Lipschitz and uniform embeddings into $\ell_\infty$

by

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Abstract. We show that there is no uniformly continuous selection of the quotient map $Q : \ell_\infty \rightarrow \ell_\infty/c_0$ relative to the unit ball. We use this to construct an answer to a problem of Benyamini and Lindenstrauss; there is a Banach space $X$ such that there is a no Lipschitz retraction of $X^{**}$ onto $X$; in fact there is no uniformly continuous retraction from $B_{X^{**}}$ onto $B_X$.

1. Introduction. It is very well-known that $c_0$ is not complemented in $\ell_\infty$; an alternate viewpoint is that there is no continuous linear selection (or right inverse) of the quotient map $Q : \ell_\infty \rightarrow \ell_\infty/c_0$. On the other hand, Aharoni and Lindenstrauss [1] showed that there is a large subspace of $\ell_\infty/c_0$ (isometric to $c_0(I)$ where $|I| = c$, the cardinality of the continuum) on which a Lipschitz selection of $Q$ exists. It is also known that $c_0$ is a Lipschitz retract of $\ell_\infty$ [15], [4]. This raises the question of whether a global Lipschitz or at least uniformly continuous selection can be found. In fact, there are set-theoretic reasons that one cannot find any selection $f$ so that $f \circ Q$ is weak*-Borel [8], which already means that any formula for such an $f$ must be rather unpleasant. It is also worth noting that every Banach space of density character $\aleph_1$ linearly and isometrically embeds into $\ell_\infty/c_0$ [16].

In this note we show that there is no selection $f$ which is uniformly continuous on the unit ball of $\ell_\infty/c_0$. We show in fact that $B_{\ell_\infty/c_0}$ cannot be uniformly embedded into $\ell_\infty$. We prove this by considering the space $\Omega_n = \omega_1^{[n]}$ of all $n$-subsets $\{\alpha_1, \ldots, \alpha_n\}$ of the countable ordinals as a graph where two subsets are adjacent if they interlace. This is the uncountable analogue of a similar graph considered in [13] in connection with uniform embeddings into reflexive spaces. One then shows that if $f : \Omega_n \rightarrow \ell_\infty$ is a map such that $\|f(\alpha) - f(\beta)\| \leq 1$ whenever $\alpha$ and $\beta$ are adjacent then there is an uncountable subset $\Theta$ of $\omega_1$ so that $f(\Theta^{[n]})$ has diameter at most one.
This is very similar to the results of [13] for maps into reflexive spaces on the countable version of this graph.

Our main application of this result is to answer a problem raised in [4, p. 183] (see also Problem 10 of [14]). We give an example of a Banach space $X$ so that there is no Lipschitz retraction of $X^{**}$ onto $X$; for a discussion of the significance of such an example in the extension theory of Lipschitz maps we refer to [14]. In fact for our example there is no uniformly continuous retraction of the unit ball $B_{X^{**}}$ onto $B_X$. The space $X$ is a Lindenstrauss space, i.e. its dual is isometric to an $L_1$-space and its bidual is therefore an injective Banach space (and in particular a 1-absolute Lipschitz retract).

We also discuss non-separable spaces which can be Lipschitz embedded into $\ell_\infty$. For example, it is shown that every Banach space with an unconditional basis of cardinality at most $\aleph_0$ Lipschitz embeds into $\ell_\infty$.

In the final section, we also show that $\ell_\infty \oplus c_0$ does not have unique Lipschitz structure and show indeed that a very wide class of spaces containing $c_0$ cannot have unique Lipschitz structure.

2. Notation and preliminaries. All Banach spaces will be real. We denote by $B_X$ the closed unit ball of a Banach space $X$ and by $\partial B_X$ the unit sphere. $I_X$ will denote the identity operator on $X$.

$X$ is called weakly compactly generated (WCG) if there is a weakly compact set $W$ such that $\bigcup_{n \in \mathbb{N}} nW$ is dense in $X$. A Markushevich basis for $X$ is a biorthogonal system $\{x_i, x_i^*\}_{i \in I}$ which is total and fundamental. We shall say that $X$ is Plichko if it has a Markushevich basis $\{x_i, x_i^*\}_{i \in I}$ so that the subset $E$ of $X^*$ given by $E = \{x^* : |\{i : x^*(x_i) \neq 0\}| \leq \aleph_0\}$ is a 1-norming subspace of $X^*$. If $E = X^*$ we say that $X$ is weakly Lindelöf determined (WLD). These are not the original definitions of Plichko and WLD spaces but are equivalent (see Theorems 5.37 and 5.63 of [10] or [23] and [22]). Note that WCG spaces are WLD and hence also Plichko. $X$ is said to have the separable complementation property (SCP) if every separable subspace of $X$ is contained in a complemented separable subspace.

For any Banach space $X$ we write $\text{dens}_X$ for the smallest cardinality of a dense subset (the same definition will be used for metric spaces). We write $\text{w}^*-\text{dens}_X$ for the smallest cardinality of a weak*-dense subset of $X^*$.

Let $M$ and $M'$ be metric spaces. If $f : M \to M'$ is any mapping we define

$$\psi_f(t) = \sup\{d(f(x), f(x')) : d(x, x') \leq t\}.$$  

$f$ is said to be Lipschitz (with Lipschitz constant $L$) if $\psi_f(t) \leq Lt$ for all $t$. The function $f$ is uniformly continuous if $\lim_{t \to 0} \psi_f(t) = 0$ and coarsely continuous if $\psi_f(t) < \infty$ for all $t$. If $f$ is injective, then $f$ is a Lipschitz (respectively uniform, respectively coarse) embedding if $f$ and $f^{-1}|_{f(M)}$ are
Lipschitz (respectively uniformly continuous, respectively coarsely continuous).

If $M$ is a metric space and $E$ is a subset of $M$ then a retraction $r : M \to E$ is a map such that $r(e) = e$ for $e \in E$.

If $M$ has a base point (labelled 0), we refer to $M$ as a pointed metric space and we define $\text{Lip}(M)$ as the Banach space of all real-valued Lipschitz maps $f : M \to \mathbb{R}$ with the usual norm,

$$\|f\|_{\text{Lip}} = \sup \left\{ \frac{|f(x) - f(x')|}{d(x, x')} : x, x' \in M, d(x, x') > 0 \right\}.$$ 

If $M = X$ is a Banach space, the base point is always the origin. The Arens–Eells space $\mathcal{E}(M)$ is defined as the closed linear span of the point evaluations $\delta_s(f) = f(s)$ in $\text{Lip}(M)^*$. The map $\delta : s \mapsto \delta_s$ is then an isometry of $M$ into $\mathcal{E}(M)$. We refer to [24] and [9] for further details (in [9] the terminology Lipschitz-free space and the notation $\mathcal{F}M$ was used). If $X$ is a Banach space there is a canonical quotient map $\beta : \mathcal{E}(X) \to X$ and $\delta$ is an isometric selection for $\beta$, i.e. $\beta \circ \delta = I_X$.

