# CURVES WITH ZERO DERIVATIVE IN $F$-SPACES

*by* N. J. KALTON

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**1. Introduction.** Let $X$ be an $F$-space (complete metric linear space) and suppose $g:[0, 1] \to X$ is a continuous map. Suppose that $g$ has zero derivative on $[0, 1]$, i.e.

$$g'(t) = \lim_{h \to 0} \frac{1}{h}(g(t+h) - g(t)) = 0$$

for $0 \le t \le 1$ (we take the left and right derivatives at the end points). Then, if $X$ is locally convex or even if it merely possesses a separating family of continuous linear functionals, we can conclude that $g$ is constant by using the Mean Value Theorem. If however $X^* = \{0\}$ then it may happen that $g$ is not constant; for example, let $X = L_p(0, 1)$ $(0 \le p < 1)$ and $g(t) = 1_{[0,t]}$ $(0 \le t \le 1)$ (the characteristic function of $[0, t]$). This example is due to Rolewicz [**6**], [**7**; p. 116].

The aim of this note is to substantiate a conjecture of Rolewicz [**7**, p. 116] that every $F$-space $X$ with trivial dual admits a non-constant curve $g:[0, 1] \to X$ with zero derivative. In fact we shall show, given any two points $x_0, x_1 \in X$, there exists a map $g:[0, 1] \to X$ with $g(0) = x_0$, $g(1) = x_1$ and

$$\lim_{|t-s| \to 0} \frac{g(t) - g(s)}{t - s} = 0 \quad \text{uniformly for} \quad 0 \le s, t \le 1.$$

To establish this result we shall need to study $X$-valued martingales. Let $\mathcal{B}$ be the $\sigma$-algebra of Borel subsets of $[0, 1)$ and let $\mathcal{F}_n$ $(n \ge 0)$ be an increasing family of *finite* sub-algebras of $\mathcal{B}$. Then a sequence of functions $u_n:[0, 1) \to X$ is an $X$-valued $F_n$-martingale if each $u_n$ is $F_n$-measurable and for $n \ge m$ we have $\mathcal{E}(u_n \mid \mathcal{F}_m) = u_m$. Here the definition of conditional expectation is the standard one with respect to Lebesgue measure $\lambda$ and there are no integration problems since each $u_n$ is finitely-valued.

It is easy to show that every $F$-space $X$ with trivial dual contains a non-constant martingale $\{u_n, \mathcal{F}_n\}$ which converges to zero uniformly. However we shall need to consider *dyadic martingales*. Let $D_{n,k} = [(k-1)/2^n, k/2^n)$ $(1 \le k \le 2^n, 0 \le n < \infty)$. Then, for $n \ge 0$, let $\mathcal{B}_n$ be the sub-algebra of $\mathcal{B}$ generated by the sets $\{D_{n,k} : 1 \le k \le 2^n\}$. A dyadic martingale is simply a $\mathcal{B}_n$-martingale. The main point of the argument will be to show that we can find non-zero dyadic martingales which converge uniformly to zero.

We note here a connection with the recent work of Roberts [**4**], [**5**] on the existence of compact convex sets without extreme points. Indeed, in a needlepoint space (see [**5**]) it would be easy to show that there are non-zero dyadic martingales which converge uniformly to zero. However there are $F$-spaces with trivial dual which contain no needlepoints [**2**].

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As usual an $F$-norm on a (real) vector space $X$ is a map $x \to \|x\|$ such that

$$\|x\|>0 \quad \text{if} \quad x \neq 0, \tag{1.0.1}$$

$$\|x+y\| \leq \|x\|+\|y\| \qquad (x, y \in X), \tag{1.0.2}$$

$$\|tx\| \leq \|x\| \qquad (|t| \leq 1), \tag{1.0.3}$$

$$\lim_{t \to 0} \|tx\| = 0 \qquad (x \in X). \tag{1.0.4}$$

The $F$-norm is said to be *strictly concave* if, for each $x \in X$ with $x \neq 0$, the map $t \to \|tx\|$ is strictly concave on $[0, \infty)$, i.e.

if $0 \leq s < t < \infty$ and $0 < a, b < 1$ with $a+b=1$ then, if $x \neq 0$,

$$\|(as+bt)x\| > a\,\|sx\| + b\,\|tx\|. \tag{1.0.5}$$

Every $F$-space can be equipped with an (equivalent) $F$-norm which is strictly concave. This follows from the results of Bessaga, Petczyński and Rolewicz [**1**]. We may give $X$ an $F$-norm $\|\cdot\|_0$ so that the map $t \to \|tx\|_0$ is concave and strictly increasing for each $x \neq 0$. Now define $\|x\| = \|x\|_0^{1/2}$.

