# CONVEXITY CONDITIONS FOR NON-LOCALLY CONVEX LATTICES

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**1. Introduction.** First we recall that a (real) quasi-Banach space $X$ is a complete metrizable real vector space whose topology is given by a quasi-norm $x \to \|x\|$ satisfying

$$\|x\| > 0 \qquad (x \in X, x \neq 0) \tag{1.1}$$

$$\|\alpha x\| = |\alpha|\,\|x\| \qquad (\alpha \in \mathbb{R}, x \in X) \tag{1.2}$$

$$\|x_1 + x_2\| \leq C(\|x_1\| + \|x_2\|) \qquad (x_1, x_2 \in X), \tag{1.3}$$

where $C$ is some constant independent of $x_1$ and $x_2$. $X$ is said to be *$p$-normable* (or *topologically $p$-convex*), where $0 < p \leq 1$, if for some constant $B$ we have

$$\|x_1 + \ldots + x_n\| \leq B(\|x_1\|^p + \ldots + \|x_n\|^p)^{1/p} \tag{1.4}$$

for any $x_1, \ldots, x_n \in X$. A theorem of Aoki and Rolewicz (see [**18**]) asserts that if in (1.3) $C = 2^{1/p-1}$, then $X$ is $p$-normable. We can then equivalently re-norm $X$ so that in (1.4) $B = 1$.

If in addition $X$ is a vector lattice and $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$ we say that $X$ is a quasi-Banach lattice. As in the case of Banach lattices [**13**] we may make the following definitions.

We shall say that $X$ satisfies an *upper $p$-estimate* if for some constant $C$ and any $x_1, \ldots, x_n \in X$ we have

$$\||x_1| \vee \ldots \vee |x_n|\| \leq C\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p}. \tag{1.5}$$

We shall say that $X$ is (*lattice*) *$p$-convex* if for some $C$ and any $x_1, \ldots, x_n \in X$

$$\left\|\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}\right\| \leq C\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p}. \tag{1.6}$$

Here the element $(|x_1|^p + \ldots + |x_n|^p)^{1/p}$ $(0 < p < \infty)$ of $X$ can be defined unambiguously exactly as for the case of Banach lattices (cf. [**13**, pp 40–41] and Popa [**17**]).

For $0 < p \leq 1$ it is trivial to see that lattice $p$-convexity implies $p$-normability and $p$-normability implies the existence of an upper $p$-estimate. In the case $p = 1$, lattice 1-convexity is equivalent to normability (i.e. $X$ is a Banach lattice). However Popa [**17**] observes that for $0 < p < 1$, the space "weak $L_p$" $L(p, \infty)$ of measurable functions on (0, 1)

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such that

$$\|f\| = \sup_{0<t<\infty} tm(|f|>t)^{1/p} < \infty$$

is $p$-normable but not lattice $p$-convex.

In this note we introduce the class of *L-convex* quasi-Banach lattices. We say that $X$ is $L$-convex if there exists $0<\varepsilon<1$ so that if $u \in X_+$ with $\|u\|=1$ and $0 \le x_i \le u$ $(1 \le i \le n)$ satisfy

$$\frac{1}{n}(x_1+\ldots+x_n) \ge (1-\varepsilon)u,$$

then

$$\max_{1\le i\le n} \|x_i\| \ge \varepsilon.$$

Roughly speaking, $X$ is $L$-convex if its order-intervals are uniformly locally convex.

It turns out that most naturally arising function spaces are $L$-convex lattices (e.g. the $L_p$-spaces, Orlicz spaces, Lorentz spaces including the spaces $L(p,\infty)$ introduced above). However we shall give examples of non $L$-convex lattices. We shall show that $X$ is $L$-convex if and only if $X$ is lattice $p$-convex for some $p>0$. If $\ell_\infty$ is not lattice finitely representable in $X$ then $X$ is necessarily $L$-convex. We also show that if $X$ is a quasi-Banach lattice linearly homeomorphic to a subspace of an $L$-convex lattice then $X$ is again $L$-convex.

$L$-convex lattices behave similarly to Banach lattices in many respects. For example if $X$ is $L$-convex and satisfies an upper $p$-estimate, then $X$ is lattice $r$-convex for any $r<p$ (compare [**13**], p. 85] and results of Maurey and Pisier [**14**], [**16**]). Also for $0<p<1$, if $X$ is $L$-convex and satisfies an upper $p$-estimate, then $X$ is $p$-normable. This is false for $p=1$; $L(1,\infty)$ is a counter-example. However an analoguous result for $1<p<2$ involving type due to Figiel and Johnson is given in [**13**, p. 88]. By contrast, in general if a quasi-Banach lattice satisfies an upper $p$-estimate, then it is $q$-normable, where $q^{-1}=p^{-1}+1$ and this result is best possible.