We will record a result here which is not needed in the sequel but represents an improvement over Proposition 4.1 of [9].

**Theorem 2.1.** Let $M$ be a pointed metric space and suppose $F \subset \mathcal{E}(M)$ is a bounded non-separable subset of density character $\aleph > \aleph_0$. If $\aleph$ has uncountable cofinality\(^{(1)}\), then there is a subset $(\mu_i)_{i \in I}$ of $F$ with $|I| = \aleph$ so that $(\mu_i)_{i \in I}$ is equivalent to the unit vector basis of $\ell_1(I)$. In particular any weakly compact subset of $\mathcal{E}(M)$ is separable.

**Proof.** If $G$ is any subset of $M$ containing 0, for $0 < \delta < 1$, we denote by $G^{[\delta]}$ the subset of $M$ of all $x$ such that $d(x, G) \leq \delta d(x, 0)$. We will first claim that there exists $0 < \delta < 1$ so that, for $G$ as above,

$$\text{(2.1)} \quad \text{if } \text{dens}(G) < \aleph, \text{ then } d(\mu, \mathcal{E}(G^{[\delta]})) > \delta \text{ for some } \mu \in F.$$ 

Indeed, if not, for $n \in \mathbb{N}$ we may pick $G_n$ containing 0 so that

$$\text{dens}(G_n) < \aleph \text{ and } d(\mu, \mathcal{E}(G_n^{[1/n]})) \leq 1/n \quad \text{for every } \mu \in F.$$ 

Let $G = \bigcup_n G_n$ and assume $\mu \in F$. We show that $\mu \in \mathcal{E}(G)$. If not there exists $f \in \text{Lip}(M)$ with $f|_G = 0$, $\|f\|_{\text{Lip}(M)} = 1$ and $\langle \mu, f \rangle = \theta \neq 0$. By picking either $f_+ = \max(f, 0)$ or $f_- = \max(-f, 0)$ we can suppose $f \geq 0$. Now define

$$f_n(x) = \min(f(x), (d(x, G) - d(x, 0)/n)_+) .$$ 

Then $\|f_n\|_{\text{Lip}(M)} \leq 1 + 1/n$. We also have

$$f(x) - 1/n \leq f_n(x) \leq f(x), \quad x \in G,$$

\(^{(1)}\) This assumption has been added by the editors.
and so \( \lim_{n \to \infty} f_n(x) = f(x) \) for \( x \in G \). This implies that \( \lim_{n \to \infty} f_n = f \) weak*. Now since \( f_n(G^{[1/n]}) = 0 \) we have
\[
\langle \mu, f_n \rangle \leq \frac{1}{n} \left( 1 + \frac{1}{n} \right)
\]
and so \( \langle \mu, f \rangle = 0 \) contrary to assumption. Thus \( F \subset \mathcal{AE}(G) \), contrary to the fact that \( G \), and hence \( \mathcal{AE}(G) \), has density character \( \aleph_\delta \), while \( F \) does not \( [2] \). This establishes \( [2] \).

Assuming \( [2.1] \) we construct a maximal family \( \{G_i, \mu_i, f_i\}_{i \in I} \) where:

(i) \( \mu_i \in F \), \( G_i \) is a countable subset of \( M \) and \( \mu_i \in \mathcal{AE}(G_i) \).
(ii) \( f_i \in \text{Lip}(M) \) with \( f_i \geq 0 \) and \( \|f_i\|_{\text{Lip}(M)} \leq 4/\delta^2 \).
(iii) \( \|\mu_i, f_i\| = 1 \) and \( \langle \mu_i, f_j \rangle = 0 \) for \( i \neq j \).
(iv) \( \text{supp } f_i = \{f_i > 0\} \subset G_i^{[\delta]} \).
(v) \( \text{supp } f_i \cap \text{supp } f_j = \emptyset \) for \( i \neq j \).

Let \( G = \bigcup_i G_i \); if \( I \) is empty we set \( G = \{0\} \).

We next argue that \( |I| = \aleph_\delta \) by contradiction. Suppose \( |I| < \aleph_\delta \). Then \( \text{dens } G < \aleph_\delta \). Hence there exists \( \mu \in F \) so that \( d(\mu, \mathcal{AE}(G_i^{[\delta]})) > \delta \). Thus there exists a function \( g \in \text{Lip}(M) \) with \( \|g\|_{\text{Lip}(M)} \leq 2/\delta \), \( g(G_i^{[\delta]}) = 0 \) and \( \langle \mu, g \rangle > 2 \). By considering either \( g_+ \) or \( g_- \) we can therefore find \( h \geq 0 \) with \( \|h\|_{\text{Lip}(M)} \leq 2/\delta \), \( h(G_i^{[\delta]}) = 0 \) and \( \|\mu, h\| = 1 \).

Take a countable set \( G' \subset M \) so that \( \mu \in \mathcal{AE}(G') \). Then we define
\[
f(x) = \max(\text{sup}(\{h(y) - 4\delta^{-2}d(x,y) : y \in G'\}, 0)).
\]
Then \( \|f\|_{\text{Lip}(M)} \leq 4\delta^{-2} \). If \( x \notin G_i^{[\delta]} \), then for any \( y \in G' \) we have \( d(x,y) > \delta d(x,0) \). Hence
\[
d(y,0) \leq d(x,0) + d(x,y) \leq (\delta^{-1} + 1)d(x,y) \leq 2\delta^{-1}d(x,y).
\]
Then \( h(y) \leq 2\delta^{-1}d(y,0) \leq 4\delta^{-2}d(x,y) \), which implies that \( \text{supp } f \subset G'_i^{[\delta]} \).

Now we can add \( (G', \mu, f) \) to the collection contradicting maximality. The conclusion is that \( |I| = \aleph_\delta \).

Finally we argue that \( (\mu_i) \) is equivalent to the unit vector basis of \( \ell_1 \).
Indeed, if \( A \) is a finite subset of \( I \) and \( (a_i)_{i \in A} \) are real numbers we define \( h \in \text{Lip}(M) \) by setting \( h = \sum_{i \in A} \epsilon_i a_i f_i \) where \( \epsilon_i a_i \langle \mu_i, f_i \rangle = |a_i| \). Then \( \|h\|_{\text{Lip}(M)} \leq 8/\delta^2 \). Indeed, suppose \( x \in \text{supp } f_i \), \( y \in \text{supp } f_j \) where \( i \neq j \). Then
\[
|h(x) - h(y)| \leq |f_i(x) - f_i(y)| + |f_j(x) - f_j(y)| \leq 8\delta^{-2}d(x,y).
\]
Other cases for \( x, y \) give better estimates. Hence
\[
\left\| \sum_{i \in A} a_i \mu_i \right\| \geq \frac{1}{8} \delta^2 \sum_{i \in A} |a_i|. \quad \blacksquare
\]