**2. Preliminary finite-dimensional results.** Suppose $N$ is a positive integer. We consider the space $\mathbb{R}^N$ with the natural co-ordinatewise partial ordering (i.e. $x \geq y$ if and only if $x_i \geq y_i$ for $1 \leq i \leq N$). We shall denote by $(e_k : 1 \leq k \leq N)$ the natural basis elements of $\mathbb{R}^N$. We shall use the idea of $\mathbb{R}^N$-valued submartingales and supermartingales; these have obvious meaning with respect to the ordering defined above. In addition, standard scalar convergence theorems can be applied co-ordinatewise to produce the same theorems for $\mathbb{R}^N$.

For $1 \leq i \leq N$, let $F_i$ be a continuous map $F_i : [0, \infty) \to [0, \infty)$ which is strictly increasing, strictly concave and satisfies $F_i(0)=0$, $F_i(1)=1$. Then $F_i$ is also subadditive since

$$F_i(s) \geq \frac{s}{s+t} F_i(s+t) \qquad (s, t > 0).$$

Hence we may define an absolute $F$-norm on $\mathbb{R}^N$ by

$$\|x\| = \sum_{i=1}^{N} F_i(|x_i|) \qquad (x \in \mathbb{R}^N). \tag{2.0.1}$$

Now, for $x \in \mathbb{R}^N$, define

$$\sigma(x) = \inf\{\max(\|y\|, \|z\|) : x = \tfrac{1}{2}(y+z)\}. \tag{2.0.2}$$

We shall need the following properties of $\sigma$.

LEMMA 2.1. (a) *If $x \in \mathbb{R}^N$ and $x \geq 0$ then there exist $y, z \in \mathbb{R}^N$ with $y \geq 0$, $z \geq 0$, $x = \frac{1}{2}(y+z)$ and $\|y\| \leq \sigma(x)$, $\|z\| \leq \sigma(x)$.*

(b) *For $x, y \in \mathbb{R}^N$,*

$$|\sigma(x) - \sigma(y)| \leq \|x-y\|, \tag{2.1.1}$$

$$\sigma(x) \leq \|x\|. \tag{2.1.2}$$

(c) *If $x \geq 0$ and $\sigma(x) = \|x\| = 1$ then, for some $k$, we have $x = e_k$.*

*Proof.* (a) is an easy consequence of a compactness argument. For (b) (2.1.1), observe that if $x=\frac{1}{2}(z+z')$ then

$$y=\tfrac{1}{2}[(z+y-x)+(z'+y-x)],$$

so that $\sigma(y)\le\sigma(x)+\|y-x\|$ and so (2.1.1) follows. (2.1.2) is an immediate consequence of the definition of $\sigma$.

We are grateful to the referee for the following short proof of (c). Suppose $x\ge 0$, $\|x\|=1$, $x_i>0$ and $x_j>0$ where $i\ne j$. We show $\sigma(x)<1$.

Since $F_i$ is concave, it has left and right derivatives at $x_i$, $\alpha_1$ and $\alpha_2$, say, with $0\le\alpha_2\le\alpha_1$. Similiarly $F_j$ has left and right derivatives at $x_j$, $\beta_1$ and $\beta_2$ with $0<\beta_2\le\beta_1$. For small $t>0$,

$$\begin{aligned}\|x+t(\beta_1e_i-\alpha_2e_j)\|&<\|x\|,\\ \|x-t(\beta_1e_i-\alpha_2e_j)\|&<\|x\|+t(-\alpha_1\beta_1+\beta_2\alpha_2)\\ &\le\|x\|.\end{aligned}$$

Hence $\sigma(x)<1$.

We conclude that if $\sigma(x)=1$ then $x=e_k$ for some $k$, $1\le k\le N$.

