QUOTIENTS OF F-SPACES

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(Received 6 October, 1976)

Let X be a non-locally convex F-space (complete metric linear space) whose dual X' separates the points of X. Then it is known that X possesses a closed subspace N which fails to be weakly closed (see [**3**]), or, equivalently, such that the quotient space X/N does not have a point separating dual. However the question has also been raised by Duren, Romberg and Shields [**2**] of whether X possesses a proper closed weakly dense (PCWD) subspace N, or, equivalently a closed subspace N such that X/N has trivial dual. In [**2**], the space H_p $(0<p<1)$ was shown to have a PCWD subspace; later in [**9**], Shapiro showed that ℓ_p $(0<p<1)$ and certain spaces of analytic function have PCWD subspaces. Our first result in this note is that every separable non-locally convex F-space with separating dual has a PCWD subspace.

It was for some time unknown whether an F-space with trivial dual could have non-zero compact endomorphisms. This problem was equivalent to the existence of a non-zero compact operator $T: X \to Y$, where X has trivial dual; for if such an operator exists, then we may suppose T has dense range and then the space $X \oplus Y$ has trivial dual and admits the compact endomorphism $(x, y) \to (0, Tx)$. Let us say that an F-space X *admits compact operators* if there is a non-zero compact operator with domain X. The most commonly arising spaces with trivial dual L_p $(0<p<1)$, do not admit compact operators ([**4**]). However in [**7**] it was shown that the spaces H_p $(0<p<1)$ possess quotients with trivial dual but admitting compact operators; equivalently there is a compact operator T with domain H_p whose kernel $T^{-1}(0)$ is a PCWD subspace of H_p. The construction depended on certain special properties of H_p. However we show here that every separable non-locally convex locally bounded F-space with a base of weakly closed neighbourhoods of zero admits a compact operator whose kernel is a PCWD subspace, and thus has a quotient with trivial dual but admitting compact operators. This result applies to H_p and to any locally bounded space with a basis; in a sense there are many examples of compact endomorphisms in spaces with trivial dual.

Let X be an F-space; we denote an F-norm (in the sense of [**3**]) defining the topology on X by $\|\cdot\|$. The following lemma is proved in [**3**] and [**8**].

LEMMA 1. *Let $|\cdot|$ be an F-norm on X which defines a topology weaker than the original topology. Suppose (x_n) is a sequence in X such that $|x_n| \to 0$ but $\|x_n\| \geq \varepsilon > 0 (n \in \mathbb{N})$. Then there is a subsequence (u_n) of (x_n) which is a strongly regular M-basic sequence; i.e. there exist continuous linear functionals $(\varphi_n : n \in \mathbb{N})$ on the closed linear span E of $(u_n : n \in \mathbb{N})$ such that*

(a) $\varphi_i(u_j) = \delta_{ij} \quad (i, j \in \mathbb{N})$,

(b) $\lim_{n\to\infty} \varphi_n(x) = 0 \quad (x \in E)$,

(c) *if $x \in E$, and $\varphi_n(x) = 0$ for all $n \in \mathbb{N}$, then $x = 0$.*

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It is well-known that most familiar properties of compact operators do not require local convexity (see e.g. [**12**]). Thus the following lemma is almost certainly known, although we include a proof for completeness. We note our later use of Lemma 2 is essentially equivalent to using a stability theorem for M-basic sequences due to Drewnowski [**1**].

LEMMA 2. *Let X and Y be F-spaces and let $A: X\to Y$ be a linear embedding (i.e. a homeomorphism onto its range). Let $K: X\to Y$ be compact; then $A+K$ has closed range.*

Proof. Let U be a balanced neighbourhood of 0 in X such that $\overline{K(U)}$ is compact in Y. Let $N=(A+K)^{-1}(0)$ and consider the quotient map $q: X\to X/N$ and the induced map $S: X/N\to Y$ so that $Sq=A+K$. We show that S is an embedding. Suppose not; then there is a sequence (x_n) in X/N such that $Sx_n\to 0$ but for some neighbourhood V of 0 in X with $V\subset \frac{1}{2}U$ we have $x_n\notin q(V)(n\in\mathbb{N})$. Select a sequence a_n with $0<a_n\le 1$ such that $a_nx_n\notin q(V)$ but $a_nx_n\in q(U)$. Then $a_nx_n=q(u_n)$, where $u_n\in U$, and there is a subsequence $(u_{p(n)})$ of (u_n) such that $Ku_{p(n)}\to y$ in Y. Hence $Au_{p(n)}\to -y$ and $u_{p(n)}\to v$, where $Av=-y$. Thus $q(v)=0$ and $q(u_{p(n)})=a_{p(n)}x_{p(n)}\to 0$, a contradiction.

