CONVEXITY CONDITIONS FOR NON-LOCALLY
CONVEX LATTICES

by N. J. KALTON†

(Received 15 November, 1982)

1. Introduction. First we recall that a (real) quasi-Banach space \( X \) is a complete
metrizable real vector space whose topology is given by a quasi-norm \( \|x\| \) satisfying

\[
\|x\| > 0 \quad (x \in X, x \neq 0) \\
\|\alpha x\| = |\alpha| \|x\| \quad (\alpha \in \mathbb{R}, x \in X) \\
\|x_1 + x_2\| \leq C(\|x_1\| + \|x_2\|) \quad (x_1, x_2 \in X),
\]

where \( C \) is some constant independent of \( x_1 \) and \( x_2 \). \( X \) is said to be \( p \)-normable (or
topologically \( p \)-convex), where \( 0 < p \leq 1 \), if for some constant \( B \) we have

\[
\|x_1 + \ldots + x_n\| \leq B (\|x_1\|^p + \ldots + \|x_n\|^p)^{1/p} \quad (1.4)
\]
for any \( x_1, \ldots, x_n \in X \). A theorem of Aoki and Rolewicz (see [18]) asserts that if in (1.3)
\( C = 2^{1/p-1} \), then \( X \) is \( p \)-normable. We can then equivalently re-norm \( X \) so that in (1.4)
\( B = 1 \).

If in addition \( X \) is a vector lattice and \( \|x\| \leq \|y\| \) whenever \( |x| \leq |y| \) we say that \( X \) is a
quasi-Banach lattice. As in the case of Banach lattices [13] we may make the following
definitions.

We shall say that \( X \) satisfies an upper \( p \)-estimate if for some constant \( C \) and any
\( x_1, \ldots, x_n \in X \) we have

\[
\|x_1 \vee \ldots \vee x_n\| \leq C \left( \sum_{i=1}^{n} \|x_i\|^p \right)^{1/p} \quad (1.5)
\]

We shall say that \( X \) is (lattice) \( p \)-convex if for some \( C \) and any \( x_1, \ldots, x_n \in X \)

\[
\left\| \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\| \leq C \left( \sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}. \quad (1.6)
\]

Here the element \( (|x_1|^p + \ldots + |x_n|^p)^{1/p} \) \((0 < p < \infty)\) of \( X \) can be defined unambiguously
exactly as for the case of Banach lattices (cf. [13, pp 40–41] and Popa [17]).

For \( 0 < p \leq 1 \) it is trivial to see that lattice \( p \)-convexity implies \( p \)-normability and
\( p \)-normability implies the existence of an upper \( p \)-estimate. In the case \( p = 1 \), lattice
\( 1 \)-convexity is equivalent to normability (i.e. \( X \) is a Banach lattice). However Popa [17]
of-fers that for \( 0 < p < 1 \), the space “weak \( L_p \)” \( L(p, \infty) \) of measurable functions on \((0, 1)\)

† Supported by NSF grant MCS-8001852.

such that
\[ \|f\| = \sup_{0 < t < \infty} tm(|f| > t)^{1/p} < \infty \]
is \(p\)-normable but not lattice \(p\)-convex.

In this note we introduce the class of \(L\)-convex quasi-Banach lattices. We say that \(X\) is \(L\)-convex if there exists \(0 < \varepsilon < 1\) so that if \(u \in X_+\) with \(\|u\| = 1\) and \(0 \leq x_i \leq u\) \((1 \leq i \leq n)\) satisfy
\[ \frac{1}{n} (x_1 + \ldots + x_n) \geq (1 - \varepsilon)u, \]
then
\[ \max_{1 \leq i \leq n} \|x_i\| \geq \varepsilon. \]

Roughly speaking, \(X\) is \(L\)-convex if its order-intervals are uniformly locally convex.

It turns out that most naturally arising function spaces are \(L\)-convex lattices (e.g. the \(L_p\)-spaces, Orlicz spaces, Lorentz spaces including the spaces \(L(p, \infty)\) introduced above). However we shall give examples of non \(L\)-convex lattices. We shall show that \(X\) is \(L\)-convex if and only if \(X\) is lattice \(p\)-convex for some \(p > 0\). If \(\ell_\infty\) is not lattice finitely representable in \(X\) then \(X\) is necessarily \(L\)-convex. We also show that if \(X\) is a quasi-Banach lattice linearly homeomorphic to a subspace of an \(L\)-convex lattice then \(X\) is again \(L\)-convex.