**2. *L*-convexity.** Before proving our basic lemma, it will be convenient to introduce some terminology. Suppose $X$ is a quasi-Banach lattice and $u \in X_+$ with $u \neq 0$. Then if we set $Y = \bigcup_{n=1}^{\infty} [-nu, nu]$ $Y$ is a sublattice of $X$; if we select $[-u, u]$ as the unit ball of $Y$ then $Y$ is an abstract $M$-space, and by a well-known theorem of Kakutani ([**13**, p. 16], [**19**, p. 104]) there is a compact Hausdorff space $\Delta$ so that $Y$ is isometrically lattice isomorphic to $C(\Delta)$. Thus we can induce a lattice homomorphism $J: C(\Delta) \to X$ so that $J$ maps the unit ball of $C(\Delta)$ onto the order interval $[-u, u]$. We call $J$ the Kakutani map associated to $u$.

LEMMA 2.1. *Let $X$ be an $L$-convex quasi-Banach lattice satisfying an upper $p$-estimate. Then*

(a) *if $0<p<r$, there is a constant $M$ so that if $x_1,\ldots,x_n \in X$ we have*

$$\left\|\left(\sum |x_i|^r\right)^{1/r}\right\| \le M\left(\sum \|x_i\|^p\right)^{1/p}.$$

(b) *If* $0<r<p$ *there is a constant* $M$ *so that if* $x_1,\ldots,x_n\in X$ *we have*

$$\left\|\left(\sum|x_i|^r\right)^{1/r}\right\|\le M\left(\sum\|x_i\|^r\right)^{1/r}.$$

*Proof.* We shall suppose $C<\infty$ and $0<\varepsilon<1$ are chosen as in (1.5) and (1.7). Without loss of generality in both parts (a) and (b) we may assume $x_i\ge 0$ $(1\le i\le n)$ and that $\|u\|=1$, where $u=(\sum|x_i|^r)^{1/r}$. Let $J:C(\Delta)\to X$ be the Kakutani map associated to $u$. Let $Jf_i=x_i$ where $0\le f_i\le 1$. Choose $\tau>0$ so that

$$1-\exp(-\tau^{-r})\ge 1-\tfrac14\varepsilon.$$

Let $(\Omega,P)$ be some probability space and let $(\xi_i:1\le i\le n)$ be independent positive random variables on $\Omega$ so that for each $i$

$$P(\xi_i>t)=t^{-r}\qquad (t\geqslant 1).$$

If $s\in\Delta$ and if $\max f_i(s)\le\tau$ then

$$\begin{aligned}P(\max \xi_if_i(s)>\tau)&=1-\prod_{i=1}^n P(\xi_i\le \tau f_i(s)^{-1})\\&=1-\prod_{i=1}^n(1-\tau^{-r}f_i(s)^r)\\&\ge 1-\prod_{i=1}^n\exp(-\tau^{-r}f_i(s)^r)\\&=1-\exp(-\tau^{-r})\\&\ge 1-\tfrac14\varepsilon.\end{aligned}\tag{2.1}$$

Here we use the fact that $J((\sum f_i^r)^{1/r})=(\sum|x_i|^r)^{1/r}=u=J1$, so that $\sum f_i(s)^r=1$ for $s\in\Delta$.

Now (2.1) holds trivially if we suppose $\max f_i(s)>\tau$. Thus we conclude

$$\int_\Omega \max_{i\le n}(\min(\xi_i(\omega)f_i(s),\tau))\,dP(\omega)\ge\tau(1-\tfrac14\varepsilon).\tag{2.2}$$

For each $k\in\mathbb{N}$ we define $\xi_{ik}$ $(1\le i\le n)$ by

$$\xi_{ik}(\omega)=\left(\frac{2^k}{m}\right)^{1/r}\quad\left(\frac{2^k}{m}\right)^{1/r}\le\xi_i(\omega)<\left(\frac{2^k}{m-1}\right)^{1/r}$$

for $m=1,2,\ldots,2^k$. Then $\lim_{k\to\infty}\xi_{ik}=\xi_i$ a.e. and for each $k\in\mathbb{N}$ the random variables $(\xi_{ik}:1\le i\le n)$ are independent and generate a finite algebra $\mathcal{A}_n$ in $\Omega$ with $2^{kn}$ atoms each of probability $2^{-kn}$. Set

$$g_k(s)=\int_\Omega\max_{i\le n}(\min(\xi_{ik}(\omega)f_i(s),\tau))\,dP(\omega).$$

