n) \leq 1 - \frac{\delta}{16} \).

Let \( B = \Omega \setminus \bigcup_{k=1}^{\infty} E_k \). Then \( \mu(B) \geq \delta/16 \). It is clear that for any \( x \in X \), \( |S_{x}(\omega)| < \infty \) \( \mu \)-a.e. on \( B \) and further if \( \|x\| = 1 \) then \( \|S_{x}X_B\|_{p, \infty} \leq M \). Hence the linear map \( T_0 : X \to L_{p, \infty}(\Omega, \mu) \) defined at \( T_0x = S_{x}X_B \) is bounded with norm \( M \) and \( \|T_0u\|_{p, \infty} \geq (\delta/16)^{1/p} \). Letting \( T = M^{-1}T_0 \) we obtain the implication (4) implies (3) for an appropriate constant.

(3) \( \implies \) (2): Clearly (3) implies that \( X \) is isomorphic to a subspace of an \( \ell_{\infty} \)-product of spaces of the type \( L_{p, \infty}(\mu) \) and this means it is crudely finitely representable in \( \ell_{\infty}(L_{p, \infty}(0, \infty)) \) and so Proposition 2.2 gives the conclusion.

(2) \( \implies \) (1): From (2) we conclude that \( X \) embeds into an ultraproduct of spaces \( L_{p, \infty}(0, \infty) \) and this is easily seen to be a \( q \)-convex quasi-Banach lattice for any \( 0 < q < p \). (Ultraproducts of quasi-Banach spaces were apparently first considered in [19]; the theory is very similar to that of ultraproducts of Banach spaces).

\[
\square
\]

3. Marcinkiewicz and Lorentz spaces. In the next theorem we characterize an upper-estimate of \( M_{p,w} \).

**Theorem 3.1.** **For any** \( 0 < p, r < \infty \), \( M_{p,w} \) **satisfies an upper r-estimate if and only if** \( W^{r/p}(t)/t \) **is pseudo-decreasing.**

**Proof.** Since \( M_{p,r,w} \), \( 0 < r < \infty \), is the \( r \)-convexification of \( M_{p,w} \), it is enough to conduct the proof only for \( r = 1 \. Suppose that \( W^{1/p}(t)/t \) is pseudo-decreasing. Then there exists a concave function equivalent to \( W^{1/p} \) (cf. Proposition 5.10 in [1]), so without loss of generality we assume that \( W^{1/p} \) is concave. Consequently, for any disjoint \( f_i \in M_{p,w} \), \( i = 1, \ldots, n \),

\[
\left\| \sum_{i=1}^{n} f_i \right\|_M = \sup_{t} W^{1/p}(d_{\sum_{i=1}^{n} f_i}(t))t = \sup_{t} W^{1/p}\left( \sum_{i=1}^{n} d_{f_i}(t) \right) t \\
\leq \sum_{i=1}^{n} \sup_{t} W^{1/p}(d_{f_i}(t))t = \sum_{i=1}^{n} \| f_i \|_M.
\]
which shows that $M_{p,w}$ has an upper 1-estimate. Conversely, assume $M_{p,w}$ has an upper 1-estimate, and take for $0 < s < t$, $n = [t/s]$,
\[ f_i = \chi_{\left(\frac{(i-1)s}{n}, \frac{is}{n}\right]}, \quad i = 1, \ldots, 2n. \]

Then $f_i^* = \chi_{(0,t/2n)}$ and $\|f_i\|_M = W^{1/p}(t/2n)$. Consequently
\[
\left\| \sum_{i=1}^{2n} f_i \right\|_M = W^{1/p}(t) \leq C \sum_{i=1}^{2n} \|f_i\|_M = C \sum_{i=1}^{2n} W^{1/p}(t/2n) = C2nW^{1/p}(t/2n) \leq C2(t/s)W^{1/p}(s),
\]
whence $W^{1/p}(t)/t \leq 2CW^{1/p}(s)/s$. \qed

**Remark 3.2.** The space $M_{p,w}$ is not order continuous, and so it contains an order copy of $\ell_\infty$ [12]. Hence it does not have any finite lower estimate neither a type $r$ for $r > 1$.

