EXTRACTA MATHEMATICAE Vol. **22**, Núm. 2, 179 – 189 (2007)
Satellite conference on Banach spaces · ICM2006

Isometric Embeddings and Universal Spaces

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Received March 20, 2007

Abstract: We show that if a separable Banach space Z contains isometric copies of every strictly convex separable Banach space, then Z actually contains an isometric copy of every separable Banach space. We prove that if Y is any separable Banach space of dimension at least 2, then the collection of separable Banach spaces which contain an isometric copy of Y is analytic non Borel.

Key words: isometrically universal space, strictly convex norm, well-founded tree.

AMS *Subject Class.* (2000): 46B04, 54H05.

1. Introduction

Many natural isometric properties (P) of norms on Banach spaces are hereditary. For such a hereditary property, the question naturally arises to know whether there is a separable space U with (P) such that the class of spaces with (P) coincide with the class of subspaces of U. A negative answer is usually easy to obtain when (P) is related with some modulus. When it is not so, topological considerations enter into play, as in [6] where a negative answer is obtained when (P) denotes strict convexity (solving a problem from [10]).

The purpose of this note is to improve on the result of [6] by showing that a separable Banach space which contains an isometric copy of every strictly convex separable space actually contains an isometric copy of every separable Banach space. A by-product of our construction is that the class of spaces which contain isometrically a given space Y is as complicated as it looks at first sight, for every space Y.

The proof consists into constructing a tree space such that every branch supports an isometrically universal space, and every well-founded subtree supports a strictly convex space. Since by Hurewicz's theorem the set of well-founded subtrees is coanalytic non Borel, the conclusion follows from the fact (from [3]) that the class of subspaces of a given separable Banach space is analytic.

It should be noted that this topological approach has been recently used (and much deepened) in isomorphic theory, where important "converse" statements have been obtained. More precisely, it was shown in [1] that several analytic families of Banach spaces have non trivial universal spaces. This new approach from [1] has been used in particular in [5] to provide an alternative proof (and an improvement) of [11] and the solution of a problem of H.P. Rosenthal [13].

The notation we use is standard. We should point out however that in the tree construction, the set of natural numbers is denoted by ω, in agreement with the usual notation in descriptive set theory.

2. Results

Here is the main technical result of this note.

Theorem 2.1. *There exists a Banach space $E(\omega^{<\omega})$ with a basis indexed by finite sequences on natural numbers, such that if $\mathbf{T}$ is a well-founded subtree of $\omega^{<\omega}$, then the subspace $E(\mathbf{T})$ consisting of vectors supported by $\mathbf{T}$ is strictly convex, and if $\mathbf{T}$ is not well-founded the subspace $E(\mathbf{T})$ contains an isometric copy of every separable Banach space.*

Proof. Our argument partly follows the lines of the proof of Propositions 3 and 5 in [6]. There are however two significant changes: in [6] the space supported by the infinite branches of $\omega^{<\omega}$ is isomorphic to l_1 and this is quite crucial in the argument. Also, strict convexity is replaced in [6] by the stronger property of uniform convexity in every direction, and a compactness argument allows to dispense with this requirement in the present work.

The following proposition provides the subspace of $E(\omega^{<\omega})$ supported by each infinite branch.

Proposition 2.2. *Let X be a Banach space with a monotone basis. There exists a Banach space F with a monotone basis, which contains an isometric copy of X, and such that*

(1) *If (U_n) denotes the sequence of partial sum operators associated with the basis (f_n) of F and (f_n^*) is the dual basis, there exist a sequence (c_n)*

of strictly positive real numbers such that, for all $y \in F$,

$$\|U_{n+1}y\|^2 \geq \|U_n y\|^2 + c_{n+1}^2 |f_{n+1}^*(y)|^2 .$$

(2) *the norm of* F *is strictly convex on the linear span of the basis vectors.*

Proof. Let (e_n) be a monotone basis of the Banach space X. We have to construct an isometric linear map T from X into a Banach space with monotone basis F, such that the norm of F is strictly convex on the space of vectors with finite support and satisfies a weighted l^2 lower estimate. The vectors (e_n) will be mapped on vectors in F with disjoint infinite support. The map T and the norm on F will be simultaneously constructed by induction.