Now let $$\pi(x)=x_1+\dots+x_n\qquad(x\in\mathbb{R}^N).$$

THEOREM 2.2. *Suppose $a\in\mathbb{R}^N$, $a\ge 0$ and $\pi(a)=1$. Then there are disjoint Borel subsets $E_1,\dots,E_N$ of $[0,1)$ with $\lambda(E_i)=a_i$ $(1\le i\le N)$ and a scalar valued dyadic supermartingale $\theta_n$ $(0\le n<\infty)$ such that*

$$0\le\theta_n(t)\le 1\qquad(0\le t<1,0\le n<\infty),\tag{2.2.1}$$

$$\lim_{n\to\infty}\theta_n(t)=0\ a.e.\tag{2.2.2}$$

*and if*

$$u_n=\mathscr{E}\Big(\sum_{i=1}^N 1_{E_i}e_i\mid\mathscr{B}_n\Big)\qquad(0\le n<\infty)\tag{2.2.3}$$

*then*

$$u_n(t)\ge\theta_n(t)a\qquad(0\le t<1,0\le n<\infty),\tag{2.2.4}$$

$$\|u_n(t)-\theta_n(t)a\|\le 1\ (0\le t<1,\ 0\le n<\infty).\tag{2.2.5}$$

*Proof.* To start observe

$$\|a\|=\sum_{i=1}^N F_i(a_i)\ge\pi(a)=1.$$

Define $\alpha_0(t)\equiv\alpha_0$ for $0\le t<1$, where $0<\alpha_0\le 1$ and $\|\alpha_0a\|=1$; then let $w_0(t)=\alpha_0a$, $0\le t<1$. We then define inductively sequences $(w_n:n\ge 0)$, $(w_n^*:n\ge 1)$, $(\alpha_n:n\ge 0)$ of functions on $[0, 1)$, where

$$w_n\ (n\geq 0) \text{ and } w_n^*\ (n\geq 1) \text{ are } \mathbb{R}^N\text{-valued and } \mathcal{B}_n\text{-measurable,} \tag{2.2.6}$$

$$\alpha_n\ (n\geq 0) \text{ is } \mathbb{R}\text{-valued and } \mathcal{B}_n\text{-measurable,} \tag{2.2.7}$$

$$\begin{aligned} w_n(t)&\geq 0 \qquad (0\leq t<1,\ n\geq 0),\\ w_n^*(t)&\geq 0 \qquad (0\leq t<1,\ n\geq 0),\\ \alpha_n(t)&\geq 0 \qquad (0\leq t<1,\ n\geq 0),\end{aligned} \tag{2.2.8}$$

$$\mathscr{E}(w_{n+1}^*\mid \mathcal{B}_n)=w_n \qquad (n\geq 0), \tag{2.2.9}$$

$$w_n(t)=w_n^*(t)+\alpha_n(t)a \qquad (0\leq t<1,\ n\geq 1), \tag{2.2.10}$$

$$\|w_n(t)\|=1 \qquad (0\leq t<1,\ n\geq 0), \tag{2.2.11}$$

$$\|w_{n+1}^*(t)\|\leq \sigma(w_n(t)) \qquad (0\leq t<1,\ n\geq 0). \tag{2.2.12}$$

Indeed suppose $w_j$, $w_j^*$ and $\alpha_j$ have been chosen for $j\leq n$. Then

$$w_n(t)=b_{n,k} \qquad (t\in D_{n,k}),$$

where $\|b_{n,k}\|=1$, and $b_{n,k}\geq 0$. Choose $y_{2k-1}, y_{2k}\geq 0$ so that $\max(\|y_{2k-1}\|, \|y_{2k}\|)=\sigma(b_{n,k})$ and $b_{n,k}=\frac{1}{2}(y_{2k-1}+y_{2k})$ (see Lemma 2.1(a)). Now define

$$w_{n+1}^*(t)=y_k \qquad (t\in D_{n+1,k}).$$

Then (2.2.9) and (2.2.12) are clear. Since

$$\|w_{n+1}^*(t)\|\leq 1 \qquad (0\leq t<1),$$

we can determine $\alpha_{n+1}$ to be $\mathcal{B}_{n+1}$-measurable so that $\alpha_{n+1}\geq 0$ and

$$\|w_{n+1}^*(t)+\alpha_{n+1}(t)a\|=1 \qquad (0\leq t<1).$$

Now define

$$w_{n+1}(t)=w_{n+1}^*(t)+\alpha_{n+1}(t)a \qquad (0\leq t<1)$$

and clearly (2.2.11) holds.