Recall that the lower and upper Boyd indices of a rearrangement invariant quasi-Banach space $X$ on $I$ are defined as follows:
\[
p(E) = \sup\{p > 0 : \text{there exists } C > 0, \|D_s\| \leq Cs^{1/p} \text{ for all } s > 1\},
\]
\[
q(E) = \inf\{q > 0 : \text{there exists } C > 0, \|D_s\| \leq Cs^{1/q} \text{ for all } 0 < s < 1\},
\]
where $D_s : X \to X, s > 0$, is the dilation operator defined as $D_s f(t) = f(t/s)$ if $t \in [0, \infty)$ and if $I = [0, 1]$ then $D_s f(t) = f(t/s)$ for $t \leq \min(1, s)$ and $D_s f(t) = 0$ for $s < t \leq 1$ [13, 14, 15].

**Theorem 3.3.** For any $0 < p < \infty$, the Boyd indices of $M_{p,w}$ are the following
\[
p(M_{p,w}) = p/\beta(W), \quad q(M_{p,w}) = p/\alpha(W).
\]

**Proof.** Let $I = [0, \infty)$. For $s > 1$,
\[
\|D_s f\|_M = \sup_t W(t)^{1/p} f^*(t/s) = \sup_u W(su)^{1/p} f^*(u),
\]
and for $r > \beta(W)$, $W(su) \leq Cs^r W(u)$. Hence
\[
\|D_s f\|_M \leq C \sup_u s^{r/p} W(u)^{1/p} f^*(u) = Cs^{r/p} \|f\|_M,
\]
which yields that $p(M_{p,w}) = p/\beta(W)$. Analogously we obtain a formula for the upper index as well as for $I = [0, 1]$. \qed

The next result is well known [17], but we provide the proof here for the sake of completeness.

**Lemma 3.4.** Let $V : \mathbb{R}_+ \to \mathbb{R}_+, V(0) = 0$ and let $V$ be concave. If $\beta(V) = 1$, then there exists a sequence $(t_n)$ of positive numbers such that for every $0 \leq a \leq 1$
\[
\lim_{n \to \infty} \frac{V(at_n)}{V(t_n)} = a.
\]
Proof. Since $\beta(V) = 1$, for all $q < 1$

$$\inf_{t > 0, 0 < a < 1} \frac{V(at)}{a^q V(t)} = 0.$$ 

Then, setting $U(t) = \frac{V(t)}{t}$, $U$ is decreasing and for all $0 < \varepsilon < 1$

$$\inf_{t > 0} \frac{\alpha^\varepsilon U(at)}{U(t)} = 0.$$ 

It follows that for every $\delta > 0$

$$\inf_{t > 0} \frac{U(\delta t)}{U(t)} \leq 1.$$ 

Indeed, if the above condition does not hold then there exists $\delta > 0$ such that

$$\theta := \inf_{t > 0} \frac{U(\delta t)}{U(t)} > 1.$$ 

Hence $\delta < 1$ and setting $\varepsilon = -\log \theta / \log \delta$, $\delta^n = \theta^{-n}$ for every $n \in \mathbb{N}$. For any $0 < a < 1$ there exists $n \in \mathbb{N} \cup \{0\}$ such that $\delta^{n+1} < a < \delta^n$. Thus for any $t > 0$ it holds

$$\frac{\alpha^\varepsilon U(at)}{U(t)} \geq (\delta^{n+1})^\varepsilon \frac{U(\delta^n t)}{U(t)} = \theta^{-n} \frac{U(\delta^n t)}{U(\delta^{n-1} t)} \cdots \frac{U(\delta t)}{U(t)} \geq \theta^{-n} \delta^\theta \theta^n = \delta > 0,$$

which is a contradiction. Therefore for every $\delta = 1/n$, $n \in \mathbb{N}$, there exists $t_n$ such that

$$1 \leq \frac{U(t_n)}{U(t)} < 1 + \frac{1}{n},$$

which implies that

$$1 \leq \frac{U(at_n)}{U(t_n)} \leq 1 + \frac{1}{n}$$

for $0 < a < 1$ and sufficiently large $n \in \mathbb{N}$. Therefore $\frac{U(at_n)}{U(t_n)} \to 1$, and hence for all $0 < a < 1$, $\frac{V(at_n)}{V(t_n)} \to a$ as $n \to \infty$. \[\square\]

Theorem 3.5. If $M_{p,w}$, $0 < p < \infty$, satisfies an upper 1-estimate and $\beta(W) \geq p$, then $L_{1,\infty}(0, 1)$ is finitely representable in $M_{p,w}$ (i.e. $M_{p,w}$ contains uniformly copies of $\ell^n_{1,\infty}$). In particular, $M_{p,w}$ does not have type 1.