\(L\)-convex lattices behave similarly to Banach lattices in many respects. For example if \(X\) is \(L\)-convex and satisfies an upper \(p\)-estimate, then \(X\) is lattice \(r\)-convex for any \(r < p\) (compare [13], p. 85 and results of Maurey and Pisier [14], [16]). Also for \(0 < p < 1\), if \(X\) is \(L\)-convex and satisfies an upper \(p\)-estimate, then \(X\) is \(p\)-normable. This is false for \(p = 1\); \(L(1, \infty)\) is a counter-example. However an analogous result for \(1 < p < 2\) involving type due to Figiel and Johnson is given in [13, p. 88]. By contrast, in general if a quasi-Banach lattice satisfies an upper \(p\)-estimate, then it is \(q\)-normable, where \(q^{-1} = p^{-1} + 1\) and this result is best possible.

2. \(L\)-convexity. Before proving our basic lemma, it will be convenient to introduce some terminology. Suppose \(X\) is a quasi-Banach lattice and \(u \in X_+\) with \(u \neq 0\). Then if we set \(Y = \bigcup_{n=1}^\infty [-nu, nu] \) \(Y\) is a sublattice of \(X\); if we select \([-u, u]\) as the unit ball of \(Y\) then \(Y\) is an abstract \(M\)-space, and by a well-known theorem of Kakutani ([13, p. 16], [19, p. 104]) there is a compact Hausdorff space \(\Delta\) so that \(Y\) is isometrically lattice isomorphic to \(C(\Delta)\). Thus we can induce a lattice homomorphism \(J: C(\Delta) \rightarrow X\) so that \(J\) maps the unit ball of \(C(\Delta)\) onto the order interval \([-u, u]\). We call \(J\) the Kakutani map associated to \(u\).

**Lemma 2.1.** Let \(X\) be an \(L\)-convex quasi-Banach lattice satisfying an upper \(p\)-estimate. Then

(a) if \(0 < p < r\), there is a constant \(M\) so that if \(x_1, \ldots, x_n \in X\) we have
\[ \left\| \left( \sum |x_i| \right)^{1/r} \right\| \leq M \left( \sum \|x_i\|^p \right)^{1/p}. \]
(b) If $0 < r < p$ there is a constant $M$ so that if $x_1, \ldots, x_n \in X$ we have

$$
\left\| \left( \sum |x_i|^r \right)^{1/r} \right\| \leq M \left( \sum \|x_i\|^r \right)^{1/r}.
$$

**Proof.** We shall suppose $C < \infty$ and $0 < \epsilon < 1$ are chosen as in (1.5) and (1.7). Without loss of generality in both parts (a) and (b) we may assume $x_i \equiv 0$ ($1 \leq i \leq n$) and that $\|u\| = 1$, where $u = \left( \sum |x_i|^r \right)^{1/r}$. Let $J : C(\Delta) \to X$ be the Kakutani map associated to $u$. Let $Jf_i = x_i$ where $0 \leq f_i \leq 1$. Choose $\tau > 0$ so that

$$1 - \exp(-\tau^r) \geq 1 - \frac{1}{2} \epsilon.
$$

Let $(\Omega, P)$ be some probability space and let $(\xi_i : 1 \leq i \leq n)$ be independent positive random variables on $\Omega$ so that for each $i$

$$P(\xi_i > t) = 1 - \epsilon \quad (t \geq 1).
$$

If $s \in \Delta$ and if $\max f_i(s) \leq \tau$ then

$$P(\max \xi f_i(s) > \tau) = 1 - \prod_{i=1}^n P(\xi_i \leq \tau f_i(s)^{-1})
$$

$$= 1 - \prod_{i=1}^n (1 - \tau^r f_i(s)^r)
$$

$$\geq 1 - \prod_{i=1}^n \exp(-\tau^r f_i(s)^r)
$$

$$= 1 - \exp(-\tau^r)
$$

$$\geq 1 - \frac{1}{2} \epsilon.
$$

(2.1)

Here we use the fact that $J((\sum f_i^r)^{1/r}) = \left( \sum |x_i|^r \right)^{1/r} = u = J1$, so that $\sum f_i(s)^r = 1$ for $s \in \Delta$.

Now (2.1) holds trivially if we suppose $\max f_i(s) > \tau$. Thus we conclude

$$\int_{\Omega} \max_{i \leq n} (\min(\xi_i(\omega)f_i(s), \tau)) \, dP(\omega) \geq \tau(1 - \frac{1}{2} \epsilon).
$$

(2.2)

For each $k \in \mathbb{N}$ we define $\xi_{ik}$ ($1 \leq i \leq n$) by

$$\xi_{ik}(\omega) = \left( \frac{2^k}{m} \right)^{1/r} \left( \frac{2^k}{m} \right)^{1/r} \leq \xi_i(\omega) < \left( \frac{2^k}{m-1} \right)^{1/r}
$$

for $m = 1, 2, \ldots, 2^k$. Then $\lim_{k \to \infty} \xi_{ik} = \xi_i$ a.e. and for each $k \in \mathbb{N}$ the random variables $(\xi_{ik} : 1 \leq i \leq n)$ are independent and generate a finite algebra $\mathcal{A}_n$ in $\Omega$ with $2^{kn}$ atoms each of probability $2^{-kn}$. Set

$$g_k(s) = \int_{\Omega} \max_{i \leq n} (\min(\xi_{ik}(\omega)f_i(s), \tau)) \, dP(\omega).
$$
Then $g_k \in C(\Delta)$ and the sequence $g_k$ is monotone increasing. From (2.2) we deduce that
\[
\lim_{k \to \infty} g_k(s) \geq \tau(1 - \frac{1}{4}\varepsilon).
\]