Observe that

$$\mathscr{E}(w_{n+1}\mid \mathcal{B}_n)=w_n+\mathscr{E}(\alpha_{n+1}\mid \mathcal{B}_n)a$$

and if $m>n$

$$\mathscr{E}(w_n\mid \mathcal{B}_n)=w_n+\left(\sum_{k=n+1}^{m}\mathscr{E}(\alpha_k\mid \mathcal{B}_n)\right)a. \tag{2.2.13}$$

Hence $w_n$ is a submartingale and it is clearly bounded. Thus $\lim_{n\to\infty} w_n(t)=w_\infty(t)$ exists almost everywhere, and $\|w_\infty(t)\|=1$ a.e.

The real-valued submartingale $(\pi \circ w_n : n \geq 0)$ is uniformly bounded and converges to $\pi \circ w_\infty$ a.e. Hence

$$\int_0^1 \pi(w_\infty(t))\,dt = \lim_{n\to\infty} \int_0^1 \pi(w_n(t))\,dt$$
$$= \int_0^1 \pi(w_0(t))\,dt + \sum_{k=1}^{\infty} \int_0^1 \alpha_k(t)\,dt$$

by (2.2.13) since $\pi(a) = 1$. Hence

$$\int_0^1 \sum_{k=1}^{\infty} \alpha_k(t)\,dt < \infty$$

and so (a.e.) $\sum \alpha_k(t) < \infty$. Thus $\alpha_n(t) \to 0$ a.e. and $\|w_{n+1}(t) - w^*_{n+1}(t)\| \to 0$ a.e. Hence $\|w^*_{n+1}(t)\| \to 1$ and $\sigma(w_n(t)) \to 1$ a.e. By Lemma 2.1(b), $\sigma$ is continuous and so (a.e.)

$$\sigma(w_\infty(t)) = \|w_\infty(t)\| = 1.$$

As $w_\infty(t) \geq 0$, we conclude that

$$w_\infty(t) = \sum_{i=1}^{N} 1_{E_i} e_i \quad \text{a.e.},$$

where $E_1, \dots, E_N$ are disjoint Borel sets with $E_1 \cup \dots \cup E_N = [0, 1)$.

Now define $u_n = \mathscr{E}(w_\infty \mid \mathscr{B}_n)$. Then, since $\{w_n\}$ is uniformly bounded and $w_n \to w_\infty$ a.e.,

$$u_n = \lim_{m\to\infty} \mathscr{E}(w_N \mid \mathscr{B}_n)$$
$$= w_n + \left( \sum_{k=n+1}^{\infty} \mathscr{E}(\alpha_k \mid \mathscr{B}_n) \right) a$$
$$= w_n + \theta_n a,$$

where $\theta_n \geq 0$ is $\mathscr{B}_n$-measurable. Since $(w_n)$ is a submartingale, $(\theta_n)$ is a supermartingale. As $u_n - w_n \to 0$ a.e., we have $\theta_n \to 0$ a.e. As $\pi(w_\infty) \leq 1$ a.e., $\pi(u_n) \leq 1$ a.e. and so $\theta_n \leq 1$ a.e. Also $\|u_n - \theta_n a\| = \|w_n\| = 1$. Finally observe

$$u_0 = (\alpha_0 + \theta_0) a$$
$$= \sum_{i=1}^{N} \lambda(E_i) e_i.$$

Hence

$$\pi(u_0) = \sum_{i=1}^{N} \lambda(E_i) = 1$$
$$= \alpha_0 + \theta_0.$$

Thus $\lambda(E_i) = a_i$ $(1 \leq i \leq N)$, and the proof is complete.

In fact we shall not use Theorem 2.2; instead we use its "finite" version.