Proof. We give the proof only for $I = [0, \infty)$. By Theorem 3.1, $W^{1/p}(t) / t$ is pseudo-decreasing. Then $W^{1/p}$ is equivalent to a concave function $V$ (cf. Proposition 5.10 in [1]), that is

$$C^{-1} V(t) \leq W^{1/p}(t) \leq CV(t)$$
for some $C > 0$ and all $t > 0$. Since $\beta(W) \geq p$, so $\beta(W^{1/p}) = \beta(V) \geq 1$. Then by Lemma 3.4, there exists a sequence $(b_j) \subset (0, \infty)$ such that
\[
\lim_{j \to \infty} \frac{V(tb_j)}{V(b_j)} = t, \quad t \in [0, 1].
\]

Letting
\[
f^{(n)}_{i,j} = \frac{n}{W^{1/p}(b_j)} D_{b_j} x_i \chi_{[\frac{i-1}{n}, \frac{i}{n})}, \quad i = 1, \ldots, n, \ j \in \mathbb{N},
\]
for any $x = (\alpha_i)_{i=1}^n$ in $n$-dimensional vector space define a linear operator $T_j$ as
\[
T_j x = \sum_{i=1}^n \alpha_i f^{(n)}_{i,j}.
\]
Then setting
\[
f(t) = \sum_{i=1}^n n\alpha_i x_i \chi_{[\frac{i-1}{n}, \frac{i}{n})} (t), \quad t > 0,
\]
we have for $(\alpha^*_i)_{i=1}^n$, a decreasing permutation of $(\alpha_i)_{i=1}^n$, and for all $j \in \mathbb{N}$,
\[
\|T_j x\|_M = \sup_t W^{1/p} (t) \left( \sum_{i=1}^n \alpha_i f^{(n)}_{i,j} (t) \right)^* = \sup_t W^{1/p} (t) (W^{-1/p}(b_j) D_{b_j} f(t))^* = \sup_t W^{1/p}(b_j) f^*(t) = \max_{i=1, \ldots, n} \left\{ n\alpha^*_i W^{1/p}(b_j) / W^{1/p}(b_i) \right\}.
\]
Hence for every $x = (\alpha_i)_{i=1}^n$ we have
\[
C^{-1} \|x\|_{1,\infty} \leq \liminf_j \|T_j x\|_M \leq \limsup_j \|T_j x\|_M \leq C \limsup_j \max_{i=1, \ldots, n} \left\{ n\alpha^*_i V(b_j i/n) / V(b_j) \right\} = C \|x\|_{1,\infty}.
\]
Notice also that for all $x \in \ell_{1,\infty}^n$, $j \in \mathbb{N}$,
\[
\|T_j x\|_M \leq nC^2 \max_{i=1, \ldots, n} \{ i\alpha^*_i \} = nC^2 \|x\|_{1,\infty}.
\]
Recall now that since $\|\cdot\|_M$ is a quasi-norm, by the Aoki-Rolewicz theorem [9], there exists $0 < r < 1$ such that $\|\cdot\|_M$ is $r$-norm. Letting
\[
\|g\|_M^r = \inf \left\{ \sum_{i=1}^m \|g_i\|_M^r : g = \sum_{i=1}^m g_i \right\},
\]
we get for some $D > 0$ and all $g \in M_{p,w}$
\[
\|g\|_M^r \leq \|g\|_M^r \leq D \|g\|_M^r.
\]
Thus the family \((\| T_j x \|_M, \| T_j y \|_M)\) is equi-continuous and uniformly bounded on the unit ball \(B_{\ell_1,\infty}^n\), and so by the Arzeli-Ascoli theorem it is compact in the space \(C(B_{\ell_1,\infty}^n)\). Thus there exists a subsequence \((j_k) \subset \mathbb{N}\) such that \(\lim_k \| T_{j_k} x \|_M \in C(B_{\ell_1,\infty}^n)\). Hence for arbitrary small \(\epsilon > 0\) and every \(n \in \mathbb{N}\) there exists \(j(n) \in \mathbb{N}\) such that for all \(x \in \ell_1^n\),

\[
\lim_k \| T_{j_k} x \|_M - \epsilon \leq \| T_{j(n)} x \|_M \leq \lim_k \| T_{j_k} x \|_M + \epsilon.
\]

Thus for any \(x \in B_{\ell_1,\infty}^n\),

\[
\| T_{j(n)} x \|_M \leq D \| T_{j(n)} x \|_M + D \lim_k \| T_{j_k} x \|_M + De \\
\leq D \limsup_j \| T_j x \|_M + De \leq DC'\| x \|_{1,\infty} + De,
\]

and

\[
\| T_{j(n)} x \|_M \geq \| T_{j(n)} x \|_M \geq \lim_k \| T_{j_k} x \|_M - \epsilon \\
\geq D^{-1} \liminf_j \| T_j x \|_M - \epsilon \geq D^{-1} C^{-1}\| x \|_{1,\infty} - \epsilon.
\]