(2) This sentence has been added by the editors. It seems that in this part of the reasoning the author uses the assumption that \( \aleph_\delta \) has uncountable cofinality.
3. A metric Ramsey theorem. We denote by $\omega_1$ the first uncountable ordinal. Let $\Omega_n = \omega_1^{[n]}$ where $n \geq 0$ denote the collection of all $n$-subsets of $\Omega_1 = \omega_1 = [0, \omega_1)$; note that in the case $n = 0$, $\Omega_0$ consists of one point, namely the empty set $\emptyset$. We write a typical element of $\Omega_n$ in the form $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ where $\alpha_1 < \cdots < \alpha_n$. If $n \geq 1$ and $A \subset \Omega_n$ we define $\partial A \subset \omega_1^{[n-1]}$ by $\{\alpha_1, \ldots, \alpha_{n-1}\} \in \partial A$ if and only if $\{\beta : \{\alpha_1, \ldots, \alpha_{n-1}, \beta\} \in A\}$ is uncountable. If $n = 1$ this amounts to the fact that $\emptyset \in \partial A$ if and only if $A$ is uncountable. We shall say that $A \subset \Omega$ is large if $\emptyset \in \partial \Omega$; otherwise $A$ is small. We say that $A$ is very large if its complement $\bar{A}$ is small.

It follows from the next lemma that if $A$ is very large it is also large.

**Lemma 3.1.** Let $(A_k)_{k=1}^\infty$ be a sequence of small subsets of $\Omega_n$. Then $\bigcup_{k \geq 1} A_k$ is also small.

**Proof.** It is trivial that $\partial \bigcup_k A_k = \bigcup_k \partial A_k$. Iterating gives the result.

**Lemma 3.2.** If $A$ is a very large subset of $\Omega_n$ there is an uncountable subset $\Theta \subset \Omega_1$ so that $\Theta^{[n]} \subset A$.

**Proof.** Let $\Theta$ be a maximal subset of $\Omega_1$ so that if $\{\alpha_1, \ldots, \alpha_k\} \in \Theta^{[k]}$ with $0 \leq k \leq n$ then $\{\alpha_1, \ldots, \alpha_k\} \notin \partial^{n-k} \bar{A}$; here we write $\partial^0 \bar{A} = \bar{A}$. Such a maximal subset exists by Zorn’s Lemma since $\emptyset$ satisfies the conditions. We show that $\Theta$ is uncountable. Assume, on the contrary, that $\Theta$ is countable. For each $\{\alpha_1, \ldots, \alpha_k\} \in \Theta^{[k]}$ with $0 \leq k \leq n - 1$ the set of $\beta$ such that $\{\alpha_1, \ldots, \alpha_k, \beta\} \in \partial^{n-k+1} \bar{A}$ is countable. Let

$$h\{\alpha_1, \ldots, \alpha_k\} = \sup\{\beta : \{\alpha_1, \ldots, \alpha_k, \beta\} \in \partial^{n-k+1} \bar{A}\}$$

so that $h\{\alpha_1, \ldots, \alpha_k\} < \omega_1$. Let $\sigma$ be the supremum of all $h\{\alpha_1, \ldots, \alpha_k\}$; since $\Theta$ is countable we have $\sigma < \omega_1$. Now $\sigma \geq \sup \Theta$ and $\Theta \cup \{\sigma + 1\}$ gives a contradiction to maximality.

**Lemma 3.3.** Let $f : \Omega_n \to \mathbb{R}$ be any mapping. Then there is an open set $U \subset \mathbb{R}$ so that $f^{-1}(U)$ is small and if $V$ is any open set with $f^{-1}(V)$ small then $V \subset U$.

**Proof.** Let $U$ be the set of all open sets $V$ so that $f^{-1}(V)$ is small. Then by the Lindelöf theorem and Lemma 3.1 $U \in \mathcal{U}$ where $U = \bigcup\{V : V \in \mathcal{U}\}$.

We will make $\Omega_n$ into a graph by declaring $\alpha \neq \beta$ to be adjacent if either

$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots \leq \alpha_n \leq \beta_n$$

or

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \beta_n \leq \alpha_n,$$

i.e. $\alpha$ and $\beta$ interlace. Then we define $d$ to be the least path metric on $\Omega_n$, which then becomes a metric space. We write $\alpha \prec \beta$ if

$$\alpha_1 \prec \cdots \prec \alpha_n < \beta_1 \prec \cdots \prec \beta_n.$$

If $\alpha \prec \beta$ then $d(\alpha, \beta) = n$ so that $\Omega_n$ has diameter $n$. Lipschitz and uniform embeddings into $\ell_\infty$
Lemma 3.4. Suppose $A$ and $B$ are large subsets of $\Omega_n$. Then there exist $\alpha \in A$ and $\beta \in B$ so that $\alpha, \beta$ interlace.

Proof. Pick $\alpha_1 \in \partial_n^{-1}A$ and then, since $\partial_n^{-1}B$ is uncountable, we may pick $\beta_1 \in \partial_n^{-1}B$ with $\beta_1 > \alpha_1$. Continuing we may pick $\alpha_2 > \beta_1$ so $\{\alpha_1, \alpha_2\} \in \partial_n^{-2}A$ and continue. \hfill \blacksquare

Lemma 3.5. Let $f : (\Omega_n, d) \to \mathbb{R}$ be any Lipschitz function with Lipschitz constant $L$. Then there exists $\xi \in \mathbb{R}$ so that $\{\alpha : |f(\alpha) - \xi| > L/2\}$ is small.

Proof. Let $U$ be the maximal open subset of $\mathbb{R}$ so that $f^{-1}(U)$ is small, given by Lemma 3.3. Let $E$ be its complement, which is necessarily nonempty and closed. It suffices to show that the diameter of $E$ given by Lemma 3.3. Let $\xi$ so that $\|f(\xi)\| = L$. In particular $\xi \in L$. Hence $f^{-1}(\xi)$ is at most $L$. Suppose $s, t \in E$ with $s - t > L$. Then we can pick $\epsilon > 0$ so small that $s - \epsilon > t + \epsilon + L$. Now the sets $f^{-1}(s - \epsilon, \infty)$ and $f^{-1}(-\infty, t + \epsilon)$ are both large. By Lemma 3.3 there exist $\alpha, \beta$ which are interlacing and so that $f(\alpha) > s - \epsilon$ and $f(\beta) < t + \epsilon$, which contradicts the definition of the Lipschitz constant. \hfill \blacksquare

Theorem 3.6. Let $f : (\Omega_n, d) \to \ell_\infty$ be a Lipschitz map with Lipschitz constant $L$. Then there exists $\xi \in \ell_\infty$ and an uncountable subset $\Theta$ of $\Omega_n$ so that

$$\|f(\alpha) - \xi\| \leq L/2, \quad \alpha \in \Theta^{[n]}.$$ 

Proof. Let $f(\alpha) = (f_n(\alpha))_{n=1}^\infty$. According to Lemma 3.5 there exists $\xi_n \in \mathbb{R}$ so that $f_n^{-1}[\xi_n - L/2, \xi_n + L/2]$ is very large. Hence if $\xi = (\xi_n)_{n=1}^\infty$, the set of $\alpha$ with $|f_n(\alpha) - \xi_n| < L/2$ for all $n$ is also very large (using Lemma 3.1). In particular $\xi \in \ell_\infty$. The proof is completed by Lemma 3.2. \hfill \blacksquare

4. Embeddings in $\ell_\infty$. Let $X$ be a Banach lattice. We will say that $X$ has the monotone transfinite sequence property (MTSP) if whenever $(x_\mu)_{\mu < \omega_1}$ is a monotone increasing transfinite sequence, then there exists $x \in X$ such that $x_\mu = x$ eventually.

