QUOTIENTS OF F-SPACES

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Let $X$ be a non-locally convex $F$-space (complete metric linear space) whose dual $X'$ separates the points of $X$. Then it is known that $X$ possesses a closed subspace $N$ which fails to be weakly closed (see [3]), or, equivalently, such that the quotient space $X/N$ does not have a point separating dual. However the question has also been raised by Duren, Romberg and Shields [2] of whether $X$ possesses a proper closed weakly dense (PCWD) subspace $N$, or, equivalently a closed subspace $N$ such that $X/N$ has trivial dual. In [2], the space $H_p$ ($0 < p < 1$) was shown to have a PCWD subspace; later in [9], Shapiro showed that $l_p$ ($0 < p < 1$) and certain spaces of analytic function have PCWD subspaces. Our first result in this note is that every separable non-locally convex $F$-space with separating dual has a PCWD subspace.

It was for some time unknown whether an $F$-space with trivial dual could have non-zero compact endomorphisms. This problem was equivalent to the existence of a non-zero compact operator $T : X \to Y$, where $X$ has trivial dual; for if such an operator exists, then we may suppose $T$ has dense range and then the space $X \oplus Y$ has trivial dual and admits the compact endomorphism $(x, y) \to (0, Tx)$. Let us say that an $F$-space $X$ admits compact operators if there is a non-zero compact operator with domain $X$. The most commonly arising spaces with trivial dual $L_p$ ($0 < p < 1$), do not admit compact operators ([4]). However in [7] it was shown that the spaces $H_p$ ($0 < p < 1$) possess quotients with trivial dual but admitting compact operators; equivalently there is a compact operator $T$ with domain $H_p$ whose kernel $T^{-1}(0)$ is a PCWD subspace of $H_p$. The construction depended on certain special properties of $H_p$. However we show here that every separable non-locally convex locally bounded $F$-space with a base of weakly closed neighbourhoods of zero admits a compact operator whose kernel is a PCWD subspace, and thus has a quotient with trivial dual but admitting compact operators. This result applies to $H_p$ and to any locally bounded space with a basis; in a sense there are many examples of compact endomorphisms in spaces with trivial dual.

Let $X$ be an $F$-space; we denote an $F$-norm (in the sense of [3]) defining the topology on $X$ by $\| \cdot \|$. The following lemma is proved in [3] and [8].

**Lemma 1.** Let $\| \cdot \|$ be an $F$-norm on $X$ which defines a topology weaker than the original topology. Suppose $(x_n)$ is a sequence in $X$ such that $|x_n| \to 0$ but $\|x_n\| \geq \epsilon > 0$ ($n \in \mathbb{N}$). Then there is a subsequence $(u_n)$ of $(x_n)$ which is a strongly regular $M$-basic sequence; i.e. there exist continuous linear functionals $(\varphi_n: n \in \mathbb{N})$ on the closed linear span $E$ of $(u_n: n \in \mathbb{N})$ such that

(a) $\varphi_i(u_j) = \delta_{ij}$ ($i, j \in \mathbb{N}$),
(b) $\lim_{n \to \infty} \varphi_n(x) = 0$ ($x \in E$),
(c) if $x \in E$, and $\varphi_n(x) = 0$ for all $n \in \mathbb{N}$, then $x = 0$. 

It is well-known that most familiar properties of compact operators do not require local convexity (see e.g. [12]). Thus the following lemma is almost certainly known, although we include a proof for completeness. We note our later use of Lemma 2 is essentially equivalent to using a stability theorem for $M$-basic sequences due to Drewnowski [1].

**Lemma 2.** Let $X$ and $Y$ be $F$-spaces and let $A : X \to Y$ be a linear embedding (i.e. a homeomorphism onto its range). Let $K : X \to Y$ be compact; then $A + K$ has closed range.

**Proof.** Let $U$ be a balanced neighbourhood of $0$ in $X$ such that $K(U)$ is compact in $Y$. Let $N = (A + K)^{-1}(0)$ and consider the quotient map $q : X \to X/N$ and the induced map $S : X/N \to Y$ so that $Sq = A + K$. We show that $S$ is an embedding. Suppose not; then there is a sequence $(x_n)$ in $X/N$ such that $Sx_n \to 0$ but for some neighbourhood $V$ of $0$ in $X$ with $V \subset \frac{1}{2}U$ we have $x_n \notin q(V)(n \in \mathbb{N})$. Select a sequence $a_n$ with $0 < a_n \leq 1$ such that $a_n x_n \notin q(V)$ but $a_n x_n \in q(U)$. Then $a_n x_n = q(u_n)$, where $u_n \in U$, and there is a subsequence $(u_{p(n)})$ of $(u_n)$ such that $Ku_{p(n)} \to y$ in $Y$. Hence $Au_{p(n)} \to -y$ and $u_{p(n)} \to v$, where $Av = -y$. Thus $q(v) = 0$ and $q(u_{p(n)}) = a_{p(n)} u_{p(n)} \to 0$, a contradiction.