It is clear now that there exists \(A > 0\) such that for all \(n \in \mathbb{N}\) and \(x \in \ell_1^n\) it holds

\[
A^{-1}\| x \|_{1,\infty} \leq \| T_{j(n)} x \|_M \leq A\| x \|_{1,\infty}.
\]

This shows that \(M_{p,w}\) contains uniformly copies of \(\ell_1^n\). \(\square\)

**Theorem 3.6.** Let \(0 < p < \infty\). The following conditions are equivalent.

1. The Hardy operator

\[
H^{(1)}f(t) = \frac{1}{t} \int_0^t f^*(s)ds \quad 0 < t \in I,
\]

is bounded in \(M_{p,w}\).

2. \(\beta(W) < p\).

3. There exists \(C > 0\) such that

\[
\int_0^t W^{-1/p} \leq Ct/W^{1/p}(t) \quad 0 < t \in I.
\]

**Proof.** Theorem 2 in [15] states that \(H^{(1)}\) is bounded in r.i. quasi-Banach space \(X\) if and only if \(p(X) > 1\). Hence in view of Theorem 3.3 we immediately obtain the equivalence of (1) and (2). In order to show that (1) is equivalent to (3), notice that \(f \in M_{p,w}\) if and only if for every \(s \in I\), \(f^*(s) \leq CW^{-1/p}(s)\). Hence \(H^{(1)}f \in M_{p,w}\)
is equivalent to inequality $H^{(1)}f(t) \leq CW^{-1/p}(t)$, that is to $\int_0^t W^{-1/p} \leq Ct/W^{1/p}(t)$ for all $0 < t \in I$. □

In the next theorem we characterize the Marcinkiewicz spaces that have type 1.

**Theorem 3.7.** Let $0 < p < \infty$. The following conditions are equivalent.

1. $M_{p,w}$ is $1$-convex, that is the space is normable.
2. $M_{p,w}$ has type 1.
3. $\beta(W) < p$.

**Proof.** It is obvious that condition (1) implies (2). Now, if we assume that $M_{p,w}$ has type 1 and $\beta(W) \geq p$ then $M_{p,w}$ satisfies an upper 1-estimate and so by Theorem 3.5 it contains copies of $\ell_{1,\infty}$ uniformly. Thus $M_{p,w}$ can not have type 1, and this contradiction proves the implication from (2) to (3). If (3) is satisfied, that is $\beta(W) < p$, then by Theorem 3.6, the Hardy operator $H^{(1)}$ is bounded in $M_{p,w}$. Then $\|H^{(1)}f\|_M$ is equivalent to the original quasi-norm in $M_{p,w}$. Moreover, $\|H^{(1)}f\|_M$ is a norm on $M_{p,w}$ since it satisfies the triangle inequality in view of the subadditivity of the operator $H^{(1)}$. Thus we showed that (1) holds, and the proof is completed. □

**Remark 3.8.** By Theorem 3.6 we see that the condition $\beta(W) < p$ is equivalent to the integral inequality (3). It is well known (cf. Theorem A in [11] and references there) that “$\beta(W) < p$” is also equivalent to another integral inequality, namely the $B_p$-condition [18], that is for all $t \in I \setminus \{0\}$ and some $C > 0$

$$\int_t^\infty s^{-p}w(s) ds \leq C t^{-p} \int_0^t w.$$  

Soria (Theorem 3.1 in [20]) proved that $M_{p,w}$ is normable if and only if $w$ satisfies the $B_p$-condition.

For any $0 < r < \infty$, the $r$-convexification of $M_{p,w}$ is $M_{pr,w}$. Hence we get the following corollary.

**Corollary 3.9.** For any $0 < p, r < \infty$, the space $M_{p,w}$ is $r$-convex if and only if $\beta(W) < p/r$.

**Remark 3.10.** As a simple conclusion we also have that $M_{p,w}$ is $L$-convex (for definition of $L$-convexity see [7]).

**Corollary 3.11.** Let $0 < p < \infty$ and $0 < r < 1$. Then the following conditions are equivalent.

1. $M_{p,w}$ has type $r$.
2. The quasi-norm $\|\cdot\|_M$ in $M_{p,w}$ is equivalent to an $r$-norm.
3. $W^{1/p}(t)/t$ is pseudo-decreasing.