Theorem 4.1. Let $X$ be a Banach lattice with the property that $B_X$ can be uniformly embedded into $\ell_\infty$. Then $X$ has (MTSP).

Proof. Let us assume $X$ fails (MTSP). Since any transfinite sequence $(x_\mu)_{\mu < \omega_1}$ is necessarily bounded we may assume it takes values in $B_X$. Let $\theta(\mu) = \sup_{\sigma > \mu} \|x_\sigma - x_\mu\|$. Then $\theta(\mu)$ is decreasing.

Let $f : B_X \to \ell_\infty$ be a uniform embedding, with inverse $g : f(B_X) \to B_X$. For each $n$ consider the map $f_n : \Omega_n \to \ell_\infty$ given by $f_n(\alpha) = f(1/n \sum_{j=1}^n x_{\alpha_j})$. If $\alpha_1 < \beta_1 < \alpha_2 < \cdots < \alpha_n < \beta_n$ we have

$$\|f_n(\alpha) - f_n(\beta)\| = \left\|\frac{1}{n} \sum_{j=1}^n (x_{\beta_j} - x_{\alpha_j})\right\| \leq \frac{1}{n} \|x_{\beta_n} - x_{\alpha_1}\| \leq 2/n.$$ 

Hence $f_n$ has Lipschitz constant $\psi_f(2/n)$.
By Theorem 3.6 we may pick an uncountable subset $\Theta_n$ of $\Omega_1$ so that
\[
\|f_n(\alpha) - f_n(\beta)\| \leq \psi_f(2/n), \quad \alpha, \beta \in \Theta_n.
\]
Hence
\[
\left\| \frac{1}{n} \sum_{j=1}^{n} x_{\beta_j} - \frac{1}{n} \sum_{j=1}^{n} x_{\alpha_j} \right\| \leq \psi_g(\psi_f(2/n)), \quad \alpha, \beta \in \Theta_n.
\]
Pick $\alpha_1 < \cdots < \alpha_n \in \Theta_n$. If $\nu > \mu > \alpha_n$ then we can find $\beta_n > \beta_{n-1} > \cdots > \beta_1 > \nu$ with $\beta_j \in \Theta_n$ for $1 \leq j \leq n$ and then
\[
\| x_\nu - x_\mu \| \leq \left\| \frac{1}{n} \sum_{j=1}^{n} x_{\beta_j} - \frac{1}{n} \sum_{j=1}^{n} x_{\alpha_j} \right\| \leq \psi_g(\psi_f(2/n)),
\]
and hence
\[
\theta(\mu) \leq \psi_g(\psi_f(2/n)), \quad \mu > \alpha_n.
\]
Applying this for every $n$, since $\lim_{n \to \infty} \psi_g(\psi_f(2/n)) = 0$ we have that $\theta(\mu) = 0$ eventually, which implies that $x_\mu$ is eventually constant. \(\blacksquare\)

**Theorem 4.2.** Let $X = \ell_\infty/c_0$ or $C[1, \omega_1]$. Then $B_X$ cannot be uniformly embedded into $\ell_\infty$.

**Proof.** Since $C[1, \omega_1]$ embeds into $\ell_\infty/c_0$ it is not really necessary to prove these separately. However it is easy to observe that both spaces fail (MTSP). For the case of $\ell_\infty/c_0$ we may, by induction, define a transfinite sequence of infinite subsets $A_\mu \subset \mathbb{N}$ so that if $\mu > \nu$ we have $A_\nu \setminus A_\mu$ infinite and $A_\mu \subset A_\nu$ modulo finite sets. Let $B_\mu$ be the complement of $A_\mu$ and then $x_\mu = Q(\chi_{B_\mu})$ where $Q : \ell_\infty \to c_0$ is the quotient map. For the case of $C[1, \omega_1]$ let $x_\mu = \chi_{[1, \mu]}$. In either case the result follows from the preceding Theorem 4.1. \(\blacksquare\)

**Remark.** In \[17\] it is shown that $C[1, \omega_1]$ does not uniformly embed into a space $c_0(I)$. In \[1\] it is shown that if $|I| = \mathfrak{c}$ then $c_0(I)$ Lipschitz embeds in $\ell_\infty$. Therefore Theorem 4.2 also implies that $C[1, \omega_1]$ does not uniformly embed into $c_0(I)$ for any set $I$. Note however that there is an injective linear map $T : C[1, \omega_1] \to c_0(I)$ where $|I| = \aleph_1$ defined by $T \varphi(\alpha) = \varphi(\alpha+1) - \varphi(\alpha)$ for $\alpha < \omega_1$ and $T \varphi(\omega_1) = \varphi(1)$. Thus there is a countable family of Lipschitz functions on $C[1, \omega_1]$ which separate the points of $C[1, \omega_1]$.

We now give a slight variation of Theorem 4.2. Let us say that a Banach lattice $X$ has the countable interpolation property if whenever $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are two sequences with $x_m \geq y_n$ whenever $m, n \in \mathbb{N}$ then there exists $z \in X$ with $x_m \geq z \geq y_n$ for all $m, n \in \mathbb{N}$. If $K$ is a compact Hausdorff space then $C(K)$ has the countable interpolation property if and only if $K$ is an F-space \[21\]; here $K$ is an F-space if the closures of two disjoint open $F_\sigma$-sets remain disjoint (see \[7\]). The space $\beta \mathbb{N} \setminus \mathbb{N}$ is an F-space so the following result implies Theorem 4.2 for the case $\ell_\infty/c_0$. 
Proposition 4.3. Let \( X \) be a Banach lattice with (MTSP) and the countable interpolation property. Then \( X \) is order-complete. In particular if \( K \) is an \( F \)-space such that \( C(K) \) has (MTSP), then \( K \) is Stonian.