**Theorem 1.** Let $X$ be a non-locally convex separable $F$-space with separating dual. Then $X$ has a PCWD subspace, and hence a non-trivial quotient with trivial dual.

**Proof.** Denote by $\mu$, the Mackey topology on $X$ (see [10]), i.e. the topology induced by all convex neighbourhoods of 0. Then $\mu$ is metrizable and may be given by an $F$-norm $|\cdot|$. Since $X$ is non-locally convex there is a sequence $(x_n)$ in $X$ such that $\|x_n\| \geq \varepsilon > 0$, but $|x_n| \to 0$. We apply Lemma 1 to deduce the existence of a strongly regular $M$-basic subsequence $(u_n)$ of $(x_n)$; as in Lemma 1, let $E$ be the closed linear span of $\{u_n : n \in \mathbb{N}\}$ and $\{\varphi_n : n \in \mathbb{N}\}$ be the biorthogonal functionals on $E$. Let $E_0$ be the closed linear span of $\{u_{2n} : n \in \mathbb{N}\}$.

Now let $\{v_n : n \in \mathbb{N}\}$ be a dense countable subset of $X$ and choose $\varepsilon_n$ such that $0 < \varepsilon_n < 1$ and $\|\varepsilon_n v_n\| \leq 2^{-n}$. Since $|u_n| \to 0$, we may find an increasing sequence $\ell(n)$ such that $|\varepsilon_n u_{2\ell(n)}| \to 0$. Now define $K : E_0 \to X$ by

$$Kx = \sum_{n=1}^{\infty} \varepsilon_n \varphi_{2\ell(n)}(x)v_n.$$  

By the Banach–Steinhaus theorem, the functionals $(\varphi_{2\ell(n)}; n \in \mathbb{N})$ are equicontinuous on $E_0$ and hence $K$ is compact (it maps the zero-neighbourhood $U = \{x : |\varphi_{2\ell(n)}(x)| \leq 1, n \in \mathbb{N}\}$ into a relatively compact set). Now let $J : E_0 \to X$ be the inclusion map and $N = (J + K)(E_0)$.

Then $N$ is closed by Lemma 2. If $N = X$, then $J + K$ is open and $(J + K)(U)$ is a neighbourhood of $0$ in $X$. Let $q : X \to X/E_0$ be the quotient map; then $q(J + K)(U) \subset qK(U)$ is relatively compact and hence $\dim X/E_0 < \infty$. However $\dim X/E_0 \geq \dim E/E_0 = \infty$ and thus $N$ is a proper closed subspace. If $x \in X$, then there is a sequence $v_{n_k} \to x$; then $\varepsilon_n^{-1}(u_{2\ell(n_k)} + \varepsilon_n v_{n_k}) \in N$ and converges to $x$ in the Mackey topology; hence $N$ is weakly dense.
REMARK. We do not know whether every non-locally convex $F$-space has a non-trivial quotient with trivial dual.

THEOREM 2. Let $B$ be a locally bounded $F$-space with a basis and let $X$ be a non-locally convex subspace of $B$. Then there is a compact operator $T$ on $X$ whose kernel is a PCWD subspace of $X$.

Thus $X$ has a quotient space with trivial dual but admitting compact operators.

Proof. The proof is a modification of the preceding theorem. Let $(e_n)$ be a basis of $B$ and let $(e'_n)$ be the linear functionals biorthogonal to $(e_n)$. Let $\| \cdot \|$ be a $p$-homogeneous norm on $B$ defining the topology, where $p < 1$. Without loss of generality we assume that the basis $(e_n)$ is monotone, i.e.

$$\sup_{1 \leq k < \infty} \left\| \sum_{n=1}^{k} a_n e_n \right\| = \left\| \sum_{n=1}^{\infty} a_n e_n \right\|.$$

Let $\sigma$ be the topology on $B$ induced by the functionals $x \to e'_n(x)$ and let $\bar{B}$ be the $\sigma$-completion of $B$. Let $B^\sigma \subset \bar{B}$ be the set of $x \in \bar{B}$ such that

$$\|x\| = \sup_{1 \leq k < \infty} \left\| \sum_{n=1}^{k} e'_n(x) e_n \right\| < \infty.$$