Then $g_k \in C(\Delta)$ and the sequence $g_k$ is monotone increasing. From (2.2) we deduce that

$$\lim_{k\to\infty} g_k(s) \geq \tau(1-\tfrac{1}{4}\varepsilon).$$

Now, by Dini's theorem, there exists $k \in \mathbb{N}$ so that $g_k(s) \geq \tau(1-\frac{1}{2}\varepsilon)$ for every $s \in \Delta$. Suppose $A \in \mathscr{A}_k$ and $P(A) \leq \frac{1}{2}\varepsilon$; then

$$\int_{\Omega\setminus A} \max_{i\leq n} (\min(\xi_{ik}(\omega) f_i(s), \tau))\, dP(\omega) \geq \tau(1-\varepsilon).$$

This implies that $(1-\varepsilon)u$ is dominated by an average of the finitely many distinct values of $\left(\tau^{-1} \max_{i\leq n} \xi_{ik}(\omega) x_i\right) \wedge u$. Thus

$$\max_{\omega\in\Omega\setminus A} \left\| \max_{i\leq n} \xi_{ik}(\omega) x_i \right\| \geq \tau\varepsilon$$

from the definition of $L$-convexity (equation (1.7)). Hence

$$P\left(\left\| \max_{i\leq n} \xi_i(\omega) x_i \right\| \geq \tau\varepsilon\right) \geq \tfrac{1}{2}\varepsilon.$$

Since $X$ satisfies an upper $p$-estimate,

$$P\left(\left(\sum_{i=1}^{n} |\xi_i(\omega)|^p \, \|x_i\|^p\right)^{1/p} \geq C^{-1}\tau\varepsilon\right) \geq \tfrac{1}{2}\varepsilon.$$

Now we consider two cases. In case (a) if $0<p<r$ then

$$\int_\Omega \sum_{i=1}^{n} |\xi_i(\omega)|^p \, \|x_i\|^p \, dP(\omega) \geq \tfrac{1}{2} C^{-p} \tau^p \varepsilon^{p+1}$$

and

$$\int_\Omega |\xi_i|^p \, dP = B < \infty.$$

Hence

$$\sum_{i=1}^{n} \|x_i\|^p \geq \tfrac{1}{2} B^{-1} C^{-p} \tau^p \varepsilon^{p+1}$$

so that (a) follows.

In case (b) pick $\alpha > 1$ so that $r\alpha > p$. Let $\eta_i = \xi_i^{p/\alpha}$ so that $P(\eta_i > t) = t^{-r\alpha/p}$ for $t \geq 1$. By Lemma 1.f.8 of [**13**, p. 86] there is a constant $B$ so that

$$\int_\Omega \left(\sum a_i^\alpha \eta_i^\alpha\right)^{1/\alpha} dP \leq B\left(\sum |a_i|^{(r\alpha)/p}\right)^{p/r\alpha}$$

for $a_1, \ldots, a_n \geq 0$. Now, for $\delta$ depending only on $C$ and $\varepsilon$,

$$\int_\Omega \left(\sum_{i=1}^{n} |\eta_i(\omega)|^\alpha (\|x_i\|^{p/\alpha})^\alpha\right)^{1/\alpha} dP \geq \delta$$

and so

$$B\left(\sum \|x_i\|^r\right)^{p/r\alpha} \geq \delta.$$

Thus (b) follows.

The next theorem should be compared with the Banach lattice case (Theorem 1.f.7 of [**13**, p. 85]).

THEOREM 2.2. *Let $X$ be a quasi-Banach lattice satisfying an upper $p$-estimate. Then the following conditions on $X$ are equivalent:*

(i) *$X$ is $L$-convex*

(ii) *$X$ is lattice $r$-convex for some $r>0$.*

(iii) *$X$ is lattice $r$-convex for every $r$, $0<r<p$.*

(i) $\Rightarrow$ (iii): This is simply Lemma 2.1 (b).

(iii) $\Rightarrow$ (ii): This is immediate.