Now, by Dini's theorem, there exists $k \in \mathbb{N}$ so that $g_k(s) \geq \tau(1 - \frac{1}{2}\varepsilon)$ for every $s \in \Delta$. Suppose $A \in \mathcal{A}_k$ and $P(A) \leq \frac{1}{2}\varepsilon$; then
\[
\int_{\Omega \setminus A} \max_{i \leq n} (\min(\xi_{ik}(\omega)f_i(s), \tau)) \, dP(\omega) \geq \tau(1 - \varepsilon).
\]

This implies that $(1 - \varepsilon)u$ is dominated by an average of the finitely many distinct values of $\left(\tau^{-1} \max_{i \leq n} \xi_{ik}(\omega)x_i\right) \wedge u$. Thus
\[
\max_{\omega \in \Omega \setminus A} \left\| \max_{i \leq n} \xi_{ik}(\omega)x_i \right\| \geq \tau\varepsilon
\]
from the definition of $L$-convexity (equation (1.7)). Hence
\[
P\left( \left\| \max_{i \leq n} \xi_{i}(\omega)x_i \right\| \geq \tau\varepsilon \right) \geq \frac{1}{2}\varepsilon.
\]

Since $X$ satisfies an upper $p$-estimate,
\[
P\left( \left( \sum_{i=1}^{n} |\xi_{i}(\omega)|^p \|x_i\|^p \right)^{1/p} \geq C^{-1}\tau\varepsilon \right) \geq \frac{1}{2}\varepsilon.
\]

Now we consider two cases. In case (a) if $0 < p < r$ then
\[
\int_{\Omega} \sum_{i=1}^{n} |\xi_{i}(\omega)|^p \|x_i\|^p \, dP(\omega) \geq \frac{1}{2}C^{-p}\tau^p\varepsilon^{p+1}
\]
and
\[
\int_{\Omega} |\xi_{i}|^p \, dP = B < \infty.
\]

Hence
\[
\sum_{i=1}^{n} \|x_i\|^p \geq \frac{1}{2}B^{-1}C^{-p}\tau^p\varepsilon^{p+1}
\]
so that (a) follows.

In case (b) pick $\alpha > 1$ so that $\alpha > p$. Let $\eta_i = \xi_{i}\eta_i^\alpha$ so that $P(\eta_i > t) = t^{-\alpha/p}$ for $t \geq 1$. By Lemma 1.f.8 of [13, p. 86] there is a constant $B$ so that
\[
\int_{\Omega} \left( \sum_{i=1}^{n} a_i^{\alpha} \eta_i^\alpha \right)^{1/\alpha} \, dP \leq B \left( \sum_{i=1}^{n} |a_i|^{\alpha/(\alpha)} \right)^{\alpha/p}
\]
for $a_1, \ldots, a_n \geq 0$. Now, for $\delta$ depending only on $C$ and $\varepsilon$,
\[
\int_{\Omega} \left( \sum_{i=1}^{n} |\eta_i(\omega)|^{\alpha} (\|x_i\|^{\alpha})^{\alpha} \right)^{1/\alpha} \, dP \geq \delta
\]
and so
\[
B \left( \sum_{i=1}^{n} \|x_i\|^p \right)^{\alpha/p} \geq \delta.
\]

Thus (b) follows.
The next theorem should be compared with the Banach lattice case (Theorem 1.f.7 of [13, p. 85]).

**Theorem 2.2.** Let $X$ be a quasi-Banach lattice satisfying an upper $p$-estimate. Then the following conditions on $X$ are equivalent:

(i) $X$ is $L$-convex

(ii) $X$ is lattice $r$-convex for some $r > 0$.

(iii) $X$ is lattice $r$-convex for every $r$, $0 < r < p$.

(i) $\Rightarrow$ (iii): This is simply Lemma 2.1 (b).

(iii) $\Rightarrow$ (ii): This is immediate.