THEOREM 2.3. *Under the same hypotheses as Theorem 2.2, given $\varepsilon>0$, there is a finite dyadic martingale $(v_0, v_1, \ldots, v_m)$ with*

$$v_0(t)=a \qquad (0\le t<1), \tag{2.3.1}$$

$$\|v_m(t)\|\le 1+\varepsilon \qquad (0\le t<1). \tag{2.3.2}$$

*For $1\le n\le m-1$, there is a positive $\mathcal{B}_n$-measurable function $\phi_n$ with $\phi_n\le 1$ and*

$$\|v_n(t)-\phi_n(t)a\|\le 1+\varepsilon \qquad (0\le t<1). \tag{2.3.3}$$

*Proof.* Suppose $0<\delta_0<\frac{1}{2}$ is chosen so that $\|2\delta_0 a\|<\frac{1}{2}\varepsilon$ and $\|(1-\delta_0)^{-1}\|<1+\frac{1}{2}\varepsilon$ whenever $\|x\|<1$.

Let $u_n$, $\theta_n$ be chosen as in Theorem 2.2 and select $m$ so that

$$\int_0^1 \theta_m(t)\,dt=\delta\le\delta_0.$$

Define

$$v_m=(1-\delta)^{-1}(u_m-\theta_m a)$$

and

$$v_n=\mathscr{E}(v_m\mid\mathcal{B}_n) \qquad (0\le n\le m).$$

Then $\|v_m\|\le 1+\varepsilon$ and

$$\begin{aligned} v_n&=(1-\delta)^{-1}(u_n-\mathscr{E}(\theta_m\mid\mathcal{B}_n)a)\\ &=(1-\delta)^{-1}(u_n-\theta_n a)+(1-\delta)^{-1}(\theta_n-\mathscr{E}(\theta_m\mid\mathcal{B}_n))a. \end{aligned}$$

Define

$$\phi_n=\theta_n-\mathscr{E}(\theta_m\mid\mathcal{B}_n) \qquad (0\le n\le m).$$

Then $0\le\phi_n\le\theta_n\le 1$ and

$$v_n-\phi_n a=(1-\delta)^{-1}(u_n-\theta_n a)+\delta(1-\delta)^{-1}\phi_n a$$

and so

$$\|v_n-\phi_n a\|\le 1+\tfrac{1}{2}\varepsilon+\tfrac{1}{2}\varepsilon=1+\varepsilon.$$

**3. Main results.** We now turn to the general infinite-dimensional problem.

LEMMA 3.1. *Suppose $X$ is an $F$-space with a strictly concave $F$-norm. Suppose $x_0\neq 0$ and that $x_0\in\operatorname{co}\{x:\|x\|\le\delta\}$. Then there is a finite dyadic martingale $u_n$ $(0\le n\le m)$ with $u_0(t)\equiv x_0$, and*

$$\|u_m(t)\|\le 2\delta \qquad (0\le t<1), \tag{3.1.1}$$

$$\|u_n(t)\|\le\|x_0\|+2\delta \qquad (0\le t<1,\ 0\le n\le m). \tag{3.1.2}$$

*Proof.* There exist $y_1, \ldots, y_N \in X$ with $y_i \neq 0$ $(1 \leq i \leq N)$, $\|y_i\| \leq \delta$ and $x_0 = a_1 y_1 + \ldots + a_N y_N$, where $a_i \geq 0$ and $a_1 + a_2 + \ldots + a_N = 1$.

For $0 \leq t < \infty$, define

$$F_i(t) = \|ty_i\| / \|y_i\|.$$

Then $F_i$ is strictly concave. Define the absolute norm on $\mathbb{R}^N$ by

$$\|b\| = \sum_{i=1}^{N} F_i(|b_i|).$$

Now, by Theorem 2.3, there is a finite $\mathbb{R}^N$-valued dyadic martingale $(v_n : 0 \leq n \leq m)$ with (taking $\varepsilon = 1$)

$$v_0(t) \equiv a = (a_1, \ldots, a_N) \qquad (0 \leq t < 1),$$
$$\|v_m(t)\| \leq 2 \qquad (0 \leq t < 1)$$

and

$$\|v_n(t) - \phi_n(t)a\| \leq 2 \qquad (0 \leq t < 1, 0 \leq n < m),$$

where $0 \leq \phi_n(t) \leq 1$. Define $T : \mathbb{R}^N \to X$ by

$$Tb = \sum_{i=1}^{N} b_i y_i.$$

Then

$$\begin{aligned} \|Tb\| &\leq \sum_{i=1}^{N} \|b_i y_i\| \\ &\leq \sum_{i=1}^{N} \|y_i\| \, F_i(|b_i|) \\ &\leq \delta \, \|b\|. \end{aligned}$$