Proof. Let \( A \) be a subset of \( X \) which is bounded above. Let us say that \( A \) has property (P) if for every countable set \( B \) with \( x \leq y \) for \( x \in A \) and \( y \in B \) there exists \( z \in X \) with \( x \leq z \leq y \) for \( x \in A, \ y \in B \). We claim that property (P) implies \( A \) has a least upper bound (or supremum). If not let \( y_1 \) be any upper bound. We construct a strictly decreasing transfinite sequence \((y_\mu)_{\mu<\omega_1}\) by transfinite induction. If \((y_\nu)_{\nu<\mu}\) has been chosen to be strictly decreasing we pick for each \( \nu \) an upper bound \( y_\nu' \leq y_\nu \) with \( y_\nu' \neq y_\nu \) and then use (P) to find an upper bound \( y_\mu \leq y_\nu' \) for all \( \nu < \mu \). This contradicts (MTSP) and shows that \( A \) has a least upper bound.

Now by the countable interpolation property, countable sets have property (P) and hence have a least upper bound. But this implies that every set \( A \) which is bounded above has property (P) and the proof is complete. \( \blacksquare \)

We will now turn to an application of the above results. We will need the following elementary lemma, which appears first in [12]; we include the proof for the convenience of the reader.

Lemma 4.4. Let \( Y \) be a Banach space and let \( Q : Y \to X \) be a quotient mapping. In order that there exists a uniformly continuous section \( f : B_X \to Y \) it is necessary and sufficient that for some \( 0 < \lambda < 1 \) there is a uniformly continuous map \( \phi : \partial B_X \to Y \) with \( \|Q(\phi(x)) - x\| \leq \lambda \) for \( x \in \partial B_X \).

Proof. We may extend \( \phi \) to \( B_X \) to be positively homogeneous and \( \phi \) remains uniformly continuous. Define \( g(x) = x - Q(\phi(x)) \), so that \( g \) is also positively homogeneous. Then \( \|g(x)\| \leq \lambda \|x\| \) and so \( \|g^n(x)\| \leq \lambda^n \|x\| \) for \( x \in B_X \). Let \( g^0(x) = x \). Let \( f(x) = \sum_{n=0}^{\infty} \phi(g^n(x)). \)

The series converges uniformly in \( x \in B_X \) and so \( f \) is uniformly continuous. Furthermore

\[
Qf(x) = \sum_{n=0}^{\infty} (g^n(x) - g^{n+1}(x)) = x. \quad \blacksquare
\]

The following theorem answers a question raised in [4, p. 183] (see also the discussion in [14]).

Theorem 4.5. There exists a (non-separable) Lindenstrauss space \( Z \) such that there is no uniformly continuous retraction of \( B_{Z^{**}} \) onto \( B_Z \). In particular there is no uniformly continuous retraction of \( Z^{**} \) onto \( Z \).
Proof. The space $Z$ is the same example that was given by Benyamini [3] as an example of a non-separable M-space which is not a $C(K)$-space.

We start by considering the quotient map $Q : \ell_\infty \to \ell_\infty / c_0$. For each $n$ we pick a maximal set $D_n$ in the interior of $B_{\ell_\infty / c_0}$ so that $\|x-x'\| \geq 1/n$ for $x,x' \in D_n$ and $x \neq x'$. Then for each $n$ we can define a map $h_n : D_n \to B_{\ell_\infty}$ with $Qh_n(x) = x$ for $x \in D_n$.

Now let $Y_n$ denote $\ell_\infty$ with the equivalent norm

$$\|y\|_{Y_n} = \max \left( \frac{1}{n} \|y\|_{\ell_\infty}, \|Qy\|_{\ell_\infty / c_0} \right).$$

Then $Y_n$ is a Lindenstrauss space; note also that $Q : Y_n \to \ell_\infty / c_0$ remains a quotient map for the usual norm on $\ell_\infty / c_0$. Let $Z = c_0(Y_n)$, which is also a Lindenstrauss space. Let us assume that there is a uniformly continuous retraction of $B_{Z^{**}}$ onto $B_Z$. Then it follows that there is a sequence of uniformly continuous rejections $g_n : Y_n^{**} \to Y_n$ which is equi-uniformly continuous, i.e. so that

$$\psi_{g_n}(t) \leq \psi(t), \quad 0 < t \leq 2,$$

where $\lim_{t \to 0} \psi(t) = 0$.

Consider the map $h_n : D_n \to Y_n$. If $x \neq x' \in D_n$ then

$$\|h_n(x) - h_n(x')\| \leq \max \left( \frac{2}{n}, \|x-x'\| \right) \leq 2\|x-x'\|.$$

Hence since $B_{Y_n^{**}}$ is a 1-absolute Lipschitz retract, there is an extension $f_n : B_{\ell_\infty / c_0} \to B_{Y_n^{**}}$ with $\text{Lip}(f_n) \leq 2$. Now if $x \in B_{\ell_\infty / c_0}$ there exists $x' \in D_n$ with $\|x-x'\| < 2/n$ (actually $1/n$ if $x$ is in the open unit ball). Thus

$$\|g_n(f_n(x)) - g_n(f_n(x'))\| \leq \psi(4/n)$$

and hence

$$\|Q(g_n \circ f_n(x)) - x\| \leq \psi(4/n) + 2/n.$$

For some choice of $n$ we have $\psi(4/n) + 2/n < 1$. By Lemma 4.4 this means there is a uniformly continuous section of the quotient map $Q : B_{\ell_\infty / c_0} \to Y_n$. Thus $B_{\ell_\infty / c_0}$ uniformly embeds into $Y_n$ which is isomorphic to $\ell_\infty$ and thus we have contradicted Theorem 4.2.

Remark. Note that $C[1,\omega_1]$ isometrically embeds into $\ell_\infty$ [16]; let $F$ be a subspace of $\ell_\infty / c_0$ isometric to $C[1,\omega_1]$. Let $E = Q^{-1}(F)$ and let $Z_0$ be the subspace of $Z$ of all sequences $(y_n)_{n=1}^\infty$ such that $y_n \in E$. Then arguing in exactly the same way shows that there is no uniformly continuous retraction of $B_{Z_0^{**}}$ onto $B_{Z_0}$. Thus $Z_0$ is an example of a Lindenstrauss space with the properties of Theorem 4.5 with the additional property that $Z_0^{*}$ is isometric to $\ell_1(I)$ where $|I| = \aleph_1$. 

5. Banach spaces which Lipschitz embed into $\ell_\infty$

**Proposition 5.1.** The Arens–Eells space $\mathcal{A}(\ell_\infty)$ of $\ell_\infty$ is linearly isometric to a closed linear subspace of $\ell_\infty$.