(ii) $\Rightarrow$ (i): We assume $r<1$. Suppose $0\leq x_i\leq u$ where $\|u\|=1$ and that

$$\frac{1}{n}(x_1+\ldots+x_n)\geq \tfrac{1}{2}u.$$

Then

$$(x_1+\ldots+x_n)\leq u^{1-r}(x_1^r+\ldots+x_n^r),$$

where the right-hand side is well-defined in $X$, cf. [**12**, pp. 41–43]. Hence

$$\tfrac{1}{2}nu\leq u^{1-r}(x_1^r+\ldots+x_n^r)$$

and so

$$(x_1^r+\ldots+x_n^r)^{1/r}\geq(\tfrac{1}{2}n)u.$$

Thus

$$(\tfrac{1}{2}n)^{1/r}\leq C\Big(\sum\|x_i\|^r\Big)^{1/r}$$

so that

$$\max_{i\leq n}\|x_i\|\geq(\tfrac{1}{2})^{1/r}C^{-1}.$$

If $r\geq 1$ the argument is simpler, since

$$(x_1^r+\ldots+x_n^r)^{1/r}\geq n^{1/r-1}(x_1+\ldots+x_n).$$

THEOREM 2.3. *Let $X$ be a quasi Banach lattice satisfying an upper $p$-estimate where $0<p<\infty$. Then*

(i) *$X$ is $q$-normable where $1/q=1/p+1$;*

(ii) *if $0<p<1$ and $X$ is $L$-convex, then $X$ is $p$-normable;*

(iii) *if $1<p<\infty$ and $X$ is $L$-convex, then $X$ is a Banach lattice.*

*Proof.* (i) We suppose (1.5) holds. Suppose $x_1,\ldots,x_n\in X_+$ and $u=x_1+\ldots+x_n$. Let $\sigma=(\|x_1\|^q+\ldots+\|x_n\|^q)^{1/q}$ and observe that

$$\begin{aligned}\|u\|&\leq\Big\|\max_{i\leq n}\sigma^q\|x_i\|^{-q}x_i\Big\|\\&\leq C\Big(\sum_{i=1}^n\sigma^{pq}\|x_i\|^{-pq}\|x_i\|^p\Big)^{1/p}\\&=C\sigma^q\Big(\sum_{i=1}^n\|x_i\|^q\Big)^{1/p}=C\sigma^{q+q/p}=C\sigma.\end{aligned}$$

(ii) This is simply Lemma 2.1 (a) with $r=1$

(iii) By Theorem 2.2 $X$ is lattice 1-convex i.e. a Banach lattice.

EXAMPLE 2.4. Let $\mathscr{A}$ be an algebra of subsets of some set $\Omega$ and let $\phi:\mathscr{A}\to\mathbb{R}$ be a normalized submeasure, i.e. $\phi$ is a set-function satisfying $\phi(\emptyset)=0$, $\phi(A)\le\phi(A\cup B)\le\phi(A)+\phi(B)$ for $A, B\in\mathscr{A}$ and $\phi(\Omega)=1$. From $\phi$ we can construct a quasi-Banach lattice $L_p(\phi)$ satisfying an upper $p$-estimate for $0<p<\infty$. If $f:\Omega\to\mathbb{R}$ is a simple $\mathscr{A}$-measurable function we define

$$\|f\|_p=\left(\int_0^\infty \phi(|f|\ge t^{1/p})\,dt\right)^{1/p}.$$

Then $\|\cdot\|_p$ is a quasi-norm; indeed

$$\begin{aligned}\|f+g\|_p^p&=\int_0^\infty \phi(|f+g|\ge t^{1/p})\,dt\\ &\le\int_0^\infty \phi(|f|\ge\tfrac{1}{2}t^{1/p})\,dt+\int \phi(|g|\ge\tfrac{1}{2}t^{1/p})\,dt\\ &\le 2^p(\|f\|_p^p+\|g\|_p^p)\end{aligned}$$

so that

$$\begin{aligned}\|f+g\|_p&\le 2^{1/p}(\|f\|_p+\|g\|_p)\qquad(0<p\le 1),\\ \|f+g\|_p&\le 2(\|f\|_p+\|g\|_p)\qquad(1\le p<\infty).\end{aligned}$$

The completion of the simple functions $S(\mathscr{A})$ with this quasi-norm is a quasi-Banach lattice $L_p(\phi)$ satisfying an upper $p$-estimate.

Suppose now $\phi$ is *pathological* ([3], [4]), that is so that whenever $0\le\lambda\le\phi$ and $\lambda$ is additive then $\lambda=0$. Then for any $\varepsilon>0$ there exist $E_1,\ldots,E_n\in\mathscr{A}$ so that $\phi(E_i)\le\varepsilon$ but $1/n\sum 1_{E_i}\ge(1-\varepsilon)1_\Omega$ ([3]). It follows quickly that $L_p(\phi)$ is not $L$-convex.

Furthermore (Talagrand [20]) $\phi$ can be chosen so that for every $n$ there exist $E_1,\ldots,E_n\in\mathscr{A}$ with $\phi(E_i)\le n^{-1}$ and $1/n\sum 1_{E_i}\ge\frac{1}{2}1_\Omega$. Suppose $L_p(\phi)$ is $q$-normable. Then

$$\tfrac{1}{2}\le\frac{C}{n}\left(\sum_{i=1}^n \|1_{E_i}\|_p^q\right)^{1/q}=Cn^{1/q-1/p-1}\qquad(n\in\mathbb{N}).$$

Hence $1/q\ge 1/p+1$ so that Theorem 2.3 (a) is best possible.