Now let $u_n = Tv_n$. Then $u_0(t) \equiv x_0$ and $\|u_m(t)\| \leq 2\delta$. Also

$$\begin{aligned} \|u_n(t)\| &\leq \|\phi_n(t) x_0\| + 2\delta \\ &\leq \|x_0\| + 2\delta. \end{aligned}$$

THEOREM 3.2. *Suppose $X$ is an $F$-space with trivial dual, and that $x_0 \in X$. Then there is a dyadic martingale $(u_n : n \geq 0)$ with $u_0(t) \equiv x_0$ and*

$$\max_{0 \leq t < 1} \|u_n(t)\| \to 0 \quad as \quad n \to \infty. \tag{3.2.1}$$

*Proof.* As explained in the introduction we may suppose that the $F$-norm on $X$ is strictly concave (passing to an equivalent $F$-norm does not affect (3.2.1)). The hypotheses guarantee that the convex hull of any neighborhood of zero is $X$. The construction is inductive, based on Lemma 3.1. To start the construction we may find a finite martingale

$(u_n : 0 \le n \le N_1)$ so that $u_0(t) = x_0$, $\|u_{N_1}(t)\| \le \frac{1}{2}\|x_0\|$ and $\|u_n(t)\| \le 2\|x_0\|$ $(1 \le n \le N_1)$, by applying Lemma 3.1 with $\delta = \frac{1}{4}\|x_0\|$ if $x_0 \ne 0$ (the case $x_0 = 0$ is trivial).

Suppose now we have defined $(u_n : 1 \le n \le N_k)$ so that

$$\|u_{N_j}(t)\| \le (\tfrac{1}{2})^j \|x_0\| \qquad (1 \le j \le k), \tag{3.2.2}$$

$$\|u_n(t)\| \le 2(\tfrac{1}{2})^j \|x_0\| \qquad (N_j < n < N_{j+1}, 1 \le j \le k-1). \tag{3.2.3}$$

We shall show how to extend to a finite dyadic martingale $(u_n : 1 \le n \le N_{k+1})$ so that (3.2.2) and (3.2.3) hold for $j \le k+1$ and $j \le k$ respectively.

We have

$$u_{N_k}(t) = y_l \qquad (t \in D_{N_k,l}).$$

For each $y_l$, there is a finite martingale $(v_n^l : 0 \le n \le M)$ with

$$\begin{aligned} v_0^l(t) &= y_l \qquad (0 \le t \le 1), \\ \|v_M^l(t)\| &\le (\tfrac{1}{2})^{k+1} \|x_0\| \qquad (0 \le t \le 1), \\ \|v_n^l(t)\| &\le \|y_l\| + (\tfrac{1}{2})^{k+1} \|x_0\| \\ &\le (\tfrac{1}{2})^{k-1} \|x_0\| \qquad (0 \le t \le 1, 0 \le n \le M). \end{aligned}$$

Here $M$ may be taken independent of $l$ by simply extending the martingale where necessary by adding further terms equal to the last term of the sequence.

Now let $N_{k+1} = N_k + M$ and define

$$u_{N_k+i} = v_1^l(2^{N_k}t - l + 1) \qquad (t \in D_{N_k,l}).$$

It is now easy to verify that conditions (3.2.2) and (3.2.3) hold where applicable. Continuing in this way we clearly have (3.2.1) for the (infinite) martingale $(u_n)$.

The step from Theorem 3.2 to our main result is a very simple one if $X$ is a quasi-Banach space or more generally is exponentially galbed (see Turpin [**8**]). In such space there is a natural correspondence between curves with uniform zero derivative and dyadic martingales converging uniformly to 0. In a general $F$-space a little more sublety is required in the proof of the main theorem.