THEOREM 1. *Let X be a non-locally convex separable F-space with separating dual. Then X has a PCWD subspace, and hence a non-trivial quotient with trivial dual.*

Proof. Denote by μ, the Mackey topology on X (see [**10**]), i.e. the topology induced by all convex neighbourhoods of 0. Then μ is metrizable and may be given by an F-norm $|\cdot|$. Since X is non-locally convex there is a sequence (x_n) in X such that $\|x_n\|\ge\varepsilon>0$, but $|x_n|\to 0$. We apply Lemma 1 to deduce the existence of a strongly regular M-basic subsequence (u_n) of (x_n); as in Lemma 1, let E be the closed linear span of $\{u_n: n\in\mathbb{N}\}$ and $\{\varphi_n: n\in\mathbb{N}\}$ be the biorthogonal functionals on E. Let E_0 be the closed linear span of $\{u_{2n};\ n\in\mathbb{N}\}$.

Now let $\{v_n: n\in\mathbb{N}\}$ be a dense countable subset of X and choose ε_n such that $0<\varepsilon_n<1$ and $\|\varepsilon_nv_n\|\le 2^{-n}$. Since $|u_n|\to 0$, we may find an increasing sequence $\ell(n)$ such that $|\varepsilon_n^{-1}u_{2\ell(n)}|\to 0$. Now define $K: E_0\to X$ by

$$Kx=\sum_{n=1}^{\infty}\varepsilon_n\varphi_{2\ell(n)}(x)v_n.$$

By the Banach–Steinhaus theorem, the functionals $(\varphi_{2\ell(n)};\ n\in\mathbb{N})$ are equicontinuous on E_0 and hence K is compact (it maps the zero-neighbourhood $U=\{x: |\varphi_{2\ell(n)}(x)|\le 1,\ n\in\mathbb{N}\}$ into a relatively compact set). Now let $J: E_0\to X$ be the inclusion map and $N=(J+K)(E_0)$.

Then N is closed by Lemma 2. If $N=X$, then $J+K$ is open and $(J+K)(U)$ is a neighbourhood of 0 in X. Let $q: X\to X/E_0$ be the quotient map; then $q(J+K)(U)\subset qK(U)$ is relatively compact and hence $\dim X/E_0<\infty$. However $\dim X/E_0\ge\dim E/E_0=\infty$ and thus N is a proper closed subspace. If $x\in X$, then there is a sequence $v_{n_k}\to x$; then $\varepsilon_{n_k}^{-1}(u_{2\ell(n_k)}+\varepsilon_{n_k}v_{n_k})\in N$ and converges to x in the Mackey topology; hence N is weakly dense.

REMARK. We do not know whether every non-locally convex F-space has a non-trivial quotient with trivial dual.

THEOREM 2. *Let B be a locally bounded F-space with a basis and let X be a non-locally convex subspace of B. Then there is a compact operator T on X whose kernel is a PCWD subspace of X.*

Thus X has a quotient space with trivial dual but admitting compact operators.

Proof. The proof is a modification of the preceding theorem. Let (e_n) be a basis of B and let (e_n') be the linear functionals biorthogonal to (e_n). Let $\|\cdot\|$ be a p-homogeneous norm on B defining the topology, where $p<1$. Without loss of generality we assume that the basis (e_n) is monotone, i.e.

$$\sup_{1\le k<\infty}\left\|\sum_{n=1}^{k} a_n e_n\right\| = \left\|\sum_{n=1}^{\infty} a_n e_n\right\|.$$

Let σ be the topology on B induced by the functionals $x \to e_n'(x)$ and let $\bar{B}$ be the σ-completion of B. Let $B^\gamma \subset \bar{B}$ be the set of $x \in \bar{B}$ such that

$$\|x\| = \sup_{1\le k<\infty}\left\|\sum_{n=1}^{k} e_n'(x)e_n\right\| < \infty.$$