Thus $B^\sigma$ is a $p$-normed space with unit ball $U = \{ x \in B^\sigma : \|x\| \leq 1 \}$. Then $U$ is $\sigma$-compact and we may define on $B^\sigma$ the topology $\beta$ which is the largest topology agreeing with $\sigma$ on each set $kU (k \in \mathbb{N})$. Then $\beta$ is a Hausdorff vector topology ([11]) and is locally $p$-convex (see e.g. [6] for the explicit form of the neighbourhoods of 0). We shall show that it is possible to choose a PCWD subspace $N$ of $X$ such that $N$ is closed in the $\beta$-topology relative to $X$. Once this is achieved $X/N$ admits a locally $p$-convex topology $\tilde{\beta}$ (the quotient $\beta$-topology) in which the unit ball is precompact. As in [7] there is a non-zero compact operator $S : X/N \to Y$, where $Y$ is a $p$-Banach space. Then if $q : X \to X/N$ is the quotient map, $Sq : X \to Y$ is compact and its kernel is PCWD.

It remains to determine $N$. Let $\| \cdot \|$ be a norm defining the Mackey topology on $X$. By assumption $\| \cdot \|$ and $\| \cdot \|$ are nonequivalent on $X$, and hence also on any closed subspace of finite codimension. Thus it is possible to construct a sequence $(x_n)$ in $X$ such that $\|x_n\| = 1$ $(n \in \mathbb{N})$, $|x_n| \to 0$ and $\lim_{n \to \infty} e'_i(x_n) = 0$ for $1 \leq i < \infty$. By standard arguments we may determine a subsequence $(y_n)$ of $(x_n)$ and a block basic sequence $(u_n)$ of $(e_n)$ such that

$$u_n = \sum_{i=p_{n-1} + 1}^{p_n} a_i e_i,$$

where $p_0 = 0 < p_1 < p_2 \ldots$, and $\|y_n - u_n\| \leq \frac{1}{16} 2^{-n}$.

Let $E$ be the closed linear span in $B$ of $\{ u_n : n \in \mathbb{N} \}$ and $E_0$ the closed linear span of $\{ u_{2n} : n \in \mathbb{N} \}$. Denote by $E^\sigma$ and $E^\sigma_0$ their respective closures in $(B^\sigma, \beta)$. Let $(u'_n)$ denote the biorthogonal functionals on $E^\sigma$; then each $u'_n$ is $\beta$-continuous in $E^\sigma$ (it is a finite linear
combination of the $e'_n$ and
\[ \|u'_n(x)u_n\| \leq 2\|x\| \quad (x \in E^\gamma), \]
so that $\|u_n\| \leq \frac{1}{2}$, $|u'_n(x)|^p \leq 4\|x\|$.

Let $(v_n)$ be a countable dense subset of $X\setminus\{0\}$ and choose $\varepsilon_n > 0$ so that
\[ \varepsilon_n = \frac{1}{16}2^{-n}\|v_n\|^{-1} \quad (n \in \mathbb{N}). \]

Then choose an increasing sequence $\ell(n)$ such that
\[ |e^{-1}_{\ell(n)}y_{2\ell(n)}| \rightarrow 0. \]

Next define $K : E^\gamma \rightarrow B^\gamma$ by
\[ Kx = \sum_{n=1}^{\infty} u'_n(x)(y_n - u_n) + \sum_{n=1}^{\infty} \varepsilon_n u'_2\ell(n)(x)v_n. \]

Thus $K$ is compact for the $p$-norm topology of $E^\gamma$ and $K(E^\gamma) \subset B$. If $\|x\| \leq 1$
\[ \|Kx\| \leq \sum_{n=1}^{\infty} |u'_n(x)|^p\|y_n - u_n\| + \sum_{n=1}^{\infty} \varepsilon_n |u'_2\ell(n)(x)|^p\|v_n\| \leq \frac{3}{2}\|x\|. \]

Let $J : E^\gamma \rightarrow B^\gamma$ be the inclusion map, and let $N = (J + K)(E_0)$. By Lemma 2, $N$ is closed in $B$. In fact it is easy to see that $N \subset X$ and an argument similar to the proof of Theorem 1 shows that $N$ is weakly dense in $X$ (note that $y_{2\ell(n)} + \varepsilon_nv_n \in N$). Suppose $N = X$; then as $(J + K)(E_0) \subset (J + K)(E) \subset X$ we have $(J + K)(E) = (J + K)(E_0)$ and there exists $x \in E$, with $x \neq 0$, such that $(J + K)x = 0$. However $\|Kx\| \leq \frac{3}{2}\|x\|$ and so this is impossible. Hence $N$ is a PCWD subspace of $X$.