(ii) $\Rightarrow$ (i): We assume $r < 1$. Suppose $0 < x_i < u$ where $\|u\| = 1$ and that

$$\frac{1}{n}(x_1 + \ldots + x_n) \geq \frac{1}{2}u.$$

Then

$$(x_1 + \ldots + x_n) \leq u^{1-r}(x_1^r + \ldots + x_n^r),$$

where the right-hand side is well-defined in $X$, cf. [12, pp. 41-43]. Hence

$$\frac{1}{2}nu \leq u^{1-r}(x_1^r + \ldots + x_n^r)$$

and so

$$(x_1^r + \ldots + x_n^r)^{1/r} \geq (\frac{1}{2}n)u.$$

Thus

$$\left(\frac{1}{2}n\right)^{1/r} \leq C \left(\sum \|x_i\|^r\right)^{1/r}$$

so that

$$\max_{i \leq n} \|x_i\| \geq \left(\frac{1}{2}\right)^{1/r} C^{-1}.$$

If $r \geq 1$ the argument is simpler, since

$$(x_1^r + \ldots + x_n^r)^{1/r} \geq n^{1/r-1}(x_1 + \ldots + x_n).$$

**Theorem 2.3.** Let $X$ be a quasi Banach lattice satisfying an upper $p$-estimate where $0 < p < \infty$. Then

(i) $X$ is $q$-normable where $1/q = 1/p + 1$;

(ii) if $0 < p < 1$ and $X$ is $L$-convex, then $X$ is $p$-normable;

(iii) if $1 < p < \infty$ and $X$ is $L$-convex, then $X$ is a Banach lattice.

**Proof.** (i) We suppose (1.5) holds. Suppose $x_1, \ldots, x_n \in X$, and $u = x_1 + \ldots + x_n$. Let

$$\sigma = (\|x_1\|^q + \ldots + \|x_n\|^q)^{1/q}$$

and observe that

$$\|u\| \leq \max_{i \leq n} \sigma^q \|x_i\|^{q-1} x_i$$

$$\leq C \left(\sum_{i=1}^n \sigma^q \|x_i\|^{q-1} \|x_i\|^p\right)^{1/p}$$

$$= C \sigma^q \left(\sum_{i=1}^n \|x_i\|^q\right)^{1/p} = C \sigma^{q + a/p} = C \sigma.$$
(ii) This is simply Lemma 2.1 (a) with \( r = 1 \)

(iii) By Theorem 2.2 \( X \) is lattice \( 1 \)-convex i.e. a Banach lattice.

**Example 2.4.** Let \( \mathcal{A} \) be an algebra of subsets of some set \( \Omega \) and let \( \phi : \mathcal{A} \to \mathbb{R} \) be a normalized submeasure, i.e. \( \phi \) is a set-function satisfying \( \phi(\emptyset) = 0 \), \( \phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B) \) for \( A, B \in \mathcal{A} \) and \( \phi(\Omega) = 1 \). From \( \phi \) we can construct a quasi-Banach lattice \( L_p(\phi) \) satisfying an upper \( p \)-estimate for \( 0 < p < \infty \). If \( f : \Omega \to \mathbb{R} \) is a simple \( \mathcal{A} \)-measurable function we define

\[
\|f\|_p = \left( \int_0^\infty \phi(|f| \geq t^{1/p}) \, dt \right)^{1/p}.
\]

Then \( \| \cdot \|_p \) is a quasi-norm; indeed

\[
\|f + g\|_p^p = \int_0^\infty \phi(|f + g| \geq t^{1/p}) \, dt \\
\leq \int_0^\infty \phi(|f| \geq \frac{1}{2} t^{1/p}) \, dt + \int \phi(|g| \geq \frac{1}{2} t^{1/p}) \, dt \\
\leq 2^p \|f\|_p^p + \|g\|_p^p
\]

so that

\[
\|f + g\|_p \leq 2^{1/p} (\|f\|_p + \|g\|_p) \quad (0 < p \leq 1),
\]

\[
\|f + g\|_p \leq 2 (\|f\|_p + \|g\|_p) \quad (1 \leq p < \infty).
\]

The completion of the simple functions \( S(\mathcal{A}) \) with this quasi-norm is a quasi-Banach lattice \( L_p(\phi) \) satisfying an upper \( p \)-estimate.

Suppose now \( \phi \) is pathological ([3], [4]), that is so that whenever \( 0 \leq \lambda \leq \phi \) and \( \lambda \) is additive then \( \lambda = 0 \). Then for any \( \epsilon > 0 \) there exist \( E_1, \ldots, E_n \in \mathcal{A} \) so that \( \phi(E_i) \leq \epsilon \) but \( 1/n \sum E_i \geq (1 - \epsilon) \Omega \) ([3]). It follows quickly that \( L_p(\phi) \) is not \( L \)-convex.