By way of contrast we observe that the space $L(p,\infty)$ is $L$-convex for $0<p<1$. In fact if $0<r<p$, $L(p,\infty)=\{f:|f|^r\in L(pr^{-1},\infty)\}$ and $L(pr^{-1},\infty)$ is a Banach lattice, i.e. is locally convex (see [5]). Hence $L(p,\infty)$ is lattice $r$-convex for $0<r<p$. As $L(p,\infty)$ satisfies an upper $p$-estimate, it is $p$-normable (see [8]).

**3. Some applications of a theorem of Bennett and Maurey.** In this section we show how a deep factorization theorem of Bennett and Maurey ([1], [2], [15]) can be used to extend a result of Krivine [12] on operators between Banach lattices (cf. [13, p. 93]). This latter result is of considerable importance in studying operators between function spaces (see [**10**]).

We start by stating that Bennett-Maurey theorem (see [**1**] or [**2**] for this statement).

THEOREM 3.1. *Let* $0<p<1$ *be fixed. Then there is a constant* $C=C(p)$ *so that whenever* $m, n \in \mathbb{N}$ *and* $T:\ell_\infty^m \to \ell_p^n$ *is a linear operator then there is a positive* $D:\ell_p^n \to \ell_1^n$ *given by* $D(\xi_i)=(d_i\xi_i)$ *so that* $\|DT\| \le \|T\|$ *and* $\sum d_i^{(-p/1-p)} \le C$.

COROLLARY 3.2. *Suppose* $0<p<1$. *Then there is a constant* $B=B(p)$ *so that if* $\Delta$, $K$ *are compact Hausdorff spaces,* $\mu$ *is a probability measure on* $K$ *and* $T:C(\Delta)\to L_p(K,\mu)$ *is a bounded linear operator, then for* $f_1,\dots,f_n \in C(\Delta)$, *we have*

$$\left\|\left(\sum_{i=1}^n |Tf_i|^2\right)^{1/2}\right\|_p \le B\|T\| \left\|\left(\sum_{i=1}^n |f_i|^2\right)^{1/2}\right\|.$$

*Proof.* Exactly as step 2 of Theorem 1.f.14 of [**1**, p. 92] this can be reduced to consideration of a map $T:\ell_\infty^m \to \ell_p^n$. Now by Theorem 3.1 we can find $D:\ell_p^n \to \ell_1^n$ so that $\|DT\|\le\|T\|$ and $D(\xi_i)=(d_i\xi_i)$ where $\sum d_i^{(-p/1-p)} \le C$. Then

$$\begin{aligned}\left\|\left(\sum |Tf_i|^2\right)^{1/2}\right\|_p^p &= \left\|D^{-1}\left(\sum |DTf_i|^2\right)^{1/2}\right\|_p^p \\ &\le \left(\sum d_i^{(-p/1-p)}\right)^{1-p} \left\|\left(\sum |f_i|^2\right)^{1/2}\right\|_1^p \\ &\le C^{1-p}K_G^p \left\|\left(\sum |DTf_i|^2\right)^{1/2}\right\|_\infty^p,\end{aligned}$$

by Theorem 1.f.14 of [**13**]. Let $B=C^{1/p-1}K_G$.

THEOREM 3.3. *Let* $Y$ *be an* $L$*-convex quasi-Banach lattice. Then there is a constant* $A$ *depending only on* $Y$ *so that whenever* $X$ *is a quasi-Banach lattice and* $T:X\to Y$ *is a bounded linear operator then for any* $x_1,\dots,x_n \in X$

$$\left\|\left(\sum_{i=1}^n |Tx_i|^2\right)^{1/2}\right\| \le A\|T\| \left\|\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}\right\|$$

*Proof.* First we observe that $Y$ is lattice $p$-convex for some $p>0$ and hence satisfies (1.6) for some $C$.

If $x_1,\dots,x_n \in X$ let $v=(\sum |Tx_i|^2)^{1/2}$ and $u=(\sum |x_i|^2)^{1/2}$. We may suppose $u, v \ne 0$. Let $J_u: C(\Delta_u)\to X$ and $J_v: C(\Delta_v)\to Y$ be associated Kakutani maps.