We pick two strictly positive decreasing sequences (a_n) and (σ_n). The choices $a_n = 10^{-1}.2^{-n}$ and $\sigma_n = 10^{-2}n^{-1}2^{-2n}$ will suffice. We require that

- $\sum a_n < 1\,,\ \ a_1 < 1/10\,,$
- $\sigma_n < a_n\,,$
- $\sigma_n \left(\sum_{j=1}^n \xi_j^2\right)^{1/2} \leq \frac{1}{2}(a_n - a_{n+1}) \left\|\sum_{j=1}^n \xi_j e_j\right\|_X\,, \qquad \xi_1, \ldots, \xi_n \in \mathbb{R}\,.$

Consider the sets $\mathbb{D}$ of pairs (n,k) where $1 \leq k \leq n$, ordered by the lexicographic ordering: $(n,k) \leq (n',k')$ if $n < n'$ or $n = n'$ and $k \leq k'$. This is of course a copy of $\mathbb{N}$ with (n,k) corresponding to $\frac{1}{2}n(n-1)+k$. We will construct a norm on $c_{00}(\mathbb{D})$. Let (f_{nk}) denote the basis vectors.

Let $E_n = [e_1, \ldots, e_n]$ and $F_n = [f_{11}, f_{21}, f_{22}, \ldots, f_{nn}]$. We define a linear map $T_n : E_n \to F_n$ by

$$T_n e_k = \frac{1 - a_k}{\left(1 - \sigma_k^2\right)^{1/2}} f_{kk} + 4 \sum_{j=k+1}^{n} a_{j-1} f_{jk}\,.$$

Denote by S_n the partial sum operators on X and $S_{n,k}$ the partial sum operators on $c_{00}(\mathbb{D})$. Let $\|\cdot\|_2$ be the standard Euclidean norm on $c_{00}(\mathbb{N})$ or $c_{00}(\mathbb{D})$.

We will construct on each F_n a monotone norm p_n with the properties:

- $p_n(T_n x) = (1 - a_n)\|x\|_X\,, \qquad x \in E_n\,,$
- $p_n(f_{jk}) = \left(1 - \sigma_n^2\right)^{1/2}\,, \qquad 1 \leq k \leq j \leq n\,,$
- $p_{n+1}(y)^2 = p_n(y)^2 + (\sigma_n^2 - \sigma_{n+1}^2)\|y\|_2^2\,, \qquad y \in F_n\,.$

To start the induction define $p_1(f_{11}) = (1-\sigma_1^2)^{1/2}$.

Now suppose p_n has been defined and we wish to define p_{n+1}. Let

$$q_n(y) = \big(p_n(y)^2 + (\sigma_n^2 - \sigma_{n+1}^2)\|y\|_2^2\big)^{1/2}\,, \qquad y \in F_n\,.$$

Note that for $x \in E_n$

$$\begin{aligned} q_n(T_n x) &\le p_n(T_n x) + \sigma_n \|T_n x\|_2 \\ &\le (1-a_n)\|x\|_X + 2\sigma_n\|x\|_2 \\ &\le (1-a_{n+1})\|x\|_X\,. \end{aligned}$$

Thus

$$(1-a_n)\|x\|_X \le q_n(T_n x) \le (1-a_{n+1})\|x\|_X\,. \tag{1}$$

Now if $y \in F_{n+1}$ we will define $p_{n+1}(y)$ as the minimum of

$$q_n(u) + (1-a_{n+1})\sum_{j=1}^{n+1}\|v_j\|_X + \big(1-\sigma_{n+1}^2\big)^{1/2}\sum_{j=1}^{n+1}|\xi_j|$$

over all $u \in F_n$, $v_1, \dots, v_{n+1} \in E_{n+1}$, $\xi_1, \dots, \xi_{n+1} \in \mathbb{R}$ so that

$$y = u + \sum_{j=1}^{n+1} S_{n+1,j} T_{n+1} v_j + \sum_{j=1}^{n+1} \xi_j f_{n+1,j}\,. \tag{2}$$

The three terms of the above equation are shown below to provide optimal representations (where the minimum is attained) for different types of vectors in F_{n+1}: if $y \in F_n$ then $y = u$ is optimal, if $y = T_{n+1}(x) \in T_{n+1}(E_{n+1})$ then $v_{n+1} = x$ provides an optimal representation, and if $y = f_{n+1,k}$ then $\xi_k = 1$ is optimal (with everything else equal to 0 in each case).