Furthermre (Talagrand [20]) \( \phi \) can be chosen so that for every \( n \) there exist \( E_1, \ldots, E_n \in \mathcal{A} \) with \( \phi(E_i) \leq n^{-1} \) and \( 1/n \sum E_i \geq \frac{1}{2} \Omega \). Suppose \( L_p(\phi) \) is \( q \)-normable. Then

\[
\frac{1}{2} \leq C \left( \frac{1}{n} \left( \sum_{i=1}^n \|E_i\|_p^q \right)^{1/q} \right)^{1/a} = Cn^{1/a - 1/p - 1} \quad (n \in \mathbb{N}).
\]

Hence \( 1/a \geq 1/p + 1 \) so that Theorem 2.3 (a) is best possible.

By way of contrast we observe that the space \( L(p, \infty) \) is \( L \)-convex for \( 0 < p < 1 \). In fact if \( 0 < r < p \), \( L(p, \infty) = \{ f : \|f^r \| \in L(pr^{-1}, \infty) \} \) and \( L(pr^{-1}, \infty) \) is a Banach lattice, i.e. is locally convex (see [5]). Hence \( L(p, \infty) \) is lattice \( r \)-convex for \( 0 < r < p \). As \( L(p, \infty) \) satisfies an upper \( p \)-estimate, it is \( p \)-normable (see [8]).

### 3. Some applications of a theorem of Bennett and Maurey.

In this section we show how a deep factorization theorem of Bennett and Maurey ([1], [2], [15]) can be used to extend a result of Krivine [12] on operators between Banach lattices (cf. [13, p. 93]). This
latter result is of considerable importance in studying operators between function spaces (see \([10]\)).

We start by stating that Bennett-Maurey theorem (see \([1]\) or \([2]\) for this statement).

**Theorem 3.1.** Let \(0 < p < 1\) be fixed. Then there is a constant \(C = C(p)\) so that whenever \(m, n \in \mathbb{N}\) and \(T: \ell_m^p \rightarrow \ell_n^p\) is a linear operator then there is a positive \(D: \ell_n^p \rightarrow \ell_m^p\) given by \(D(\xi) = (d_i \xi)\) so that \(\|DT\| \leq \|T\|\) and \(\sum d_i^{-p(1-p)} \leq C\).

**Corollary 3.2.** Suppose \(0 < p < 1\). Then there is a constant \(B = B(p)\) so that if \(\Delta, K\) are compact Hausdorff spaces, \(\mu\) is a probability measure on \(K\) and \(T: C(\Delta) \rightarrow L_p(K, \mu)\) is a bounded linear operator, then for \(f_1, \ldots, f_n \in C(\Delta)\), we have

\[
\left\| \left( \sum_{i=1}^n |Tf_i|^2 \right)^{1/2} \right\|_p \leq B \|T\| \left\| \left( \sum_{i=1}^n |f_i|^2 \right)^{1/2} \right\|_p
\]

**Proof.** Exactly as step 2 of Theorem 1.f.14 of \([1, p. 92]\) this can be reduced to consideration of a map \(T: \ell_m^p \rightarrow \ell_n^p\). Now by Theorem 3.1 we can find \(D: \ell_n^p \rightarrow \ell_m^p\) so that \(\|DT\| \leq \|T\|\) and \(D(\xi) = (d_i \xi)\) where \(\sum d_i^{-p(1-p)} \leq C\). Then

\[
\left\| \left( \sum_{i=1}^n |Tf_i|^2 \right)^{1/2} \right\|_p \leq \left( \sum_{i=1}^n |Df_i|^2 \right)^{1/2} \leq C \left\| \left( \sum_{i=1}^n |f_i|^2 \right)^{1/2} \right\|_p
\]

by Theorem 1.f.14 of \([13]\). Let \(B = C^{1/p-1}K_0\).

**Theorem 3.3.** Let \(Y\) be an \(L\)-convex quasi-Banach lattice. Then there is a constant \(A\) depending only on \(Y\) so that whenever \(X\) is a quasi-Banach lattice and \(T: X \rightarrow Y\) is a bounded linear operator then for any \(x_1, \ldots, x_n \in X\)

\[
\left\| \left( \sum_{i=1}^n |Tx_i|^2 \right)^{1/2} \right\| \leq A \|T\| \left\| \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2} \right\|
\]

**Proof.** First we observe that \(Y\) is lattice \(p\)-convex for some \(p > 0\) and hence satisfies (1.6) for some \(C\).

If \(x_1, \ldots, x_n \in X\) let \(v = (\sum |Tx_i|^2)^{1/2}\) and \(u = (\sum |x_i|^2)^{1/2}\). We may suppose \(u, v \neq 0\). Let \(J_v: C(\Delta_v) \rightarrow X\) and \(J_u: C(\Delta_u) \rightarrow Y\) be associated Kakutani maps.