**Proof.** For each $m \in \mathbb{N}$ we may pick a countable collection of functions $\{f_{mn}\}_{n=1}^\infty$ in $\text{Lip}(\ell_\infty^m)$ with $\|f_{mn}\|_{\text{Lip}(\ell_\infty^m)} \leq 1$ and such that

$$
\|\mu\|_{\mathcal{A}(\ell_\infty^m)} = \sup_n \langle \mu, f_{mn} \rangle, \quad \mu \in \mathcal{A}(\ell_\infty^m).
$$

Now for each finite subset $J = \{k_1, \ldots, k_m\} \subset \mathbb{N}$ we define the natural quotient $Q_J : \ell_\infty \to \ell_\infty^m$ by $Q\xi = (\xi_{k_j})_{j=1}^m$. We consider the countable family $\Phi$ of Lipschitz functions $\varphi = \varphi_{J,n}$ of the form

$$
\varphi_{J,n}(\xi) = f_{mn}(Q_J \xi).
$$

Clearly $\|\varphi_{J,n}\|_{\text{Lip}(\ell_\infty)} \leq 1$ for all $J$ and $n$.

Now suppose $\mu \in \mathcal{A}(\ell_\infty)$, with $\mu \neq 0$, has finite support $\xi_1, \ldots, \xi_n \in \ell_\infty$ and let $F$ be the linear span of $\xi_1, \ldots, \xi_n$. Then $\|\mu\|_{\mathcal{A}(\ell_\infty)} = \|\mu\|_{\mathcal{A}(F)}$.

Given $\epsilon > 0$ we can find a finite subset $J$ of $\mathbb{N}$, with $|J| = m$, say, so that

$$
\|Q_J \xi\| > (1 - \epsilon)\|\xi\|, \quad \xi \in F.
$$

Hence if $\xi = \sum_{j=1}^n a_j \delta_{\xi_j}$,

$$
\left\| \sum_{j=1}^n a_j \delta_{Q_J \xi_j} \right\|_{\mathcal{A}(\ell_\infty^m)} > (1 - \epsilon)\|\mu\|_{\mathcal{A}(\ell_\infty)}.
$$

Thus we can find $n$ so that

$$
\langle \mu, \varphi_{J,n} \rangle > (1 - \epsilon)\|\mu\|_{\mathcal{A}(\ell_\infty)}.
$$

Thus the map

$$
\mu \mapsto (\langle \mu, \varphi_{J,n} \rangle)_{J,n}
$$

defines an isometry of $\mathcal{A}(\ell_\infty)$ into $\ell_\infty(S)$ where $S$ is a countable set. ■

**Lemma 5.2.** In order that a Banach space $X$ Lipschitz embeds into $\ell_\infty$ it is necessary and sufficient that there is a Lipschitz function $g : X \to \ell_\infty$ with the property that

$$
\|g(u) - g(v)\| \geq 1, \quad \|u - v\| \geq 1.
$$

**Proof.** Let $Q_+$ denote the set of positive rationals and define $G : X \to \ell_\infty(Q_+ \times \mathbb{N})$ by $G(x)(q,n) = q^{-1}g(qx)_n$. Clearly $G$ has Lipschitz constant at most $\text{Lip}(g)$. If $u, v \in X$ we have

$$
\|G(u) - G(v)\| = \sup_{t>0} t^{-1} \|g(tu) - g(tv)\|.
$$

If $u \neq v$ let $t = \|u - v\|^{-1}$ and we have

$$
\|G(u) - G(v)\| \geq \|u - v\|.
$$

Thus $G$ is a Lipschitz embedding. ■
Theorem 5.3. Let $X$ be a Banach space. Then the following conditions on $X$ are equivalent:

(i) $X$ uniformly embeds into $\ell_\infty$.
(ii) $X$ coarsely embeds into $\ell_\infty$.
(iii) $X$ Lipschitz embeds into $\ell_\infty$.
(iv) $\mathcal{E}(X)$ linearly embeds into $\ell_\infty$.
(v) There is a subspace $Z$ of $\ell_\infty$ and a linear surjection $Q : Z \to X$ which admits a Lipschitz selection $\varphi : X \to Z$.
(vi) There is a subspace $Z$ of $\ell_\infty$ and a linear surjection $Q : Z \to X$ which admits a uniformly continuous selection $\varphi : X \to Z$.
(vii) There is a subspace $Z$ of $\ell_\infty$ and a linear surjection $Q : Z \to X$ which admits a coarsely continuous selection $\varphi : X \to Z$.

Proof. (i) or (ii) $\Rightarrow$ (iii). Let $f : X \to \ell_\infty$ be either a coarse embedding or a uniform embedding. Then, in either case, we may find $0 < a, b, c < \infty$ such that

$$\|x - y\| \leq a \Rightarrow \|f(x) - f(y)\| \leq b$$

and

$$\|f(x) - f(y)\| \leq 5b \Rightarrow \|x - y\| \leq c.$$

Pick a maximal subset $S$ of $X$ such that $\|x - y\| \geq a$ if $x, y \in S$ and $x \neq y$. Then $f|_S$ has Lipschitz constant bounded by $\sup_{\delta \geq a} \omega(\delta)/\delta \leq 2b/a$. Since $\ell_\infty$ is a 1-absolute Lipschitz retract we may find a function $h : X \to \ell_\infty$ with Lipschitz constant at most $2b/a$ and $h|_S = f|_S$. Now assume $\|u - v\| \geq c + 2a$.

Then there exist $x, y \in S$ with $\|x - u\|, \|y - v\| < a$ and hence $\|x - y\| > c$. Thus $\|f(x) - f(y)\| > 5b$. Also $\|h(x) - f(u)\| < 2b$ and $\|h(y) - f(v)\| < 2b$.

Hence $\|h(u) - h(v)\| > b$. If we let $g(x) = b^{-1}h((a + 2c)x)$ then $g$ satisfies the hypotheses of Lemma 5.2 and so (iii) is proved.

(iii) $\Rightarrow$ (iv). $\mathcal{E}(X)$ linearly embeds into $\mathcal{E}(\ell_\infty)$; then use Proposition 5.1.

(iv) $\Rightarrow$ (v). Take $Z = \mathcal{E}(X)$ (cf. [9]).

(v) $\Rightarrow$ (vi) and (vii) is trivial. Clearly (vi) $\Rightarrow$ (i) and (vii) $\Rightarrow$ (ii). \(\blacksquare\)

Let $|I| = c$. Then $\ell_1(I)$ linearly isometrically embeds into $\ell_\infty$. On the other hand $c_0(I) 2$-Lipschitz embeds into $\ell_\infty$ by the result of [1]. The following theorem shows that we also have a Lipschitz embedding for spaces such as $\ell_p(I)$ where $1 < p < \infty$.

Theorem 5.4. Let $X$ be a Banach space with a 1-unconditional basis $(e_i)_{i \in I}$ where $|I| \leq c$. Then $X$ 2-Lipschitz embeds in $\ell_\infty$.