First observe that if y has a representation of type (2) then

$$S_{nn}(y) = u + T_n\left(\sum_{j=1}^{n+1} v_j\right)$$

so that

$$q_n(S_{nn}(y)) \le q_n(u) + (1-a_{n+1})\sum_{j=1}^{n+1}\|v_j\|_X\,,$$

and hence we deduce that

$$q_n(y) = p_{n+1}(y)\,, \qquad y \in F_n\,, \tag{3}$$

and

$$p_{n+1}(S_{kj}(y)) \le p_{n+1}(y)\,, \qquad y \in F_{n+1}\,, \quad 1 \le k \le n\,, \quad 1 \le j \le k\,. \tag{4}$$

It follows easily from (4) and the definition of p_{n+1} that it is in fact monotone on F_{n+1}. It also follows from (3) that

$$p_{n+1}(f_{jk}) = \left(1 - \sigma_{n+1}^2\right)^{1/2}\,, \qquad 1 \le j \le n\,, \quad 1 \le k \le j\,.$$

We now must prove that

$$p_{n+1}(T_{n+1}x) = (1 - a_{n+1})\|x\|_X \qquad x \in E_{n+1}.$$

Suppose $y = T_{n+1}x$ has an optimal representation in the form of (2). Then

$$T_n x = S_{nn} y = u + \sum_{j=1}^{n+1} T_n v_j\,.$$

Hence

$$u = T_n \left(x - \sum_{j=1}^{n+1} v_j \right).$$

Let $z = \sum_{j=1}^{n+1} v_j$. It follows from optimality that we have $x - z \in E_n$. Thus

$$1 - a_n)\|x - z\|_X \le q_n(u) \le (1 - a_{n+1})\|x - z\|_X\,.$$

Let us write

$$w = \sum_{j=1}^{n+1} S_j v_j\,.$$

Again, $x - w \in E_n$ and

$$y = T_n(x - w) + T_{n+1}w + \sum_{j=1}^{n+1} \xi_j f_{n+1,j}\,.$$

It follows that

$$q_n(u) \le q_n(T_n(x - w))\,.$$

This implies that

$$(1-a_n)\|x-z\|_X \le (1-a_{n+1})\|x-w\|_X\,.$$

Now

$$(T_{n+1}-T_n)(x-w)=\sum_{j=1}^{n+1}\xi_j f_{n+1,j}$$

so that

$$\sum_{j=1}^{n+1}|\xi_j| \ge 4a_n\|x-w\|_X\,.$$

Hence

$$\begin{aligned}
p_{n+1}(y) &\ge (1-a_n)\|x-z\|_X+(1-a_{n+1})\|z\|_X+4\left(1-\sigma_{n+1}^2\right)^{1/2}a_n\|x-w\|_X\\
&\ge (1-a_n)\|x-z\|_X+(1-a_{n+1})\|z\|_X+2a_n\frac{1-a_n}{1-a_{n+1}}\|x-z\|_X\\
&\ge (1-a_n)\|x-z\|_X+(1-a_{n+1})\|z\|_X+a_n\|x-z\|_X\\
&\ge (1-a_{n+1})\|x\|_X\,.
\end{aligned}$$

This shows that $p_{n+1}(T_{n+1}x)\ge(1-a_{n+1})\|x\|_X$. The converse inequality is automatic.

Finally we must show that

$$p_{n+1}(f_{n+1,k})=\left(1-\sigma_{n+1}^2\right)^{1/2}\,,\qquad 1\le k\le n+1\,.$$

Then if $k<n+1$, we assume an optimal representation of $y=f_{n+1,k}$ in form (2) and we have

$$1=f_{n+1,k}^*\left(\sum_{j=k}^{n+1}T_{n+1}v_j\right)+\xi_k$$

or

$$1=4a_n e_k^*\left(\sum_{j=k}^{n+1}v_j\right)+\xi_k$$

where $4a_n<\frac{1}{2}$. Thus

$$1\le\frac{1}{2}\left\|\sum_{j=k}^{n+1}v_j\right\|_X+|\xi_k|\,.$$

If $k < n+1$ we therefore have

$$1 \leq \frac{1}{2}\sum_{j=1}^{n+1} \|v_j\|_X + |\xi_k| \leq \left(1-\sigma_{n+1}^2\right)^{-1/2} p_{n+1}(f_{n+1,k})$$

from which it follows that $p_{n+1}(f_{n+1,k}) = \left(1-\sigma_{n+1}^2\right)^{1/2}$.

If $k = n+1$ then $f_{n+1,n+1} = (1-a_{n+1})^{-1}\left(1-\sigma_{n+1}^2\right)^{1/2} T_{n+1}e_{n+1}$ and so

$$p_{n+1}(f_{n+1,n+1}) = \left(1-\sigma_{n+1}^2\right)^{1/2} .$$

This completes the inductive construction.