It remains to show that $N$ is relatively $\beta$-closed. Consider first $N^\gamma = (J + K)(E^\gamma)$. To show that $N^\gamma$ is $\beta$-closed in $B^\gamma$ it is only necessary to show that $N^\gamma \cap U$ is $\sigma$-closed. Suppose $z_n \in E^\gamma$, $\|(J + K)z_n\| \leq 1$ and $(J + K)(z_n) \rightarrow u(\beta)$. Then as $\|(J + K)z_n\| \geq \frac{1}{2}\|z_n\|$ we have $\|z_n\| \leq 2$; by passing to a subsequence we may suppose $z_n \rightarrow z(\sigma)$ and $z \in E^\gamma$. Since each $u'_n$ is $\sigma$-continuous on $E^\gamma$, $K$ is continuous on bounded sets for the $\sigma$-topology. Hence $(J + K)z_n \rightarrow (J + K)z(\sigma)$ and $(J + K)z = u$. Thus $N^\gamma$ is $\beta$-closed in $B^\gamma$. Now consider $N^\gamma \cap B = N$; if $x \in N^\gamma \cap B$, then $x = (J + K)u$, $u \in E^\gamma$ and $u = x - Ku \in B \cap E^\gamma = E$. Thus $x \in N$ and so $N = N^\gamma \cap B$ is $\beta$-closed in $B$ and hence in $X$, as required.

**Corollary.** Every non-locally convex separable locally bounded F-space with a base of weakly closed neighbourhoods of 0 has a quotient with trivial dual which admits compact operators.

**Proof.** These spaces are precisely those isomorphic to subspaces of a locally bounded F-space with a basis (see [5], Theorem 7.4).

**Remarks.** (1) There exist locally bounded non-locally convex F-spaces with bases such that every non-trivial quotient admits compact operators. Indeed any pseudo-reflexive space as constructed in [8] has this property, as it possesses a topology $\beta$ in
which the unit ball is compact and which defines the same closed subspaces as the original topology.

(2) There exist separable locally bounded non-locally convex $F$-spaces $X$ with separating duals such that every compact operator on $X$ has a weakly closed kernel. To see this construct an Orlicz function $\varphi$ on $[0, \infty)$ so that

(a) $x^{-1}\varphi(x)$ is increasing,
(b) $\lim_{x \to \infty} x^{-1}\varphi(x) = \infty$,
(c) $\liminf_{x \to \infty} x^{-1}\varphi(x) = 1$,
(d) $\sup_{x=1} \varphi(2x) \varphi(x) < \infty$.

Then $L_\varphi(0,1)$ is a locally bounded non-locally convex space with separating dual and, by the results of [4], every compact operator on $L_\varphi$ factors through the inclusion map $L_\varphi \subset L_1$. Hence the kernel of any compact operator is weakly closed.

(3) In the proof of the theorem the Mackey topology on $X$ may be replaced by any strictly weaker metrizable topology. In particular if $X$ is locally $p$-convex ($p < 1$) but not locally $r$-convex for any $r > p$, then we may consider the topology induced by all absolutely $r$-convex neighbourhoods of 0 ($r > p$). We then obtain by the same arguments the following result.

**Theorem 3.** Let $X$ be a closed subspace of a locally bounded $F$-space with a basis. Suppose $X$ is locally $p$-convex ($p < 1$) but not locally $r$-convex for any $r > p$. Then there is a non-zero compact operator $T : X \to Y$ (where $Y$ is a locally bounded $F$-space) such that $X/\ker T$ admits no operators into any locally $r$-convex space for $r > p$.

Let $X = \ell_p$ and $U$ be the closed unit ball of $\ell_p$. If we construct $T$ as in the theorem then $T(U)$ is a compact $p$-convex subset of $Y$, which cannot be linearly embedded into a locally $r$-convex space. For if $S$ is such an embedding (i.e. $S$ is linear on the linear span of $T(U)$ and continuous on $T(U)$), then $ST$ is continuous on $\ell_p$ and $\ker (ST) \subset \ker T$. By the theorem $ST = 0$. On the other hand $T(U)$ can be linearly embedded in a locally $p$-convex space (see [6]).

**Note added in proof.** (See the Remark preceding Theorem 2): An example of a non-locally convex $F$-space with no non-trivial quotient with trivial dual has been constructed by J. Roberts (Springer Lecture Notes No. 604, pp. 76–81). Similar examples were also obtained independently by M. Ribe and the author.

**References**


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