Thus B^γ is a p-normed space with unit ball $U=\{x\in B^\gamma : \|x\|\le 1\}$. Then U is σ-compact and we may define on B^γ the topology β which is the largest topology agreeing with σ on each set $kU (k\in \mathbb{N})$. Then β is a Hausdorff vector topology (**[11]**) and is locally p-convex (see e.g. **[6]** for the explicit form of the neighbourhoods of 0). We shall show that it is possible to choose a PCWD subspace N of X such that N is closed in the β-topology relative to X. Once this is achieved X/N admits a locally p-convex topology $\tilde{\beta}$ (the quotient β-topology) in which the unit ball is precompact. As in **[7]** there is a non-zero compact operator $S: X/N \to Y$, where Y is a p-Banach space. Then if $q: X\to X/N$ is the quotient map, $Sq: X\to Y$ is compact and its kernel is PCWD.

It remains to determine N. Let $|\cdot|$ be a norm defining the Mackey topology on X. By assumption $|\cdot|$ and $\|\cdot\|$ are nonequivalent on X, and hence also on any closed subspace of finite codimension. Thus it is possible to construct a sequence (x_n) in X such that $\|x_n\|=1$ $(n\in\mathbb{N})$, $|x_n|\to 0$ and $\lim_{n\to\infty} e_i'(x_n)=0$ for $1\le i<\infty$. By standard arguments we may determine a subsequence (y_n) of (x_n) and a block basic sequence (u_n) of (e_n) such that

$$u_n = \sum_{i=p_{n-1}+1}^{p_n} a_i e_i,$$

where $p_0=0<p_1<p_2\dots$, and $\|y_n-u_n\|\le \frac{1}{16}2^{-n}$.

Let E be the closed linear span in B of $\{u_n : n\in\mathbb{N}\}$ and E_0 the closed linear span of $\{u_{2n} : n\in\mathbb{N}\}$. Denote by E^γ and E_0^γ their respective closures in (B^γ,β). Let (u_n') denote the biorthogonal functionals on E^γ; then each u_n' is β-continuous in E^γ (it is a finite linear combination of the e_n') and

$$\|u_n'(x)u_n\| \leq 2\|x\| \quad (x \in E^\gamma),$$

so that as $\|u_n\| \geq \frac{1}{2}$, $|u_n'(x)|^p \leq 4\|x\|$.

Let (v_n) be a countable dense subset of $X\backslash\{0\}$ and choose $\varepsilon_n > 0$ so that

$$\varepsilon_n^p = \tfrac{1}{16}2^{-n}\|v_n\|^{-1} \quad (n \in \mathbb{N}).$$

Then choose an increasing sequence $\ell(n)$ such that

$$|\varepsilon_n^{-1} y_{2\ell(n)}| \to 0.$$

Next define $K: E^\gamma \to B^\gamma$ by

$$Kx = \sum_{n=1}^{\infty} u_n'(x)(y_n - u_n) + \sum_{n=1}^{\infty} \varepsilon_n u_{2\ell(n)}'(x)v_n.$$

Thus K is compact for the p-norm topology of E^γ and $K(E^\gamma) \subset B$. If $\|x\| \leq 1$

$$\|Kx\| \leq \sum_{n=1}^{\infty} |u_n'(x)|^p \|y_n - u_n\| + \sum_{n=1}^{\infty} \varepsilon_n^p |u_{2\ell(n)}'(x)|^p \|v_n\|$$
$$\leq \tfrac{1}{2}\|x\|.$$

Let $J: E^\gamma \to B^\gamma$ be the inclusion map, and let $N = (J+K)(E_0)$. By Lemma 2, N is closed in B. In fact it is easy to see that $N \subset X$ and an argument similar to the proof of Theorem 1 shows that N is weakly dense in X (note that $y_{2\ell(n)} + \varepsilon_n v_n \in N$). Suppose $N = X$; then as $(J+K)(E_0) \subset (J+K)(E) \subset X$ we have $(J+K)(E) = (J+K)(E_0)$ and there exists $x \in E$, with $x \neq 0$, such that $(J+K)x = 0$. However $\|Kx\| \leq \frac{1}{2}\|x\|$ and so this is impossible. Hence N is a PCWD subspace of X.