Now we can define a norm on $c_{00}(\mathbb{D})$ by letting

$$\|x\| = \sup_{k\geq m} p_k(x)\,, \qquad x \in F_m\,.$$

Then if $x \in F_m$ we have

$$\|x\| = \left(p_m(x)^2 + \sigma_m^2\|x\|_2^2\right)^{1/2} .$$

Clearly this norm has all the desired properties including strict convexity on the linear span (but not the closed linear span) of the basis vectors. We define $T : X \to F$ (where F is the closure on $c_{00}(\mathbb{D})$ in this norm) by

$$Tx = \lim_{n\to\infty} T_n S_n x\,.$$

Then it is easy to verify that T is an isometry.

We switch to labelling $\mathbb{D}$ as $\mathbb{N}$ and the $S_{n,k}$ as U_n. The c_n's are readily obtained from the σ_n's, and this concludes the proof. ∎

Let us mention that in the case $X = C([0,1])$ equipped with its natural norm (which is of course sufficient for proving the proposition), the space X cannot contain any of the subspaces F_n. This follows from the fact that contractively complemented subspaces of X are not strictly convex. It is therefore quite natural that every nonzero vector from X has infinite support in F.

We now proceed to the tree construction, where we follow the lines of [6] but simplify it with a compactness argument.

Let $\omega^{<\omega}$ denote the set of finite sequences of natural numbers, and ω^ω the set of infinite sequences. Each $\sigma \in \omega^\omega$ is seen as an infinite branch of the tree

$\omega^{<\omega}$. Set $X = C([0,1])$ equipped with its natural norm, let F be the space obtained by applying Proposition 2.2 to X, and denote by $\|\,.\,\|_F$ the norm we constructed in the proof. For any $\sigma \in \omega^\omega$, we let T_σ be the operator from $c_{00}(\omega^{<\omega})$ to F defined by

$$T_\sigma(x) = \sum_{s<\sigma} x(s) f_{|s|}\,.$$

We denote $\sigma^* = \{s \in \omega^{<\omega};\ s \not< \sigma\}$ and we define norms on $c_{00}(\omega^{<\omega})$ by

$$\|x\|_\sigma^2 = \sum_{s\in\sigma^*} c_{|s|}^2 (x(s))^2 + \|T_\sigma(x)\|_F^2$$

and take

$$|||x||| = \sup_\sigma \|x\|_\sigma$$

as in [6].

By condition (1) in Proposition 2.2 we can add finite branches $\{s \leq (n_1,\dots,n_k)\}$ (terminating at some node) and the corresponding $\|x\|_\sigma$ in the definition whithout changing the value of $|||x|||$. Now the set of all branches (finite or not) is a compact set in the topology induced by $2^{(\omega^{<\omega})}$ and if $x \in c_{00}$ the map $\sigma \to \|x\|_\sigma$ is continuous (actually, locally constant). By a uniform approximation argument it is also continuous when x is in the closure of c_{00} with respect to $|||\,.\,|||$, which we denote by $E(\omega^{<\omega})$. Hence for all $x \in E(\omega^{<\omega})$ one has

$$|||x||| = \max_\sigma \|x\|_\sigma\,.$$

Now if $\mathbf{T}$ is any subtree of $\omega^{<\omega}$, we denote by $E(\mathbf{T})$ the subspace of $E(\omega^{<\omega})$ consisting of vectors supported by $\mathbf{T}$.

Assume first that $\mathbf{T}$ is well-founded, and pick $x \neq 0$ in $E(\mathbf{T})$. There is σ such that $|||x||| = \|x\|_\sigma$. Since σ meets the subtree $\mathbf{T}$ in a finite branch, condition (2) in Proposition 2.2 shows that the norm $\|\,.\,\|_\sigma$ restricted to $E(\mathbf{T})$ is strictly convex, and it follows that for all $y \neq 0$ in $E(\mathbf{T})$ one has

$$2|||x||| < |||x+y||| + |||x-y|||$$

and we have shown that the norm $|||\,.\,|||$ is strictly convex when restricted to $E(\mathbf{T})$.

Assume now that $\mathbf{T}$ is not well-founded and let σ be an infinite branch of $\mathbf{T}$, It follows from condition (1) in Proposition 2.2 that the map T_σ is an isometry from $E(\sigma)$ onto F, and it follows that when $\mathbf{T}$ is not well-founded then $E(\mathbf{T})$ equipped with the norm $|||\,.\,|||$ contains an isometric copy of X and thus an isometric copy of every separable Banach space.