If \(f_1, \ldots, f_m \in C(\Delta_u)\),

\[
\left\| J_v \left( \sum_{i=1}^m |f_i|^p \right)^{1/p} \right\| \leq C \left( \sum_{i=1}^m |J_v f_i|^p \right)^{1/p}.
\]

As \(J_v\) is positive this implies that for some \(s \in K\)

\[
\left( \sum_{i=1}^m |f_i(s)|^p \right)^{1/p} \leq C \|v\|^{-1} \left( \sum_{i=1}^m |J_v f_i|^p \right)^{1/p}.
\]
Now by a standard Hahn-Banach separation argument there is a probability measure \( \mu \) on \( \Delta_v \) so that for \( f \in C(\Delta_v) \),
\[
\int_{\Delta_v} |f|^p \, d\mu \leq C^n \|v\|^{-p} \|J_uf\|^p.
\]

For \( x \in X_v \) define \( Sx \in L_p(\Delta_v, \mu) \) by
\[
Sx = \sup_n J^{-1}(x \wedge n u)
\]
and extend \( S \) linearly. Then \( S \) is a lattice-homomorphism and \( \|S\| \leq C\|v\|^{-1} \).

Now consider \( STJ_u : C(\Delta_u) \to L_p(\Delta_v, \mu) \). By Theorem 3.2, if \( f_1, \ldots, f_n \in C(\Delta_u) \) are chosen so that \( J_uf_i = x_i \),
\[
\left( \sum_{i=1}^n |STJ_u f_i|^2 \right)^{1/2} \leq B \|STJ_u\| \left( \sum_{i=1}^n |f_i|^2 \right)^{1/2}
\]
where \( B \) depends only on \( p \).

Now, since \( S \) is a lattice-homomorphism,
\[
\left( \sum_{i=1}^n |STJ_u f_i|^2 \right)^{1/2} = \|S\left( \sum_{i=1}^n |T x_i|^2 \right)^{1/2} = \|Sv\| = 1.
\]

On the other hand \( \left( \sum_{i=1}^n |f_i|^2 \right)^{1/2} = 1 \) and so
\[
1 \leq B\|STJ_u\| \leq BC\|v\|^{-1} \|T\| \|u\|
\]
so that
\[
\|v\| \leq A \|T\| \|u\|
\]
where \( A = BC \).

Applying Theorem 3.3 in the case \( X = \ell^\infty_\infty \) we obtain the following result.

**Corollary 3.4.** Suppose \( Y \) is an \( L \)-convex quasi-Banach lattice. Then there is a constant \( A \) so that if \( y_1, \ldots, y_n \in Y \) then
\[
\left( \sum_{i=1}^n |y_i|^2 \right)^{1/2} \leq A \sup_{|ai| = 1} \|a_1 y_1 + \ldots + a_n y_n\|.
\]

**Proof.** Apply the theorem to the map \( T : \ell^\infty_\infty \to Y \) given by \( Te_i = y_i \) where \( \{e_i\} \) are the basis vectors in \( \ell^\infty_\infty \).

**Example 3.5.** We do not know whether the conclusions of Theorem 3.3 or Corollary 3.4 characterize \( L \)-convex lattices. However we can give an example to show that both are false without the \( L \)-convexity assumption.

Our example will be of the form of an \( \ell^\infty_\infty \)-product of spaces of the type \( L_1(\phi_n) \), where each \( \phi_n \) is a submeasure. We then need only produce \( \phi_n \) to show that there is no uniform constant \( A \) valid for each \( n \).
Let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$ i.e.

$$S^{n-1} = \{(\xi_1, \ldots, \xi_n) : \xi_1^2 + \ldots + \xi_n^2 = 1\}.$$ 

Let $\mathcal{A}$ be the algebra of all subsets of $S^{n-1}$.

If $a \in \mathbb{R}^n$ and $a \neq 0$ let $B_a \in \mathcal{A}$ be defined by $B_a = \{\xi : a \cdot \xi \neq 0\}$. For any set $a^{(1)}, \ldots, a^{(n-1)} \in \mathbb{R}^n \setminus \{0\}$ there exists $\xi \in S^{n-1}$ so that $a^{(1)} \cdot \xi = \ldots = a^{(n-1)} \cdot \xi = 0$ so that $\bigcup_{i=1}^{n-1} B_{a^{(i)}} \neq S^{n-1}$. Define $\phi_n : \mathcal{A} \to \mathbb{R}$ by

$$\phi_n(A) = \frac{1}{n} \inf \left\{ k : A \subseteq \bigcup_{i=1}^{k} B_{a^{(i)}} \right\}.$$ 

Then $\phi_n$ is a normalized submeasure.

Let $f_i(\xi) = \xi_i$. Then if $|a_i| \leq 1$, $|a_1 f_1 + \ldots + a_n f_n| \leq \sqrt{n} 1_{B_{a^{(i)}}}$. Hence

$$\|a_1 f_1 + \ldots + a_n f_n\| \leq \sqrt{n} \cdot \frac{1}{n} = n^{-1/2}.$$ 

However $(f_1^2 + \ldots + f_n^2)^{1/2} = 1$ and $\|1\| = 1$.