Proof. We will consider the case when $I = \mathbb{R}$, i.e. the basis is indexed by $\mathbb{R}$. Let $(e_i^*)_{i \in \mathbb{R}}$ denote the biorthogonal functionals. If $x \in X$ we write $x(t) = e_i^*(x)$. Suppose $a, b, c \in \mathbb{Q}^n$ for some $n \in \mathbb{N}$. We write $a = (a_1, \ldots, a_n)$ and so on; we denote by $-a$ the sequence $(-a_1, \ldots, -a_n)$. We then define
a subset $U(a, b, c) \subset \mathbb{R}^n$ by $(t_1, \ldots, t_n) \in U(a, b, c)$ if $b_j < t_j < c_j$ for $j = 1, \ldots, n$, $t_1 < \cdots < t_n$ and 

$$\left\| \sum_{j=1}^{n} a_j e_{t_j}^* \right\|_{X^*} \leq 1.$$ 

Note that the set $U(a, b, c)$ can be empty.

For $t \in \mathbb{R}$ we write $t_+ = \max(t, 0)$ and $t_- = t_+ - t$. We next define $f_{a, b, c} : X \to \mathbb{R}$ by $f_{a, b, c} \equiv 0$ if $U(a, b, c)$ is empty and otherwise 

$$f_{a, b, c}(x) = \sup \left\{ \sum_{j=1}^{n} (a_j x(t_j))_+ : (t_1, \ldots, t_n) \in U(a, b, c) \right\}.$$ 

It follows from the definition that each $f_{a, b, c}$ is Lipschitz with constant at most one and $f_{a, b, c}(0) = 0$.

Suppose $x, y \in X$ and $\epsilon > 0$. Then we may find a finite subset $\{t_1, \ldots, t_n\}$ of $\mathbb{R}$ with $t_1 < \cdots < t_n$ so that 

$$\left\| x - \sum_{j=1}^{n} x(t_j) e_{t_j} \right\| < \epsilon/6, \quad \left\| y - \sum_{j=1}^{n} y(t_j) e_{t_j} \right\| < \epsilon/6.$$ 

Then pick $a_1, \ldots, a_n \in \mathbb{Q}$ so that $\left\| \sum_{j=1}^{n} a_j e_{t_j}^* \right\| \leq 1$ and 

$$\sum_{j=1}^{n} a_j (x(t_j) - y(t_j)) > \|x - y\| - \epsilon/3.$$ 

Finally pick $b_j, c_j$ rationals such that $b_1 < t_1 < c_1 < b_2 < t_2 < \cdots < t_n < c_n$.

It is clear that 

$$\left| f_{a, b, c}(x) - \sum_{j=1}^{n} (a_j x(t_j))_+ \right| < \epsilon/6$$ 

and 

$$\left| f_{-a, b, c}(x) - \sum_{j=1}^{n} (a_j x(t_j))_- \right| < \epsilon/6.$$ 

Thus 

$$\left| f_{a, b, c}(x) - f_{-a, b, c}(x) - \sum_{j=1}^{n} a_j x(t_j) \right| < \epsilon/3.$$ 

Similarly 

$$\left| f_{a, b, c}(y) - f_{-a, b, c}(y) - \sum_{j=1}^{n} a_j y(t_j) \right| < \epsilon/3.$$ 

Hence 

$$f_{a, b, c}(x) - f_{a, b, c}(y) - f_{-a, b, c}(x) + f_{-a, b, c}(y) > \|x - y\| - \epsilon.$$
and either
\[ |f_{a,b,c}(x) - f_{a,b,c}(y)| > \frac{1}{2}(\|x - y\| - \epsilon) \]
or
\[ |f_{-a,b,c}(x) - f_{-a,b,c}(y)| > \frac{1}{2}(\|x - y\| - \epsilon). \]

This shows that the map \( F(x) = (f_{a,b,c}(x))_{(a,b,c)\in\cup_n(\mathbb{Q}_n)^3} \) defines a 2-Lipschitz embedding of \( X \) into \( \ell_\infty \).

**Remark.** The constant 2 is optimal for \( c_0(\mathbb{R}) \). Indeed suppose \( F : c_0(\mathbb{R}) \rightarrow \ell_\infty \) satisfies \( F(0) = 0 \) and
\[ \|x - y\| \leq \|F(x) - F(y)\| \leq \lambda \|x - y\|, \; x, y \in c_0(\mathbb{R}), \]
where \( \lambda < 2 \). We may define the sets \( A_t = \{ n : |F(e_t)_n - F(-e_t)_n| > \lambda \} \). It is then clear that the sets \( \{ A_t : t \in \mathbb{R} \} \) are disjoint and non-empty, producing a contradiction. It would, however, be interesting to know the best constant for an embedding of \( \ell_p(I) \) where \( 1 < p < \infty \).

We also note that we do not know whether every reflexive space (or even WCG space) of density character \( \epsilon \) (or even \( \aleph_1 \)) can be Lipschitz embedded into \( \ell_\infty \).

### 6. Applications of pull-back constructions.

Our final section is inspired by a result of Cabello Sánchez and Castillo \[5\]. We will need a preliminary lemma:

**Lemma 6.1.** Let \( 0 \rightarrow X \rightarrow Y_1 \rightarrow Z_1 \rightarrow 0 \) be a short exact sequence of Banach spaces and suppose there is a Lipschitz selection \( \varphi : Z_1 \rightarrow Y_1 \) of the quotient map \( Y_1 \rightarrow Z_1 \). Let \( T : Z \rightarrow Z_1 \) be a bounded linear operator and let \( 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \) be the short exact sequence obtained by the pull-back construction. Then there is a Lipschitz selection \( \psi : Z \rightarrow Y \) of the quotient map \( Y \rightarrow Z \). Thus \( Y \) is Lipschitz isomorphic to \( X \oplus Z \).

**Proof.** We have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & X & \rightarrow & Y_1 & \rightarrow & Z_1 & \rightarrow & 0 \\
& & \uparrow T & & \uparrow I_x & & \uparrow T & & \\
0 & \rightarrow & X & \rightarrow & Y & \rightarrow & Z & \rightarrow & 0 \\
\end{array}
\]

We may define a map \( \theta : Z \rightarrow Y \), which is a selection for the quotient \( Y \rightarrow Z \) and such that \( T \circ \theta = \varphi \circ T \). We must show that \( \theta \) is Lipschitz. Assume \( \varphi \) is Lipschitz with constant \( L \). If \( z_1, z_2 \in Z \) we may determine \( y_1, y_2 \in Y \) with \( \|y_1 - y_2\| \leq 2\|x_1 - x_2\| \) and \( QZy_1 = z_1, \; QZy_2 = x_2 \) where \( QZ \) is the quotient map. Then \( \theta(z_j) - y_j \in X \) and \( \theta(z_j) - y_j = \varphi(Tz_j) - Ty_j \)
for $j = 1, 2$. Thus

$$
\|\theta(z_1) - \theta(z_2)\| \leq 2\|z_1 - z_2\| + \|\theta(z_1) - y_1 - \theta(z_2) + y_2\|
$$

$$
\leq 2\|z_1 - z_2\| + \|\varphi(Tz_1) - \varphi(Tz_2)\| + \|\tilde{T}(y_1 - y_2)\|
$$

$$
\leq (2 + \|\tilde{T}\| + L\|T\|)\|z_1 - z_2\|.\]