If $f_1,\dots,f_m \in C(\Delta_v)$,

$$\left\|J_v\left(\sum_{i=1}^n |f_i|^p\right)^{1/p}\right\| \le C\left(\sum_{i=1}^n \|J_vf_i\|^p\right)^{1/p}.$$

As $J_v$ is positive this implies that for some $s\in K$

$$\left(\sum_{i=1}^m |f_i(s)|^p\right)^{1/p} \le C\|v\|^{-1}\left(\sum_{i=1}^m \|J_vf_i\|^p\right)^{1/p}.$$

Now by a standard Hahn-Banach separation argument there is a probability measure $\mu$ on $\Delta_v$ so that for $f \in C(\Delta_v)$,

$$\int_{\Delta_v} |f|^p \, d\mu \leq C^p \|v\|^{-p} \|J_v f\|^p.$$

For $x \in X_+$ define $Sx \in L_p(\Delta_v, \mu)$ by

$$Sx = \sup_n J^{-1}(x \wedge nv)$$

and extend $S$ linearly. Then $S$ is a lattice-homomorphism and $\|S\| \leq C\|v\|^{-1}$.

Now consider $STJ_u : C(\Delta_u) \to L_p(\Delta_v, \mu)$. By Theorem 3.2, if $f_1, \ldots, f_n \in C(\Delta_u)$ are chosen so that $J_u f_i = x_i$,

$$\left\| \left( \sum_{i=1}^n |STJ_u f_i|^2 \right)^{1/2} \right\|_p \leq B \|STJ_u\| \left\| \left( \sum_{i=1}^n |f_i|^2 \right)^{1/2} \right\|,$$

where $B$ depends only on $p$.

Now, since $S$ is a lattice-homomorphism,

$$\left\| \left( \sum |STJ_u f_i|^2 \right)^{1/2} \right\|_p = \left\| S\left( \sum_{i=1}^n |Tx_i|^2 \right)^{1/2} \right\| = \|Sv\| = 1.$$

On the other hand $(\sum |f_i|^2)^{1/2} = 1$ and so

$$1 \leq B\|STJ_u\| \leq BC\|v\|^{-1} \|T\| \|u\|$$

so that

$$\|v\| \leq A \|T\| \|u\|,$$

where $A = BC$.

Applying Theorem 3.3 in the case $X = \ell_\infty^n$ we obtain the following result.

Corollary 3.4. *Suppose $Y$ is an $L$-convex quasi-Banach lattice. Then there is a constant $A$ so that if $y_1, \ldots, y_n \in Y$ then*

$$\left\| \left( \sum_{i=1}^n |y_i|^2 \right)^{1/2} \right\| \leq A \sup_{|a_i| \leq 1} \|a_1 y_1 + \ldots + a_n y_n\|.$$

*Proof.* Apply the theorem to the map $T : \ell_\infty^n \to Y$ given by $Te_i = y_i$, where $\{e_i\}$ are the basis vectors in $\ell_\infty^n$.

Example 3.5. We do not know whether the conclusions of Theorem 3.3 or Corollary 3.4 characterize $L$-convex lattices. However we can give an example to show that both are false without the $L$-convexity assumption.

Our example will be of the form of an $\ell_\infty$-product of spaces of the type $L_1(\phi_n)$, where each $\phi_n$ is a submeasure. We then need only produce $\phi_n$ to show that there is no uniform constant $A$ valid for each $n$.

Let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$ i.e.

$$S^{n-1}=\{(\xi_1,\ldots,\xi_n):\xi_1^2+\ldots+\xi_n^2=1\}.$$

Let $\mathscr{A}$ be the algebra of all subsets of $S^{n-1}$.

If $a\in\mathbb{R}^n$ and $a\neq 0$ let $B_a\in\mathscr{A}$ be defined by $B_a=\{\xi:a\,.\,\xi\neq 0\}$. For any set $a^{(1)},\ldots,a^{(n-1)}\in\mathbb{R}^n\setminus\{0\}$ there exists $\xi\in S^{n-1}$ so that $a^{(1)}\,.\,\xi=\ldots=a^{(n-1)}\,.\,\xi=0$ so that $\bigcup_{j=1}^{n-1} B_{a(j)}\neq S^{n-1}$. Define $\phi_n:\mathscr{A}\to\mathbb{R}$ by

$$\phi_n(A)=\frac{1}{n}\inf\left\{k:A\subset\bigcup_{j=1}^{k}B_{a(j)}\right\}.$$

Then $\phi_n$ is a normalized submeasure.