### 4. Further conditions for $L$-convexity

Our first result in this section shows that a wide class of quasi-Banach lattices are automatically $L$-convex. We say that $\ell_\infty$ is lattice finitely representable in $X$ if given $\epsilon > 0$ and $n \in \mathbb{N}$ there exist $x_i \geq 0$ (1 $\leq i \leq n$) so that $x_i \wedge x_j = 0$ (i $\neq j$), $\|x_i\| = 1$ (1 $\leq i \leq n$) and whenever $a_1, \ldots, a_n \in \mathbb{R}$

$$\|a_1 x_1 + \ldots + a_n x_n\| \leq (1 + \epsilon) \max_{1 \leq i \leq n} |a_i|.$$ 

If $\ell_\infty$ is not lattice finitely representable in $X$, then there exists $c > 1$ and $n \in \mathbb{N}$ so that for any sequence $(x_1, \ldots, x_n)$ of disjoint elements we have

$$\|x_1 + \ldots + x_n\| \geq c \min_{1 \leq i \leq n} \|x_i\|.$$ 

It then follows quickly by induction that for every $d > 1$ there exists $N \in \mathbb{N}$ so that for disjoint $x_1, \ldots, x_N$,

$$\|x_1 + \ldots + x_N\| \geq d \min_{1 \leq i \leq N} \|x_i\|.$$ 

We remark that if $F$ is an Orlicz function satisfying the $\Delta_2$-condition then $\ell_\infty$ is not lattice finitely representable in the Orlicz space $L_F(0, 1)$; equally $\ell_\infty$ is not lattice finitely representable in the Lorentz space $L(p, q)$ if $0 < q \leq \infty$ (cf. [5]).

**Theorem 4.1.** Let $X$ be a quasi-Banach lattice such that $\ell_\infty$ is not lattice finitely representable in $X$. Then $X$ is $L$-convex.

**Proof.** We can and do suppose $X$ is $p$-normed; that is for suitable $0 < p < 1$

$$\|x_1 + \ldots + x_n\| \leq \left(\|x_1\|^p + \ldots + \|x_n\|^p\right)^{1/p},$$ 

for $x_1, \ldots, x_n \in X$. 

Fix $N \in \mathbb{N}$ so that for any sequence of disjoint elements $(x_1, \ldots, x_N)$ we have

$$\|x_1 + \ldots + x_N\| \geq 6^{1/p} \min_{i \leq N} \|x_i\|.$$ 

Then fix $\varepsilon$, $0 < \varepsilon < 1$ so that $\varepsilon < \frac{1}{32} e^{-1}$ and $\varepsilon < (1/32) e^{-2} N^{-1}$. Suppose that $u \in X_+$, with $0 \leq x_i \leq u$ and $(1/m)(x_1 + \ldots + x_m) \geq (1 - \frac{1}{2} \varepsilon) u$.

Let $J : C(\Delta) \to X$ be the Kakutani map associated to $u$. We claim first that $J$ is exhaustive; that is if $\{f_i : i \in \mathbb{N}\}$ is a uniformly bounded disjoint sequence in $C(\Delta)$ then $Jf_i \to 0$. This follows easily from the hypothesis on $X$. Now by a theorem of Thomas [29] (cf. also [7], [9]), there is a regular $X$-valued measure $\mu$ defined on the Borel sets $\beta$ of $\Delta$ so that

$$Jf = \int f \, d\mu \quad (f \in C(\Delta)).$$

We remark that $\co \mu(\beta)$ is bounded and so there is no difficulty in defining the integral of any bounded Borel function. It is easy to see that $\mu(\Delta) = u$ and $\mu$ is monotone; that is $0 \leq \mu(A) \leq \mu(B)$ whenever $A \subset B$.

Let $\phi : B \to \mathbb{R}$ be defined by $\phi(A) = \|\mu(A)\|^p$. Then $\phi$ is a submeasure. We shall show that $\phi$ satisfies the hypotheses of [11, Lemma 3.1]. If $A_1, \ldots, A_N$ are disjoint sets, then $\mu(A_1), \ldots, \mu(A_N)$ are disjoint in $X$ and so

$$1 \geq \|\mu(A_1 \cup \ldots \cup A_N)\|^p \geq 6 \min \|\mu(A_i)\|^p,$$

so that $\min \phi(A_i) \leq \frac{1}{6}$. Hence if $A_1, \ldots, A_n$ are disjoint, then, as required,

$$\sum_{i=1}^n \phi(A_i) \leq N + \frac{1}{6} n. \quad (3.1)$$

Choose $g_i$ $(1 \leq i \leq m)$ so that $Jg_i = x_i$. Let $B_i = \{g_i \geq \frac{1}{2}\}$. Then