It remains to show that N is relatively β-closed. Consider first $N^\gamma = (J+K)(E^\gamma)$. To show that N^γ is β-closed in B^γ it is only necessary to show that $N^\gamma \cap U$ is σ-closed. Suppose $z_n \in E^\gamma$, $\|(J+K)z_n\| \leq 1$ and $(J+K)(z_n) \to u(\beta)$. Then as $\|(J+K)z_n\| \geq \frac{1}{2}\|z_n\|$ we have $\|z_n\| \leq 2$; by passing to a subsequence we may suppose $z_n \to z(\sigma)$ and $z \in E^\gamma$. Since each (u_n') is σ-continuous on E^γ, K is continuous on bounded sets for the σ-topology. Hence $(J+K)z_n \to (J+K)z(\sigma)$ and $(J+K)z = u$. Thus N^γ is β-closed in B^γ. Now consider $N^\gamma \cap B \supset N$; if $x \in N^\gamma \cap B$, then $x = (J+K)u$, $u \in E^\gamma$ and $u = x - Ku \in B \cap E^\gamma = E$. Thus $x \in N$ and so $N = N^\gamma \cap B$ is β-closed in B and hence in X, as required.

COROLLARY. *Every non-locally convex separable locally bounded F-space with a base of weakly closed neighbourhoods of* 0 *has a quotient with trivial dual which admits compact operators.*

Proof. These spaces are precisely those isomorphic to subspaces of a locally bounded F-space with a basis (see [**5**], Theorem 7.4).

REMARKS. (1) There exist locally bounded non-locally convex F-spaces with bases such that every non-trivial quotient admits compact operators. Indeed any pseudo-reflexive space as constructed in [**8**] has this property, as it possesses a topology β in which the unit ball is compact and which defines the same closed subspaces as the original topology.

(2) There exist separable locally bounded non-locally convex F-spaces X with separating duals such that every compact operator on X has a weakly closed kernel. To see this construct an Orlicz function φ on $[0, \infty)$ so that

(a) $x^{-\frac{1}{2}}\varphi(x)$ is increasing,
(b) $\limsup_{x\to\infty} x^{-1}\varphi(x) = \infty$,
(c) $\liminf_{x\to\infty} x^{-1}\varphi(x) = 1$,
(d) $\sup_{x\geq 1} \frac{\varphi(2x)}{\varphi(x)} < \infty$.

Then $L_\varphi(0, 1)$ is a locally bounded non-locally convex space with separating dual and, by the results of [**4**], every compact operator on L_φ factors through the inclusion map $L_\varphi \subsetneq L_1$. Hence the kernel of any compact operator is weakly closed.

(3) In the proof of the theorem the Mackey topology on X may be replaced by any strictly weaker metrizable topology. In particular if X is locally p-convex ($p<1$) but not locally r-convex for any $r>p$, then we may consider the topology induced by all absolutely r-convex neighbourhoods of 0 ($r>p$). We then obtain by the same arguments the following result.

THEOREM 3. *Let X be a closed subspace of a locally bounded F-space with a basis. Suppose X is locally p-convex ($p<1$) but not locally r-convex for any $r>p$. Then there is a non-zero compact operator $T: X \to Y$ (where Y is a locally bounded F-space) such that $X/\ker T$ admits no operators into any locally r-convex space for $r>p$.*

Let $X = \ell_p$ and U be the closed unit ball of ℓ_p. If we construct T as in the theorem then $\overline{T(U)}$ is a compact p-convex subset of Y, which cannot be linearly embedded into a locally r-convex space. For if S is such an embedding (i.e. S is linear on the linear span of $\overline{T(U)}$ and continuous on $\overline{T(U)}$), then ST is continuous on ℓ_p and $\ker(ST) \subset \ker T$. By the theorem $ST=0$. On the other hand $\overline{T(U)}$ can be linearly embedded in a locally p-convex space (see [**6**]).

Note added in proof. (See the Remark preceding Theorem 2): An example of a non-locally convex F-space with no non-trivial quotient with trivial dual has been constructed by J. Roberts (Springer Lecture Notes No. 604, pp. 76–81). Similar examples were also obtained independently by M. Ribe and the author.

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