THEOREM 3.3. *Suppose $X$ is an $F$-space with trivial dual and that $x_0, x_1 \in X$. Then there is a curve $g : [0, 1] \to X$ with $g(0) = x_0$, $g(1) = x_1$ and*

$$\lim_{|t-s| \to 0} \frac{g(t) - g(s)}{t - s} = 0 \quad \textit{uniformly for} \quad 0 \le s, t \le 1. \tag{3.3.1}$$

*In particular $g'(t) = 0$ for $0 \le t \le 1$.*

*Proof.* It suffices to suppose $x_0 = 0$. Then there is a dyadic martingale $(u_n : n \ge 0)$ with

$$u_0(t) = x_1 \qquad (0 \le t < 1),$$

$$\max_{0 \le t < 1} \|u_n(t)\| = \varepsilon_n \to 0.$$

Choose $N_0=0$. Since each $u_n$ has finite range it is possible to choose a strictly increasing sequence of positive integers $(N_k : k \geq 1)$ so that

$$\|2^{N_j-N_k}(u_k(t)-u_{k-1}(t))\| \leq 2^{j-k}\varepsilon_j \tag{3.3.2}$$

for $0 \leq j \leq k-1$, $0 \leq t < 1$. Each $t \in [0, 1)$ has a unique binary expansion

$$t = \sum_{j=1}^{\infty} \tau_j 2^{-j},$$

where each $\tau_j$ is zero or one and $\tau_j = 0$ infinitely often. Now define

$$v_k(t) = u_k\left(\sum_{j=1}^{k} \tau_{N_j} 2^{-j}\right).$$

(Recall that $u_k$ is constant on the interval $\sum_{j=1}^{k} \tau_{N_j} 2^{-j} \leq t < \sum_{j=1}^{k} \tau_{N_j} 2^{-j} + 2^{-k}$.) Then we observe that $v_k$ is a $\mathcal{B}_{N_k}$-martingale, with

$$\max_{0 \leq t < 1} \|v_k(t)\| = \varepsilon_k,$$

$$\mathscr{E}(v_k \mid \mathcal{B}_0) = \int_0^1 v_k(t)\, dt = \int_0^1 u_k(t)\, dt = x_1.$$

In fact we observe that

$$\mathscr{E}(v_k \mid \mathcal{B}_{N_k-1}) = v_{k-1}. \tag{3.3.3}$$

For $k \geq 1$ and $0 \leq t \leq 1$, we define

$$g_k(t) = \int_0^t v_k(s)\, ds$$

(the integrand is simple). Then each $g_k$ is continuous and from (3.3.3) we have

$$g_k(t) = g_{k-1}(t) \quad \text{if} \quad 2^{N_k-1} t \in \mathbb{Z}.$$

Now suppose that $0 < t < 1$ and that $2l \leq 2^{N_k} t < 2l+1$, where $l$ is an integer. Then

$$\begin{aligned} g_k(t) - g_{k-1}(t) &= \int_{2l/2^{N_k}}^{t} (v_k(s) - v_{k-1}(s))\, ds \\ &= (t - 2l(2^{-N_k}))(v_k(t) - v_{k-1}(t)). \end{aligned} \tag{3.3.4}$$

Equally, if $2l+1 \leq 2^{N_k} t < 2l+2$,

$$g_k(t) - g_{k-1}(t) = ((2l+2)2^{-N_k} - t)(v_k(t) - v_{k-1}(t)). \tag{3.3.5}$$

Combining these results, we have

$$\begin{aligned} \|g_k(t) - g_{k-1}(t)\| &\leq \max_{0 \leq t < 1} \|2^{-N_k}(v_k(t) - v_{k-1}(t))\| \\ &= \max_{0 \leq t < 1} \|2^{-N_k}(u_k(t) - u_{k-1}(t))\| \\ &\leq 2^{-k}\varepsilon_0. \end{aligned}$$

Hence $(g_k)$ converges uniformly to a continuous function $g$ on $[0, 1]$, and $g(0)=0$, $g(1)=x_1$.

Now suppose $0\le s<t\le 1$. Then there is a least integer $n$ so that for some integer $l$ we have $2^n s\le l<l+1\le 2^n t$. Clearly $2^{n-1}t-2^{n-1}s<2$ and $2^n t-2^n s\ge 1$. Hence $2^{-n}\le t-s<4\,.\,2^{-n}$ and $n\ge \log_2 1/(t-s)$.