$$\frac{1}{m} \sum_{i=1}^m 1_{B_i} \geq (1 - \varepsilon) 1_{\Delta}.$$

From Lemma 3.1 and Proposition 2.3 of [11] we deduce (taking $r = 3$ in the statement of the lemma)

$$\frac{1}{m} \sum_{i=1}^m \phi(B_i) \geq 1 - 3 \cdot \frac{1}{6} - N(2e^2)^{1/2} \varepsilon^{1/2} \geq \frac{1}{4}$$

so that

$$\max_{1 \leq i \leq m} \phi(B_i) \geq \frac{1}{4}.$$

Hence

$$\max \|x_i\| \geq \frac{1}{2} \|u\|^{1/p} \geq \varepsilon,$$

so that $X$ is $L$-convex.
**Theorem 4.2.** Let $Y$ be an $L$-convex quasi-Banach lattice and let $X$ be a quasi-Banach lattice linearly homeomorphic to a subspace of $Y$. Then $X$ is $L$-convex.

**Proof.** We shall suppose $Y$ is lattice $p$-convex for some $p$, $0 < p \leq 1$ satisfying equation (1.6), i.e.

$$\left\| \left( \sum |y_i|^p \right)^{1/p} \right\| \leq C \left( \sum \|y_i\|^p \right)^{1/p}$$

for $y_1, \ldots, y_n \in Y$. We also suppose that the conclusion of Theorem 3.3 holds with constant $A < \infty$. Let $T : X \to Y$ be a linear operator so that

$$B^{-1} \|x\| \leq \|Tx\| \leq B \|x\| \quad (x \in X),$$

for some constant $B < \infty$.

If $X$ is not $L$-convex, then given $\delta > 0$ we can find $u \in X_+$ with $\|u\| = 1$ and $0 \leq x_i \leq u$ $(1 \leq i \leq n)$ so that $(1/n) (x_1 + \ldots + x_n) \geq (1 - \delta) u$ and $\|x_i\| \leq \delta$ $(1 \leq i \leq n)$.

Let $y_i = Tx_i$. Then

$$\left\| \left( \sum |y_i|^p \right)^{1/p} \right\| \leq C \left( \sum \|y_i\|^p \right)^{1/p} \leq CB \left( \sum \|x_i\|^p \right)^{1/p} \leq CBn^{1/p} \delta.$$

On the other hand

$$\left\| \left( \sum |x_i|^2 \right)^{1/2} \right\| \leq \left\| \left( \sum |y_i|^2 \right)^{1/2} \right\| \leq An^{1/2} \|u\| = An^{1/2}.$$

Let $v_1 = \delta^{-1} n^{-1/p} \left( \sum |y_i|^p \right)^{1/p}$ and $v_2 = n^{-1/2} \left( \sum |y_i|^2 \right)^{1/2}$. Let $\theta = p(2-p)^{-1}$. Then

$$\delta^{-\theta} n^{-1} \sum |y_i| \leq v_1 v_2^{1-\theta}.$$

[This is easily seen by using a Kakutani map to represent the elements of $Y$ as functions.] Hence

$$n^{-1} \sum |y_i| \leq \delta^\theta (\theta v_1 + (1 - \theta) v_2) \leq \delta^\theta (v_1 + v_2)$$

and so if $C'$ is the constant occurring in equation (1.3) for quasi-norms,

$$\left\| n^{-1} \sum |y_i| \right\| \leq \delta^\theta C'(A + CB).$$

Now

$$\left\| \sum |y_i| \right\| = \left\| \sum |Tx_i| \right\| = \left\| T \left( \sum x_i \right) \right\| \geq B^{-1} \left\| \sum x_i \right\|.$$

Hence

$$(1 - \delta) \leq \delta^\theta BC'(A + CB).$$

For small enough $\delta$ this is a contradiction and so $X$ is $L$-convex.

**Conjecture.** If $Y$ is lattice $p$-convex where $0 < p < 1$, then $X$ is lattice $p$-convex.

We remark that the conjecture is true for $p = 1$ trivially and for $0 < p < 2$, if we assume $\ell_\infty$ is not lattice finitely representable in $X$. The proof of this latter statement is the
same as of Theorem 1.d.7 of [12, p. 51] (see also Johnson, Maurey, Schechtman and Tzafriri [6]).

REFERENCES

2. G. Bennett, Lectures on matrix transformations of $\ell^p$ spaces, in Notes in Banach spaces (H. E. Lacey, ed.) (University of Texas Press, Austin, Texas, 1980).
3. J. P. R. Christensen, Some results with relation to the control measure problem, in Vector space measures and applications II. Lecture Notes in Mathematics No 645 (Springer-Verlag, 1978).
15. B. Maurey Théorèmes de factorisation pour les operateurs linéaires à valeurs dans un espace $L_p$ (Asterisque No 11, 1974).