Let $f_i(\xi)=\xi_i$. Then if $|a_i|\leq 1$, $|a_1f_1+\ldots+a_nf_n|\leq\sqrt{n}1_{B_{(a)}}$. Hence

$$\|a_1f_1+\ldots+a_nf_n\|\leq\sqrt{n}\,.\,\frac{1}{n}=n^{-1/2}.$$

However $(f_1^2+\ldots+f_n^2)^{1/2}\equiv 1$ and $\|1\|=1$.

**4. Further conditions for $L$-convexity.** Our first result in this section shows that a wide class of quasi-Banach lattices are automatically $L$-convex. We say that $\ell_\infty$ is *lattice finitely representable* in $X$ if given $\varepsilon>0$ and $n\in\mathbb{N}$ there exist $x_i\geq 0$ $(1\leq i\leq n)$ so that $x_i\wedge x_j=0$ $(i\neq j)$, $\|x_i\|=1$ $(1\leq i\leq n)$ and whenever $a_1,\ldots,a_n\in\mathbb{R}$

$$\|a_1x_1+\ldots+a_nx_n\|\leq(1+\varepsilon)\max_{1\leq i\leq n}|a_i|.$$

If $\ell_\infty$ is not lattice finitely representable in $X$, then there exists $c>1$ and $n\in\mathbb{N}$ so that for any sequence $(x_1,\ldots,x_n)$ of disjoint elements we have

$$\|x_1+\ldots+x_n\|\geq c\min_{1\leq i\leq n}\|x_i\|.$$

It then follows quickly by induction that for every $d>1$ there exists $N\in\mathbb{N}$ so that for disjoint $x_1,\ldots,x_N$,

$$\|x_1+\ldots+x_N\|\geq d\min_{1\leq i\leq N}\|x_i\|.$$

We remark that if $F$ is an Orlicz function satisfying the $\Delta_2$-condition then $\ell_\infty$ is not lattice finitely representable in the Orlicz space $L_F(0,1)$; equally $\ell_\infty$ is not lattice finitely representable in the Lorentz space $L(p,q)$ if $0<q<\infty$ (cf. [**5**]).

THEOREM 4.1. *Let $X$ be a quasi-Banach lattice such that $\ell_\infty$ is not lattice finitely representable in $X$. Then $X$ is $L$-convex.*

*Proof.* We can and do suppose $X$ is p-normed; that is for suitable $0<p<1$

$$\|x_1+\ldots+x_n\|\leq(\|x_1\|^p+\ldots+\|x_n\|^p)^{1/p},$$

for $x_1,\ldots,x_n\in X$.

Fix $N\in\mathbb{N}$ so that for any sequence of disjoint elements $(x_1,\ldots,x_N)$ we have

$$\|x_1+\ldots+x_N\|\geq 6^{1/p}\min_{i\leq N}\|x_i\|.$$

Then fix $\varepsilon$, $0<\varepsilon<1$ so that $\varepsilon<\frac{1}{2}(\frac{1}{4})^{1/p}$ and $\varepsilon<(1/32)e^{-2}N^{-1}$. Suppose that $u\in X_+$, with $0\leq x_i\leq u$ and $(1/m)(x_1+\ldots+x_m)\geq(1-\frac{1}{2}\varepsilon)u$.

Let $J:C(\Delta)\to X$ be the Kakutani map associated to $u$. We claim first that $J$ is exhaustive; that is if $\{f_i:i\in\mathbb{N}\}$ is a uniformly bounded disjoint sequence in $C(\Delta)$ then $Jf_i\to 0$. This follows easily from the hypothesis on $X$. Now by a theorem of Thomas [**29**] (cf. also [**7**], [**9**]), there is a regular $X$-valued measure $\mu$ defined on the Borel sets $\beta$ of $\Delta$ so that

$$Jf=\int f\,d\mu \qquad (f\in C(\Delta)).$$

We remark that co $\mu(\beta)$ is bounded and so there is no difficulty in defining the integral of any bounded Borel function. It is easy to see that $\mu(\Delta)=u$ and $\mu$ is monotone; that is $0\leq\mu(A)\leq\mu(B)$ whenever $A\subset B$.

Let $\phi:B\to\mathbb{R}$ be defined by $\phi(A)=\|\mu(A)\|^p$. Then $\phi$ is a submeasure. We shall show that $\phi$ satisfies the hypotheses of [**11**, Lemma 3.1]. If $A_1,\ldots,A_N$ are disjoint sets, then $\mu(A_1),\ldots,\mu(A_N)$ are disjoint in $X$ and so

$$1\geq\|\mu(A_1\cup\ldots\cup A_N)\|^p\geq 6\min\|\mu(A_i)\|^p,$$

so that $\min\phi(A_i)\leq\frac{1}{6}$.