Now suppose $N_{k-1}\le n<N_k$, where $1\le k<\infty$. Suppose $l_1$ is the least integer not less than $2^n s$ and $l_2$ is the greatest integer not greater than $2^n t$. Then

$$2^n(g_{k-1}(t)-g_{k-1}(l_2 2^{-n}))=(2^n t-l_2)v_{k-1}(l_2 2^{-n}),$$
$$2^n(g_{k-1}(l_1 2^{-n})-g_{k-1}(s))=(l_1-2^n s)v_{k-1}(l_1 2^{-n}),$$
$$2^n(g_{k-1}(i2^{-n})-g_{k-1}((i-1)2^{-n}))=v_{k-1}((i-1)2^{-n}).$$

Hence

$$\|2^n(g_{k-1}(l_2 2^{-n})-g_{k-1}(l_1 2^{-n}))\|\le (l_2-l_1)\varepsilon_{k-1}$$

and

$$\|2^n(g_{k-1}(t)-g_{k-1}(s))\|\le (l_2-l_1+2)\varepsilon_{k-1}.$$

However $l_2-l_1\le 2^n(t-s)<4$ so that $l_2-l_1+2\le 5$. Hence

$$\|2^n(g_{k-1}(t)-g_{k-1}(s))\|\le 5\varepsilon_{k-1}. \tag{3.3.6}$$

Now

$$2^n(g_k(t)-g_{k-1}(t))=2^{n-N_k}\rho(v_k(t)-v_{k-1}(t)),$$

where $0\le\rho\le 1$, by (3.3.4) and (3.3.5). Hence

$$\|2^n(g_k(t)-g_{k-1}(t))\|\le \varepsilon_k+\varepsilon_{k-1}. \tag{3.3.7}$$

A similar inequality holds for $s$.

If $r>k$

$$2^n(g_r(t)-g_{r-1}(t))=2^{n-N_r}\rho(v_r(t)-v_{r-1}(t)),$$

where $0\le\rho\le 1$, and so

$$\begin{aligned}\|2^n(g_r(t)-g_{r-1}(t))\| &\le \|2^{N_k-N_r}(v_r(t)-v_{r-1}(t))\|\\ &\le \max_{0\le t<1}\|2^{N_k-N_r}(u_r(t)-u_{r-1}(t))\|\\ &\le 2^{k-r}\varepsilon_k\end{aligned}$$

by (3.3.2). Hence

$$\|2^n(g(t)-g_k(t))\|\le\Big(\sum_{r>k}2^{k-r}\Big)\varepsilon_k=\varepsilon_k. \tag{3.3.8}$$

A similar inequality holds for $s$.

Combining (3.3.6), (3.3.7) and (3.3.8) and the similar results for $s$ we obtain

$$\|2^n(g(t)-g(s))\| \leq 7\varepsilon_{k-1}+4\varepsilon_k$$

and hence

$$\left\|\frac{g(t)-g(s)}{t-s}\right\| \leq 7\varepsilon_{k-1}+4\varepsilon_k,$$

where $N_k > \log_2 1/(t-s)$. Hence $g$ has the properties specified in the theorem.

Every $F$-space $X$ has a unique maximal linear subspace with trivial dual; this subspace is closed. Let us call this maximal subspace the *core* of $X$. If core $(X)=\{0\}$, it does not necessarily follow that $X$ has a separating dual; for a detailed investigation of related ideas see Ribe [**3**]. We conclude with a simple corollary.

COROLLARY 3.4. *Suppose $X$ is an $F$-space and $x \in X$. In order that there exists a curve $g:[0,1] \to X$ with $g(0)=0$, $g(1)=x$ and $g'(t)=0$ for $0 \leq t \leq 1$ it is necessary and sufficient that $x \in \text{core}(X)$.*

*Proof.* If $x \in \text{core}(X)$ the existence of $g$ is given by Theorem 3.3. Suppose conversely such a $g$ exists and let $Y$ be the closed linear span of $\{g(t): 0 \leq t \leq 1\}$. Suppose $\phi$ is a continuous linear functional on $Y$. Then $(\phi \circ g)'(t)=0$ $(0 \leq t \leq 1)$ and hence by the Mean Value Theorem $\phi(g(t))=0$ $(0 \leq t \leq 1)$. Thus $\phi=0$ and so $Y \subset \text{core}(X)$; in particular $x \in \text{core}(X)$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MISSOURI-COLUMBIA
COLUMBIA
MISSOURI 65211
U.S.A.