For asserting that this space $E(\omega^{<\omega})$ satisfies our initial requirements, it now suffices to observe that any increasing map from $\omega^{<\omega}$ to the set of natural numbers provides an ordering of the unit vectors which makes it into a basis of $E(\omega^{<\omega})$. ∎

The next result is of course the motivation for the whole construction. It provides a positive answer to Problems 2 and 3 of [6].

COROLLARY 2.3. *Let Z be a separable Banach space which contains an isometric copy of every strictly convex separable Banach space. Then Z is isometrically universal, that is, Z contains an isometric copy of every separable Banach space.*

Proof. The proof follows the lines of [2], as in [6].

We rely on the frame displayed in [3], and equip the set S of closed linear subspaces of $E(\omega^{<\omega})$ with the Effros-Borel structure, which makes it a standard Borel space. It is easy to check (see Lemma 8 of [6]) that the map $\mathbf{T} \to E(\mathbf{T})$ from the compact set of trees on ω to S is Borel.

The relation of isometric embedding is analytic (see Lemma 7 in [6]) and thus the set A of trees $\mathbf{T}$ such that $E(\mathbf{T})$ is isometric to a subspace of Z is analytic. Our assumption on Z together with Theorem 2.1 implies that every well-founded tree belongs to A. By a classical result (see [9]) the set of well-founded trees is *not* analytic. Hence there is some $\mathbf{T_0} \in A$ which is not well-founded. But then $E(\mathbf{T_0})$ is isometric to a subspace of Z and it is isometrically universal, and this concludes the proof. ∎

Let us recall along these lines that a problem which goes back to S. Rolewicz (Problem IX.9.4 in [12]) asks whether a separable Banach space which contains isometric copies of every *finite dimensional* Banach space is isometrically universal.

We note that the non existence of an isometrically universal space in the class of separable strictly convex spaces, and in some other isometric classes, readily follows from the coanalytic non Borel complexity of the corresponding collection of spaces (see [2], [4], [8]). A number of similar statements to Corollary 2.3, claiming that universal spaces for given classes of norms are actually universal for all separable spaces. are likely to be true.

Corollary 3.3 from [3] asserts in particular that if Y is an infinite dimensional Banach space, the set of separable Banach spaces which contain an isomorphic copy of Y is analytic non Borel. Our next corollary claims that this is true as well in the isometric case, even if Y is finite dimensional. Of course, the word "set" below and the relevant Borel structure refer to Bossard's theory from [3].

Corollary 2.4. *Let Y be a separable Banach space of dimension at least 2. Then the set of separable Banach spaces which contain an isometric copy of Y is analytic non Borel.*

Proof. Assume first that Y is not strictly convex. Then Theorem 2.1 implies that Y isometrically embeds into $E(\mathbf{T})$ if and only if $\mathbf{T}$ is not well-founded, and the set of non well-founded trees is analytic non Borel.

On the other hand, if Y is a 2-dimensional strictly convex space, then it is well-known that Y is not isometric to a subspace of an isometric predual of l_1. Indeed this is an easy consequence of the fact that l_1 equipped with its natural norm has countably many extreme points, through a Baire category argument applied to the sphere of Y.

In particular, if Y is strictly convex of dimension at least 2, then Y is not isometric to a subspace of a $C(K)$ space with K countable. On the other hand Y is isometric to a subspace of $C(L)$ where L is any uncountable metric compact set. Thus the conclusion easily follows from Hurewicz's theorem asserting that the set of uncountable compact subsets of $[0, 1]$ is analytic non Borel for the Hausdorff metric. ∎

A straightforward consequence of Corollary 2.4 is that if Y is a separable Banach space of dimension greater than 1, and X is a separable Banach space which contains an isometric copy of every separable space which does not contain an isometric copy of Y, then X contains an isometric copy of Y. In other words, the class of separable spaces which do not contain an isometric copy of Y has no universal element. This is the isometric version of Corollary 3.6 from [3].

This Corollary 2.4 really means that there is no way to characterize isometric embeddability of Y into a separable Banach space X which would be less complex than simply claiming the existence of the embedding. Note that by [7] the existence of a non necessarily linear isometric embedding from a separable Banach space into a Banach space forces the existence of a linear one.

Acknowledgements

This work has been completed during the Cáceres Conference in September 2006. The authors are glad to thank the organizers, and in particular J.M.F. Castillo, for the pleasant conditions and excellent working atmosphere of this meeting.

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