**Remark.** The same proof works for uniformly continuous selections. A simpler proof works for push-outs.

We now follow the argument of Cabello Sánchez and Castillo [5]. We start with the short exact sequence constructed by Johnson and Lindenstrauss [11] and later used by Aharoni and Lindenstrauss [1] as a counterexample to the Lipschitz isomorphism problem for non-separable Banach spaces:

$$
0 \rightarrow c_0 \rightarrow JL_{\infty} \rightarrow c_0(I) \rightarrow 0
$$

where $|I| = c$, $Y$ embeds in $\ell_\infty$ and there is a 2-Lipschitz selection of the quotient map of $Y$ onto $c_0(I)$. If we consider the inclusion map $T : \ell_2(I) \rightarrow c_0(I)$ and perform the pull-back construction we obtain a short exact sequence

$$
0 \rightarrow c_0 \rightarrow JL_2 \rightarrow \ell_2(I) \rightarrow 0.
$$

Here $JL_2$ denotes another Johnson–Lindenstrauss space first constructed in [11], as an example in the theory of WCG spaces. $JL_2$ is a not a WCG space and no non-separable subspace of $JL_2$ linearly embeds $\ell_\infty$. An immediate conclusion, using Theorem 5.4 and Lemma 6.1, is:

**Proposition 6.2.** $JL_2$ is Lipschitz isomorphic to $c_0 \oplus \ell_2(I)$ and Lipschitz embeds into $\ell_\infty$.

Continuing we note (following [5]) that there is a quotient map $S : \ell_\infty \rightarrow \ell_2(I)$ (first proved by Rosenthal [19], see also [10, p. 141]). Performing the pull-back operation again we obtain the commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & c_0 & \rightarrow & JL_{\infty} & \rightarrow & c_0(I) & \rightarrow & 0 \\
& & & & \uparrow l_{c_0} & & \uparrow T & & \uparrow 0 \\
0 & \rightarrow & c_0 & \rightarrow & JL_2 & \rightarrow & \ell_2(I) & \rightarrow & 0 \\
& & & & \uparrow l_{c_0} & & \uparrow \tilde{S} & & \uparrow S \\
0 & \rightarrow & c_0 & \rightarrow & CC & \rightarrow & \ell_\infty & \rightarrow & 0
\end{array}
$$

**Proposition 6.3.** The Banach space $CC$ is linearly isomorphic to a closed linear subspace of $\ell_\infty$ and Lipschitz isomorphic to $\ell_\infty \oplus c_0$. However it is not linearly isomorphic to $\ell_\infty \oplus c_0$. 
Proof. CC is isomorphic to a subspace of $\ell_\infty$ because $JL_\infty$ embeds into $\ell_\infty$ and $CC$ is a subspace of $JL_\infty \oplus \ell_\infty$. The fact that it is not isomorphic to $\ell_\infty \oplus c_0$ is shown in [5], but we include the details. Thus $JL_2$ is a quotient of $CC$ but every operator from $\ell_\infty$ to $JL_2$ is weakly compact ([20]); but since $JL_2$ is not WCG it cannot be a quotient of $\ell_\infty \oplus c_0$. However $CC$ is Lipschitz isomorphic to $\ell_\infty \oplus c_0$ by an application of Lemma 6.1.

Remark. This proposition shows that $\ell_\infty \oplus c_0$ does not have unique Lipschitz structure. Other examples of $C(K)$-spaces with non-unique Lipschitz structure have been given in [1], [6] and [2]. We do not know if $CC$ (which is an $L_\infty$-space) is linearly isomorphic to any $C(K)$-space.

We do not know, however, if $\ell_\infty$ has unique Lipschitz structure.

We conclude by showing that Proposition 6.2 can be generalized considerably:

Theorem 6.4. Let $X$ be a Plichko space containing a subspace isomorphic to $c_0$ and a non-separable subspace $X_0$ with $\aleph_0 < w^*-\text{dens } X_0 \leq \text{dens } X_0 \leq c$. Then $X$ fails to have unique Lipschitz structure.

In particular if $X$ is a non-separable WLD space containing a subspace isomorphic to $c_0$ then $X$ fails to have unique Lipschitz structure.

Proof. If $X$ is Plichko, then $X$ has the separable complementation property [18] and so every copy of $c_0$ in $X$ is complemented. In particular $X$ is linearly isomorphic to $X \oplus c_0$.

Let $(x_j, x_j^*)_{j \in J}$ be a Markushevich basis for $X$. Then there is a subset $I_0 \subset J$ with $\aleph_0 < |I| \leq c$ such that $X_0 \subset [x_i]_{i \in I_0}$. Let $I \supset I_0$ be a set with $|I| = c$. Define the map $T : X \to c_0(I)$ by $Tx(i) = x_i^*(x)$ for $i \in I_0$ and $Tx(i) = 0$ for $i \in I \setminus I_0$.

We now construct the pull-back from the Johnson–Lindenstrauss sequence $0 \to c_0 \to JL_\infty \to c_0(I) \to 0$ as before:

$$
\begin{array}{cccccc}
0 & \rightarrow & c_0 & \rightarrow & JL_\infty & \rightarrow & c_0(I) & \rightarrow & 0 \\
\uparrow_{I_{c_0}} & & \uparrow_T & & \downarrow_{\tilde{T}} & & \downarrow_T \\
0 & \rightarrow & c_0 & \rightarrow & Z & \rightarrow & X & \rightarrow & 0
\end{array}
$$

We claim that the pull-back sequence cannot split. Indeed, if $c_0$ is complemented in $Z$ it follows there exists a bounded operator $S : X \to JL_\infty$ so that $QS = T$ where $Q : JL_\infty \to c_0(I)$ is the quotient map. The linear functionals $(x_i^*)_{i \in I_0}$ separate the points of $X_0$ and so $T|_{X_0}$ is injective: hence $S$ is injective and thus $w^*-\text{dens } X_0 = \aleph_0$, which gives a contradiction. Hence $Z$ fails (SCP) and is not linearly isomorphic to $X$. However $Z$ is Lipschitz isomorphic to $X \oplus c_0$ and hence to $X$ using Lemma 6.1.
If $X$ is WLD we may take $X_0$ to be any subspace with $\text{dens } X_0 = \aleph_1$. Then $X_0$ is also WLD and hence (see [10, p. 181]) $\text{dens } X_0 = w^*-\text{dens } X_0 = \aleph_1$.

**Remark.** Thus, for example $X = c_0 \oplus Y$, where $Y$ is any non-separable reflexive space, fails to have unique Lipschitz structure. The theorem also applies to $C[1,\omega_1]$ which is a Plichko space since $\text{dens } C[1,\omega_1] = w^*-\text{dens } C[1,\omega_1] = \aleph_1$. Thus $C[1,\omega_1]$ fails to have unique Lipschitz structure.

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