**Proof.** The equivalence of (1) and (2) is a result of Theorem 4.2 in [6]. By the Kalton’s result (Theorem 2.3 (ii) in [7]) it follows also that for $0 < r < 1$, if a quasi-normed space $(X, \|\cdot\|)$ is $L$-convex, then $\|\cdot\|$ is an $r$-norm if $X$ satisfies an upper $r$-estimate. This and Theorem 3.1 provide the equivalence of the last two conditions. □

We end the paper with conditions on when type 1 and upper 1-estimate are equivalent in quasi-Banach lattices. We then illustrate the obtained result in Lorentz spaces, providing examples of Lorentz spaces with type 1 that are not normable.
THEOREM 3.12. Let $X$ be a quasi-Banach lattice that is $r$-convex and $q$-concave for some $0 < r < q < \infty$. Then $X$ has type 1 if and only if it satisfies an upper 1-estimate.

Proof. By the Khintchine's inequality for scalars [14], for any $0 < s < \infty$, and $x_i \in X$, $i = 1, \ldots, n$, we have for some $A_s, B_s > 0$,

$$A_s \left( \sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \leq \left( \int_0^1 \left| \sum_{i=1}^{n} r(t)x_i \right|^s dt \right)^{1/s} \leq B_s \left( \sum_{i=1}^{n} |x_i|^2 \right)^{1/2}. $$

Then by the monotonicity of the quasi-norm and its $r$-convexity and $q$-concavity, we get the following generalized Khintchine's inequality in $X$

$$A_r \left\| \sum_{i=1}^{n} |x_i|^2 \right\|^{1/2} \leq \left\| \int_0^1 \left| \sum_{i=1}^{n} r(t)x_i \right|^r dt \right\|^{1/r} \leq C^{(r)} \left( \int_0^1 \left\| \sum_{i=1}^{n} r(t)x_i \right\|^r dt \right)^{1/r} \leq C^{(r)} C_{(q)} \left\| \left( \int_0^1 \sum_{i=1}^{n} r(t)x_i \right|^q dt \right\|^{1/q} \leq C^{(r)} C_{(q)} B_q \left\| \sum_{i=1}^{n} |x_i|^2 \right\|^{1/2}. $$

Assuming now that $X$ satisfies an upper 1-estimate, we get by Lemma 2.1 in [7] that for some $C > 0$ and any $x_i \in X$, $i = 1, \ldots, n$,

$$\left\| \sum_{i=1}^{n} |x_i|^2 \right\|^{1/2} \leq C \sum_{i=1}^{n} \|x_i\|. $$

Hence by the generalized Khintchine's inequality and the Kahane's inequality [6],

$$\int_0^1 \left\| \sum_{i=1}^{n} r(t)x_i \right\| dt \leq B_1 \left\| \left( \sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \right\| \leq B_1 C \sum_{i=1}^{n} \|x_i\|, $$

which finishes the proof. \qed

Applying now Theorem 3.12 and well known characterizations of convexity and concavity of $\Lambda_{p,w}$ [11] we get the following description of $\Lambda_{p,w}$ with type 1.

THEOREM 3.13. Let $w$ be a weight function such that $0 < \alpha(W) \leq \beta(W) < \infty$. Then the following conditions are equivalent.

1. $\Lambda_{p,w}$ has type 1.
2. $\Lambda_{p,w}$ satisfies an upper 1-estimate.
3. $W(t)/t^p$ is pseudo-decreasing and $p \geq 1$.

Proof. By the assumption $0 < \alpha(W) \leq \beta(W) < \infty$ it follows by Theorems 2 and 6 in [11] that $\Lambda_{p,w}$ is $r$-convex and $q$-concave for some $0 < r < q < \infty$. Applying now Theorem 3.12, the conditions (1) and (2) are equivalent. The equivalence of (2) and (3) is a direct consequence of a characterization of upper 1-estimate of $\Lambda_{p,w}$ (Theorem 3 in [11]). \qed
Remark 3.14. The characterization of the Lorentz spaces $\Lambda_{p,w}$ with type 1 differs substantially from that of $M_{p,w}$. There are Lorentz spaces with type 1 that are not normable. By Theorem A in [11], $\Lambda_{p,w}$, $1 < p < \infty$, is normable if and only if $\beta(W) < p$. Now, letting $w(t) = t^{p-1}$, $1 < p < \infty$, we have $W(t) = t^p/p$, and so $\beta(W) = p$ and $W(t)/t^p$ is pseudo-decreasing. Thus the space $L_{1,p} := \Lambda_{p,w}$ is not normable (see also [3]), but it has type 1 by Theorem 3.13.

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