Hence if $A_1,\ldots,A_n$ are disjoint, then, as required,

$$\sum_{i=1}^{n}\phi(A_i)\leq N+\tfrac{1}{6}n. \tag{3.1}$$

Choose $g_i$ $(1\leq i\leq m)$ so that $Jg_i=x_i$. Let $B_i=\{g_i\geq\frac{1}{2}\}$. Then

$$\frac{1}{m}\sum_{i=1}^{m}1_{B_i}\geq(1-\varepsilon)1_\Delta.$$

From Lemma 3.1 and Proposition 2.3 of [**11**] we deduce (taking $r=3$ in the statement of the lemma)

$$\frac{1}{m}\sum_{i=1}^{m}\phi(B_i)\geq 1-3\,.\,\tfrac{1}{6}-N(2e^2)^{1/2}\varepsilon^{1/2}\geq\tfrac{1}{4}$$

so that

$$\max_{1\leq i\leq M}\phi(B_i)\geq\tfrac{1}{4}.$$

Hence

$$\max_{1\leq i\leq m}\|x_i\|\geq\tfrac{1}{2}(\tfrac{1}{4})^{1/p}\geq\varepsilon,$$

so that $X$ is $L$-convex.

THEOREM 4.2. *Let $Y$ be an $L$-convex quasi-Banach lattice and let $X$ be a quasi-Banach lattice linearly homeomorphic to a subspace of $Y$. Then $X$ is $L$-convex.*

*Proof.* We shall suppose $Y$ is lattice $p$-convex for some $p$, $0<p\leq 1$ satisfying equation (1.6), i.e.

$$\left\|\left(\sum |y_i|^p\right)^{1/p}\right\|\leq C\left(\sum \|y_i\|^p\right)^{1/p}$$

for $y_1,\ldots,y_n\in Y$. We also suppose that the conclusion of Theorem 3.3 holds with constant $A<\infty$. Let $T:X\to Y$ be a linear operator so that

$$B^{-1}\|x\|\leq\|Tx\|\leq B\|x\| \qquad (x\in X),$$

for some constant $B<\infty$.

If $X$ is not $L$-convex, then given $\delta>0$ we can find $u\in X_+$ with $\|u\|=1$ and $0\leq x_i\leq u$ $(1\leq i\leq n)$ so that $(1/n)\ (x_1+\ldots+x_n)\geq(1-\delta)u$ and $\|x_i\|\leq\delta$ $(1\leq i\leq n)$.

Let $y_i=Tx_i$. Then

$$\left\|\left(\sum |y_i|^p\right)^{1/p}\right\|\leq C\left(\sum \|y_i\|^p\right)^{1/p}\leq CB\left(\sum \|x_i\|^p\right)^{1/p}\leq CBn^{1/p}\delta.$$

On the other hand

$$\left\|\left(\sum |y_i|^2\right)^{1/2}\right\|\leq A\left\|\left(\sum |x_i|^2\right)^{1/2}\right\|\leq An^{1/2}\|u\|=An^{1/2}.$$

Let $v_1=\delta^{-1}n^{-1/p}(\sum|y_i|^p)^{1/p}$ and $v_2=n^{-1/2}(\sum|y_i|^2)^{1/2}$. Let $\theta=p(2-p)^{-1}$. Then

$$\delta^{-\theta}n^{-1}\sum|y_i|\leq v_1^\theta v_2^{1-\theta}.$$

[This is easily seen by using a Kakutani map to represent the elements of $Y$ as functions.] Hence

$$n^{-1}\sum|y_i|\leq\delta^\theta(\theta v_1+(1-\theta)v_2)\leq\delta^\theta(v_1+v_2)$$

and so if $C'$ is the constant occurring in equation (1.3) for quasi-norms,

$$\left\|n^{-1}\sum|y_i|\right\|\leq\delta^\theta C'(A+CB).$$

Now

$$\left\|\sum|y_i|\right\|=\left\|\sum|Tx_i|\right\|\geq\left\|\left|T\left(\sum x_i\right)\right|\right\|\geq B^{-1}\left\|\sum x_i\right\|.$$

Hence

$$(1-\delta)\leq\delta^\theta BC'(A+CB).$$

For small enough $\delta$ this is a contradiction and so $X$ is $L$-convex.

*Conjecture.* If $Y$ is lattice $p$-convex where $0<p<1$, then $X$ is lattice $p$-convex.

We remark that the conjecture is true for $p=1$ trivially and for $0<p<2$, if we assume $\ell_\infty$ is not lattice finitely representable in $X$. The proof of this latter statement is the same as of Theorem 1.d.7 of [**12**, p. 51] (see also Johnson, Maurey, Schechtman and Tzafriri [**6**]).

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