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Distances between Banach spaces

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Abstract. The main object of the paper is to study the distance between Banach spaces introduced by Kadets. For Banach spaces X and Y, the Kadets distance is defined to be the infimum of the Hausdorff distance $d(B_X, B_Y)$ between the respective closed unit balls over all isometric linear embeddings of X and Y into a common Banach space Z. This is compared with the Gromov-Hausdorff distance which is defined to be the infimum of $d(B_X, B_Y)$ over all isometric embeddings into a common metric space Z. We prove continuity type results for the Kadets distance including a result that shows that this notion of distance has applications to the theory of complex interpolation.

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1. Introduction

The standard notion of distance between two Banach spaces is the *Banach-Mazur distance* which is defined by

$$d_{BM}(X, Y) = \log \inf\{\|T\|\|T^{-1}\|: T: X \to Y \text{ is an isomorphism}\}.$$

(It is usual to omit the logarithm, but for consistency we will include it). The Banach-Mazur distance is only finite when X and Y are isomorphic. The main object of this paper is to study a measure of distance we call the Kadets distance and certain related notions of distance. The Kadets distance has natural applications in interpolation theory which we explain.

We recall that if Z is a Banach space and X and Y are closed subspaces of Z the *gap* or *opening* $\Lambda(X, Y)$ is defined as the Hausdorff distance between the closed unit balls B_X and B_Y of X and Y i.e.

$$\Lambda(X, Y) = \max\{\sup_{y \in B_Y} d(y, B_X), \sup_{x \in B_X} d(x, B_Y)\}.$$

If X and Y are arbitrary Banach spaces we define the *Kadets distance*

$$d_K(X,Y) = \inf_{Z,U,V} \Lambda(UX, VY)$$

where the infimum is taken over all Banach spaces Z and all linear isometric embeddings $U: X \to Z$ and $V: Y \to Z$.

This distance was apparently introduced by Kadets [15] who proved for example that $\lim_{p\to 2} d_K(l_p, l_2) = 0$. However the basic idea seems to be implicit in some earlier work of Krein, Krasnoselskii and Milman [20], Brown [5] and Douady [10]. The second author studied the notion in [22] and proved that d_K satisfies the triangle law but that there are non-isomorphic Banach spaces X and Y for which $d_K(X,Y) = 0$ (thus d_K is a "pseudo-metric"). In the same paper there is a completeness result: if (X_n) is a sequence of Banach spaces Cauchy with respect to d_K then there is a Banach space X so that $\lim_{n\to\infty} d_K(X_n, X) = 0$.

There is a series of papers studying the general problem of the identifying properties which are stable under small perturbations in the Kadets distance (see [2], [5], [10], [20], [22], [23], [24] and [25]). Precisely a property $\mathscr{P}$ is called *stable* if there exists $\varepsilon > 0$ so that if X has $\mathscr{P}$ and $d_K(X,Y) < \varepsilon$ then Y has $\mathscr{P}$. We refer to the survey article [24] Chapter 6: a partial list of stable properties includes reflexivity, super-reflexivity, B-convexity (nontrivial Rademacher type), the Banach-Saks property, the alternate-signs Banach Saks property and the property of not containing l_1.

The Kadets distance is clearly related to the notion of Gromov-Hausdorff distance between metric spaces (see [11], [27]; the precise definition is given in Section 2). It is natural to introduce the Gromov-Hausdorff distance between two Banach spaces X and Y as

$$d_{GH}(X,Y) = \inf_{Z,U,V} d(UB_X, VB_Y)$$

where the infimum is taken over all isometries $U: X \to Z$ and $V: Y \to Z$ into a common metric space Z (here $d(UB_X, VB_Y)$ is the Hausdorff distance between UB_X and VB_Y.) Thus the Gromov-Hausdorff distance is simply the nonlinear analogue of the Kadets distance. It is not difficult to see that this definition coincides with computing the standard Gromov-Hausdorff distance between the unit balls B_X, B_Y as metric spaces. Let us remark at this point that a related global notion was considered in [4] and [12].

In this paper we first compare these two notions of distance. It is worth pointing out that one must distinguish between the case of complex scalars and real scalars, because there are examples ([3], [17], [31]) of complex Banach spaces which are real-isometric but not even complex-isomorphic. Gromov-Hausdorff distance cannot distinguish complex structures.

We show that (for real scalars) while Gromov-Hausdorff distance is not equivalent to the Kadets distance, the two notions are equivalent if one restricts to Banach spaces which are nice enough. For example if X is B-convex (i.e. has non-trivial Rademacher type) or if X^* embeds into an $\mathscr{L}_1$-space then $d_{GH}(X_n, X) \to 0$ implies $d_K(X_n, X) \to 0$. If X is isomorphic to either c_0 or l_∞ then $d_{GH}(X_n, X) \to 0$ implies $d_{BM}(X_n, X) \to 0$. On the other hand $d_{GH}(l_p, l_1) \to 0$ as $p \to 1$ while $d_K(l_p, l_1) = 1$ if $p > 1$. The precise identification of the class on which the two distances are equivalent is related to the notion of a $\mathscr{K}$-space introduced in [16] (see [18]) and to the theory of twisted sums.

In fact for real scalars, Gromov-Hausdorff distance is equivalent to a notion of distance analogous to the Kadets distance but allowing the superspace Z to be a quasi-Banach space.

These results are developed in Section 3, after some preliminary results in Section 2. In Section 4, we then apply our techniques to prove a number of continuity-type results for the Kadets metric. For example we show that in an obvious sense the map $X \to X^*$ is continuous for the Kadets metric, and even more one has $d_K(X^*, Y^*) \leq 2\, d_K(X, Y)$ for any pair of Banach spaces. We also show that if (X_0, X_1) is a complex Banach couple and $X_\theta = [X_0, X_1]_\theta$ are the spaces obtained by the (Calderoń) method of complex interpolation (cf. [6]) then the map $\theta \to X_\theta$ is continuous for the Kadets distance for $0 < \theta < 1$. This result is closely related to recent work on uniform homeomorphisms between the unit balls of two Banach spaces using complex interpolation methods (cf. [8]). We give precise estimates here and obtain the estimate for $1 < p, q < \infty$,

$$d_K(l_p, l_q) \leq 2\, \frac{\sin(\pi|1/p - 1/q|/2)}{\sin(\pi(1/p + 1/q)/2)}$$

which improves earlier estimates (see [15], [22]). We remark that in [23] or [24] (pp. 292, 303) there is a lower estimate $d_K(l_p, l_q) \geqq \frac{1}{2}(2^{1/p} - 2^{1/q})$.

Finally in Section 5, we make some remarks on the topology of the pseudo-metric space of all Banach spaces with a given density character with either notion of distance. We point out that results on stability or openness of some property lead automatically to results on complex interpolation spaces, and also show that the continuity results of the previous section lead to new stable or open properties. We identify the component of l_1 for the Kadets distance and raise the question of identifying the components of l_2 and c_0. We do not know if the set of separable Banach spaces is connected for the Gromov-Hausdorff distance. We also show that if $1 < p \neq 2 < \infty$ the set of spaces isomorphic to l_p is non-separable for both notions of distance.

2. Gromov-Hausdorff distance and Kadets distance

We first recall the notion of Gromov-Hausdorff distance between metric spaces. It will be convenient to expand the definition to include pseudo-metric spaces. We recall that if M is a set, a pseudo-metric on M is a map $d: M \times M \to [0, \infty)$ which